

LOWER BOUNDS FOR SOME FACTORABLE MATRICES

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We determine the lower bounds for classes of Rhaly matrices, considered as bounded linear operators on ℓ^p . We improve on and provide correct proofs of the results of the first author (1990).

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The following conjecture was posed by Axler and Shields. Let $\{x_n\}$ be a monotone decreasing sequence of nonnegative numbers, C the Cesàro matrix of order one. What is the best constant K for which $\|Cx\|_2 \geq K\|x\|_2$ for all such sequences $\{x_n\}$? Lyons [3] determined that the best constant is $\pi/\sqrt{6}$. This result was extended to ℓ^p spaces for $p > 1$ by Bennett [1]. In [1], Bennett established the following result, where $B(\ell^p)$ denotes the set of bounded linear operators on ℓ^p .

THEOREM 1. *Let $\{x_n\}$ be a monotone decreasing nonnegative sequence, let $A \in B(\ell^p)$ with nonnegative entries, and $1 < p < \infty$. Then*

$$\|Ax\|_p \geq L\|x\|_p, \quad (1)$$

where

$$L^p := \inf_r (r+1)^{-1} \sum_{j=0}^{\infty} \left(\sum_{k=0}^r a_{jk} \right)^p = \inf_r f(r), \quad \text{say.} \quad (2)$$

For $A = C$, the minimum occurs at $f(0)$, which is the sum of the p th power of the first column of C . The proof of the result of Bennett is relatively easy, when contrasted with the task of finding L^p for a particular matrix, or class of matrices.

A factorable matrix is a lower triangular matrix whose nonzero entries a_{nk} can be written in the form $a_n b_k$, where a_n depends only on n and b_k depends only on k . Rhaly [4–6] defined three classes of matrices, all of which are factorable.

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In [8], Rhoades investigated the lower bounds question for the Rhaly matrices, and obtained some partial results. In this paper, we return to the study of the Rhaly matrices, as special cases of factorable matrices. This paper differs from [8] in three important respects. First, one general result is proved, and the theorems of [8] then follow as special cases. Second, the results of [8] are extended to all $p > 1$. Third, as an application of the general procedure developed here, we are able to provide a new proof of [7, Theorem 1] as well as to verify the conjecture that, for the weighted mean methods with $p_n = (n+1)^\alpha$, $\alpha \geq 1$, $L^p = f(0)$.

For any sequence $\{x_n\}$, the forward difference operator Δ is defined by $\Delta x_n = x_n - x_{n+1}$, and $\Delta^{m+1} x_n = \Delta(\Delta^m x_n)$.

THEOREM 2. *Let A be a factorable matrix with positive entries, row sums t_n , and $\{a_n\}$ monotone decreasing. Then sufficient conditions for $f(0) = L^p$ are that*

$$\Delta y_r^p < 0, \quad \Delta^2 y_r^p > 0, \quad (3)$$

$$\Delta^2 \left(\frac{1}{\Delta y_r^p} \right) \leq 0, \quad (4)$$

where $y_r = t_r/a_r$,

$$\lim_{r \rightarrow \infty} \frac{a_{r+1}^p \Delta y_{r+1}^p}{\Delta^2 y_r^p} \geq 0, \quad (5)$$

$$t_0 + 2\Delta y_0^p \sum_{j=1}^{\infty} a_j^p \leq 0. \quad (6)$$

Proof. With

$$\begin{aligned} t_n &= a_n \sum_{k=0}^n b_k, & y_n &= \frac{t_n}{a_n}, \\ f(r) &= \frac{1}{r+1} \sum_{j=0}^{\infty} \left(\sum_{k=0}^r a_{jk} \right)^p \\ &= \frac{1}{r+1} \left[\sum_{j=0}^r \left(\sum_{k=0}^j a_{jk} \right)^p + \sum_{j=r+1}^{\infty} \left(\sum_{k=0}^r a_{jk} \right)^p \right] \\ &= \frac{1}{r+1} \left[\sum_{j=0}^r t_j^p + y_r^p \sum_{j=r+1}^{\infty} a_j^p \right], \end{aligned} \quad (7)$$

$$f(r) - f(r+1) = \frac{1}{(r+1)(r+2)} \sum_{j=0}^r t_j^p - \frac{t_{r+1}^p}{r+2} + \frac{y_r^p a_{r+1}^p}{r+1} + \Delta \left(\frac{y_r^p}{r+1} \right) \sum_{j=r+2}^{\infty} a_j^p.$$

Note that

$$-\frac{t_{r+1}^p}{r+2} + \frac{y_r^p a_{r+1}^p}{r+1} = \frac{y_r^p a_{r+1}^p}{r+1} - \frac{(a_{r+1} y_{r+1})^p}{r+2} = a_{r+1}^p \Delta \left(\frac{y_r^p}{r+1} \right). \quad (8)$$

Thus

$$f(r) - f(r+1) = \frac{1}{(r+1)(r+2)} \sum_{j=0}^r t_j^p + \Delta \left(\frac{y_r^p}{r+1} \right) \sum_{j=r+1}^{\infty} a_j^p. \quad (9)$$

Define

$$\begin{aligned} g(r) &= (r+1)(r+2)[f(r) - f(r+1)] \\ &= \sum_{j=0}^r t_j^p + (r+1)(r+2)\Delta \left(\frac{y_r^p}{r+1} \right) \sum_{j=r+2}^{\infty} a_j^p. \end{aligned} \quad (10)$$

Then

$$\begin{aligned} g(r) - g(r+1) &= -t_{r+1}^p + (r+2) \left[(r+1)\Delta \left(\frac{y_r^p}{r+1} \right) - (r+3)\Delta \left(\frac{y_{r+1}^p}{r+2} \right) \right] \sum_{j=r+2}^{\infty} a_j^p \\ &\quad + (r+1)(r+2)\Delta \left(\frac{y_r^p}{r+1} \right) a_{r+1}^p. \end{aligned} \quad (11)$$

But

$$\begin{aligned} &(r+1)\Delta \left(\frac{y_r^p}{r+1} \right) - (r+3)\Delta \left(\frac{y_{r+1}^p}{r+2} \right) \\ &= (r+1) \left[\frac{y_r^p}{r+1} - \frac{y_{r+1}^p}{r+2} \right] - (r+3) \left[\frac{y_{r+1}^p}{r+2} - \frac{y_{r+2}^p}{r+3} \right] \\ &= y_r^p - \frac{(r+1)y_{r+1}^p}{r+2} - \frac{(r+3)y_{r+1}^p}{r+2} + y_{r+2}^p = \Delta^2 y_r^p, \\ &-t_{r+1}^p + (r+1)(r+2)\Delta \left(\frac{y_r^p}{r+1} \right) a_{r+1}^p \\ &= -(a_{r+1}y_{r+1})^p + (r+1)(r+2) \left(\frac{y_r^p}{r+1} - \frac{y_{r+1}^p}{r+2} \right) a_{r+1}^p \\ &= a_{r+1}^p [-y_{r+1}^p + (r+2)y_r^p - (r+1)y_{r+1}^p] = a_{r+1}^p (r+2)\Delta y_r^p. \end{aligned} \quad (12)$$

Thus

$$g(r) - g(r+1) = (r+2)a_{r+1}^p (\Delta y_{r+1})^p + (r+2)\Delta^2 y_r^p \sum_{j=r+1}^{\infty} a_j^p. \quad (13)$$

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Define

$$\begin{aligned}
 h(r) &= \frac{g(r) - g(r+1)}{(r+2)\Delta^2 y_r^p} = \frac{a_{r+1}^p \Delta y_r^p}{\Delta^2 y_r^p} + \sum_{j=r+2}^{\infty} a_j^p, \\
 h(r) - h(r+1) &= \frac{a_{r+1}^p \Delta y_r^p}{\Delta^2 y_r^p} + a_{r+2}^p \left(1 - \frac{\Delta y_{r+1}^p}{\Delta^2 y_{r+1}^p} \right) \\
 &= \frac{1}{\Delta^2 y_r^p \Delta^2 y_{r+1}^p} [a_{r+1}^p \Delta^2 y_{r+1}^p \Delta y_r^p + a_{r+2}^p \Delta^2 y_r^p (-\Delta y_{r+2}^p)] \\
 &= \frac{\Delta y_r^p \Delta y_{r+2}^p}{\Delta^2 y_r^p \Delta^2 y_{r+1}^p} \left[\frac{a_{r+1}^p \Delta^2 y_{r+1}^p}{\Delta y_{r+2}^p} - a_{r+2}^p \left(1 - \frac{\Delta y_{r+1}^p}{\Delta y_r^p} \right) \right],
 \end{aligned} \tag{14}$$

and $h(r) - h(r+1) \geq 0$ if and only if

$$a_{r+1}^p \left(\frac{\Delta y_{r+1}^p}{\Delta y_{r+2}^p} - 1 \right) - a_{r+2}^p \left(1 - \frac{\Delta y_{r+1}^p}{\Delta y_r^p} \right) \geq 0. \tag{15}$$

Since $\{a_r\}$ is monotone decreasing, it is sufficient to have

$$\frac{\Delta y_{r+1}^p}{\Delta y_{r+2}^p} - 1 \geq 1 - \frac{\Delta y_{r+1}^p}{\Delta y_r^p}; \tag{16}$$

that is,

$$\Delta y_{r+1}^p \left(\frac{1}{\Delta y_r^p} + \frac{1}{\Delta y_{r+2}^p} \right) \geq 2. \tag{17}$$

Using (3), $\Delta y_{r+1}^p > \Delta y_{r+2}^p$ and $\Delta y_{r+1}^p < \Delta y_r^p$. Since $\Delta y_r^p < 0$, the above inequality is equivalent to (4). Thus h is monotone decreasing in r . From (5), h is nonnegative, so g is monotone decreasing in r . From (6), $g(0)$ is negative, so that f is monotone increasing in r . \square

LEMMA 3. Define sequences $\{u(r)\}$ and $\{v(r)\}$ by $u(r) = 1/\Delta v(r)$. Then $\Delta^2 u(r)$ can be written in the form

$$\Delta^2 u(r) = \frac{1}{\Delta v(r)} \left[\frac{2\Delta^2 v(r)\Delta^2 v(r+1)}{\Delta v(r+1)\Delta v(r+2)} - \frac{\Delta^3 v(r)}{\Delta v(r+2)} \right]. \tag{18}$$

Proof. The equation $u(r) = 1/\Delta v(r)$ implies that

$$\Delta u(r)\Delta v(r) + u(r+1)\Delta^2 v(r) = 0, \tag{19}$$

or

$$\Delta u(r) = -\frac{u(r+1)\Delta^2 v(r)}{\Delta v(r)}. \tag{20}$$

Hence

$$\Delta^2 u(r)\Delta v(r) + 2\Delta u(r+1)\Delta^2 v(r) + u(r+2)\Delta^3 v(r) = 0, \tag{21}$$

or

$$\Delta^2 u(r) = \frac{1}{\Delta v(r)} \left[-2\Delta^2 v(r) \left(-\frac{u(r+2)\Delta^2 v(r+2)}{\Delta v(r+1)} \right) - \frac{\Delta^3 v(r)}{\Delta v(r+2)} \right]. \tag{22}$$

□

LEMMA 4. Suppose that $v \in C^3[0, \infty)$. If, for all $0 < t < 1$, $p > 1$, one has

- (a) $v' > 0$,
- (b) $v'' > 0$,
- (c) $2(v'')^2 - v'v''' > 0$,

then $\Delta^2 v(r) \leq 0$.

Proof. Conditions (a) and (b) imply that $\Delta v(r) < 0$ and $\Delta^2 v(r) > 0$. Therefore, from (18), $\Delta^2 u(r) \leq 0$. □

The Rhaly generalized Cesáro matrices [4] are factorable matrices with nonzero entries $a_n = t^n/(n+1)$, $b_k = t^{-k}$, where $0 < t < 1$. If $t = 1$, the matrix reduces to C .

THEOREM 5. Let $p > 1$. Then, for the Rhaly generalized Cesáro matrices, $L^p = f(0)$ for $t_0 \leq t < 1$, where t_0 satisfies

$$1 - 2 \left[\left(\frac{1+t_0}{t_0} \right)^p - 1 \right] \sum_{j=1}^{\infty} \left(\frac{t_0^j}{j+1} \right)^p = 0. \tag{23}$$

Proof. First, we will show that conditions (a)–(c) of Lemma 4 are satisfied.

Clearly

$$t_n = \frac{1 - t^{n+1}}{(n+1)(1-t)}, \quad y_n = \frac{1 - t^{n+1}}{t^n(1-t)}. \tag{24}$$

Thus

$$\begin{aligned} v(r) &= \frac{1}{(1-t)^p} \left(\frac{1-t^{r+1}}{t^r} \right)^p, \\ v'(r) &= \frac{pt^{-r} \log(1/t)}{(1-t)^p} (t^{-r} - t)^{p-1}, \\ v''(r) &= p(1-t)^{-p} (t^{-r} - t)^{p-2} (pt^{-2r} - t^{-r+1}) \left(\log \left(\frac{1}{t} \right) \right)^2, \end{aligned} \tag{25}$$

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and (a) and (b) are satisfied.

$$\begin{aligned}
 v'''(r) &= p(1-t)^{-p} \left(\log \left(\frac{1}{t} \right) \right)^2 [(p-2)(t^{-r}-t)^{p-3} (-t^{-r} \log t) (pt^{-2r} - t^{-r+1}) \\
 &\quad + (t^{-r}-t)^{p-2} (-2pt^{-2r} \log t + t^{-r+1} \log t)] \\
 &= p(1-t)^{-p} (t^{-r}-t)^{p-3} t^{-3r} \left(\log \left(\frac{1}{t} \right) \right)^3 \\
 &\quad \times [(p-2)(p-t^{r+1}) + (t^{-r}-t)(2pt^r - t^{2r+1})].
 \end{aligned} \tag{26}$$

It then follows that

$$\begin{aligned}
 2(v'')^2 - v'v''' &= 2 \left[p(1-t)^{-p} (t^{-r}-t)^{p-2} (pt^{-2r} - t^{-r+1}) \left(\log \left(\frac{1}{t} \right) \right)^2 \right]^2 \\
 &\quad - p(1-t)^{-p} t^{-r} \log \left(\frac{1}{t} \right) (t^{-r}-t)^{p-1} \\
 &\quad \times p(1-t)^{-p} (t^{-r}-t)^{p-3} t^{-3r} \left(\log \left(\frac{1}{t} \right) \right)^3 \\
 &\quad \times [p^2 - (3p-1)t^{r+1} + t^{2r+2}] \\
 &= p^2 t^{-4r} (1-t)^{-2p} \left(\log \left(\frac{1}{t} \right) \right)^4 (t^{-r}-t)^{2p-4} w(r, p),
 \end{aligned} \tag{27}$$

where

$$w(r) = p^2 - (p+1)t^{r+1} + t^{2r+2}. \tag{28}$$

Note that $w'(r) > 0$ for $p \geq 1$. Therefore, w is monotone increasing in r . Since $w(0) > 0$, w is positive for $0 < t < 1$, $r \geq 0$, and condition (4) is satisfied. Thus h is monotone decreasing in r :

$$\begin{aligned}
 \lim_{r \rightarrow \infty} h(r) &= \lim_{r \rightarrow \infty} \left(\frac{t^{r+1}}{r+2} \right)^p \\
 &= \frac{[(1/(1-t)^p) \{ (1-t^{r+2}/t^{r+1})^p - (1-t^{r+3}/t^{r+2})^p \}]}{[(1/(1-t)^p) \{ (1-t^{r+1}/t^r)^p - 2(1-t^{r+2}/t^{r+1})^p + (1-t^{r+3}/t^{r+2})^p \}]}
 \end{aligned} \tag{29}$$

Since the limit of the quantity in brackets is $(t^p - 1)/(t^p - 1)^2$, $\lim_{r \rightarrow \infty} h(r) = 0$, and condition (5) of Theorem 2 is satisfied; that is, g is monotone decreasing in p ,

$$\begin{aligned}
 g(0) &= t_0^p + 2[y_0^p - y_1^p] \sum_{j=1}^{\infty} a_j^p \\
 &= 1 - 2 \left[\left(\frac{1+t}{t} \right)^p - 1 \right] \sum_{j=1}^{\infty} \left(\frac{t^j}{j+1} \right)^p = q(t, p), \quad \text{say,} \\
 \frac{\partial q}{\partial t} &= -2p \left(\frac{1+t}{t} \right)^{p-1} \left(-\frac{1}{t^2} \right) \sum_{j=1}^{\infty} \left(\frac{t^j}{j+1} \right)^p \\
 &\quad - 2 \left[\left(\frac{1+t}{t} \right)^p - 1 \right] \sum_{j=1}^{\infty} \frac{p j t^{j-1}}{(j+1)^p} \\
 &= 2p \sum_{j=1}^{\infty} \left(\frac{t^j}{j+1} \right)^p \left[\left(\frac{1+t}{t} \right)^{p-1} \frac{1}{t^2} - \left(\left(\frac{1+t}{t} \right)^p - 1 \right) \frac{j}{t} \right].
 \end{aligned} \tag{30}$$

Since the coefficient of j is negative, the quantity in brackets is monotone decreasing in j . For $j = 1$, it becomes

$$\begin{aligned}
 &\frac{1}{t^2} \left[\left(\frac{1+t}{t} \right)^{p-1} + t - t \left(\frac{1+t}{t} \right)^p \right] \\
 &= \frac{1}{t^2} \left[\left(\frac{1+t}{t} \right)^{p-1} (1 - 1 - t) + t \right] = \frac{1}{t} \left[1 - \left(\frac{1+t}{t} \right)^{p-1} \right] < 0,
 \end{aligned} \tag{31}$$

and q is monotone decreasing in t .

$$q(1, p) = 1 - 2[2^p - 1] \sum_{j=1}^{\infty} \frac{1}{(j+1)^p}. \tag{32}$$

The function in brackets is convex in p for $p > 1$. So also is the series. Therefore, so is the product. Multiplying by -1 yields a concave function. Since 1 is also concave, $q(1, p)$ is a concave function of p for $p > 1$. Since

$$\lim_{p \rightarrow \infty} q(1, p) = 0, \tag{33}$$

$g(0)$ is negative for those values of $t > t_0$, where t_0 satisfies (23); that is, (6) of Theorem 2, and $L^p = f(0)$. □

The Rhaly s -Cesàro matrices [5] are factorable matrices with nonzero entries $a_n = (n+1)^{-s}$, $s > 1$, and each $b_k = 1$. Thus $t_n = (n+1)^{s-1}$ and $y_n = n+1$.

THEOREM 6. *For the Rhaly s -Cesàro matrices, $L^p = f(\infty)$ for $p, s > 1$.*

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Proof. First, we will show that conditions (a)–(c) of Lemma 4 are satisfied,

$$\begin{aligned} v(r) &= (r+1)^p, & v'(r) &= p(r+1)^{p-1}, \\ v''(r) &= p(p-1)(r+1)^{p-2}, \end{aligned} \quad (34)$$

and conditions (a) and (b) are satisfied,

$$v'''(r) = p(p-1)(p-2)(r+1)^{p-3}. \quad (35)$$

Therefore

$$\begin{aligned} 2(v'')^2 - v'v''' &= 2(p(p-1)(r+1)^{p-2})^2 - p(r+1)^{p-1}q(p-1)(p-2)(r+1)^{p-3} \\ &= p^2(p-1)(r+1)^{2p-4}[2(p-1) - (p-2)] \\ &= p^3(p-1)(r+1)^{2p-4} > 0, \end{aligned} \quad (36)$$

and condition (c) is satisfied,

$$0 \leq h(r) \leq \sum_{j=r+1}^{\infty} \frac{1}{(j+1)^{sp}}. \quad (37)$$

Therefore $\lim_{r \rightarrow \infty} h(r) = 0$ and condition (5) of Theorem 2 is satisfied. Thus g is monotone decreasing in r ,

$$g(r) = \sum_{j=0}^r \frac{1}{(j+1)^{(s-1)p}} + (r+1)(r+2)[(r+1)^{s-1} - (r+2)^{s-1}] \sum_{j=r+1}^{\infty} \frac{1}{(j+1)^{ps}}. \quad (38)$$

It then follows that

$$\lim_{r \rightarrow \infty} g(r) = \sum_{j=0}^{\infty} \frac{1}{(j+1)^{(p-1)s}} > 0, \quad (39)$$

and so $L^p = f(\infty)$. □

The Rhaly terraced matrices [6] are factorable matrices with each $b_k = 1$ and $a_n = a_n$, where $\{a_n\}$ is a monotone decreasing positive sequence such that $\lim(n+1)a_n$ exists. Clearly $t_n = (n+1)a_n$ and $y_n = n+1$.

THEOREM 7. *For the Rhaly terraced matrices, $L^p = f(0)$ for $p > 1$.*

Proof. Since $y_n = n+1$, the first part of the proof of Theorem 6 applies and h is monotone decreasing in r ,

$$h(r) = \frac{a_{r+1}^p [(r+2)^p - (r+3)^p]}{(r+1)^p - 2(r+2)^p + (r+3)^p} + \sum_{j=r+1}^{\infty} a_j^p \leq \sum_{j=r+1}^{\infty} a_j^p, \quad (40)$$

and $\lim h(r) = 0$, so that g is monotone decreasing in r ,

$$\begin{aligned} g(0) &= a_0^p + 2[1 - 2^{p-1}] \sum_{j=1}^{\infty} a_j^p \\ &= a_0^p - 2(2^{p-1} - 1)a_1^p - 2(2^{p-1} - 1) \sum_{j=2}^{\infty} a_j^p < 0, \end{aligned} \tag{41}$$

since $\{a_n\}$ is monotone decreasing. Therefore $L^p = f(0)$. □

A weighted mean matrix is a factorable matrix with $a_n = 1/P_n$, $b_k = p_k$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{k=0}^n p_k$.

THEOREM 8. *Let (\bar{N}, p_n) be a weighted mean method with the $\{p_n\}$ nondecreasing. Then*

$$1 + (r + 1) \left(\frac{P_{r+1}}{P_r} \right)^p - (r + 2) \left(\frac{P_{r+2}}{P_{r+1}} \right)^p \geq 0 \quad \text{for each } r \geq 0, p > 1, \tag{42}$$

is a sufficient condition for $L^p = f(0)$.

Proof. Since a weighted mean matrix has row sums one, from (9),

$$f(r) - f(r + 1) = \frac{1}{r + 2} + \Delta \left(\frac{P_r^p}{r + 1} \right) \sum_{j=r+1}^{\infty} \frac{1}{P_j^p}. \tag{43}$$

Thus

$$g(r) = \frac{f(r) - f(r + 1)}{\Delta(P_r^p/(r + 1))} = \frac{1}{(r + 2)\Delta(P_r^p/(r + 1))} + \sum_{j=r+1}^{\infty} \frac{1}{P_j^p}, \tag{44}$$

$$\begin{aligned} g(r) - g(r + 1) &= \frac{1}{(r + 2)\Delta(P_r^p/(r + 1))} - \frac{1}{(r + 3)\Delta(P_{r+1}^p/(r + 2))} + \frac{1}{P_{r+1}^p} \\ &= \frac{1}{P_{r+1}^p(r + 2)(r + 3)\Delta(P_r^p/(r + 1))\Delta(P_{r+1}^p/(r + 2))} m(r), \end{aligned} \tag{45}$$

where

$$\begin{aligned} m(r) &= P_{r+1}^p(r + 3)\Delta \left(\frac{P_{r+1}^p}{r + 2} \right) - P_{r+1}^p(r + 2)\Delta \left(\frac{P_r^p}{r + 1} \right) + (r + 2)(r + 3)\Delta \left(\frac{P_r^p}{r + 1} \right)\Delta \left(\frac{P_{r+1}^p}{r + 2} \right) \\ &= P_{r+1}^p \left[(r + 3) \left(\frac{P_{r+1}^p}{r + 2} - \frac{P_{r+2}^p}{r + 3} \right) - (r + 2) \left(\frac{P_r^p}{r + 1} - \frac{P_{r+1}^p}{r + 2} \right) \right] \\ &\quad + (r + 2)(r + 3) \left(\frac{P_r^p}{r + 1} - \frac{P_{r+1}^p}{r + 2} \right) \left(\frac{P_{r+1}^p}{r + 2} - \frac{P_{r+2}^p}{r + 3} \right) \end{aligned}$$

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$$\begin{aligned}
&= P_{r+1}^p \left[\left(\frac{r+3}{r+2} \right) P_{r+1}^p - P_{r+2}^p - \left(\frac{r+2}{r+1} \right) P_r^p + P_{r+1}^p \right] \\
&\quad + \left(\frac{r+3}{r+1} \right) P_r^p P_{r+1}^p - \left(\frac{r+3}{r+2} \right) (P_{r+1}^p)^2 - P_r^p P_{r+2}^p \left(\frac{r+2}{r+1} \right) + P_{r+1}^p P_{r+2}^p \\
&= P_{r+1}^p \left[\left(\frac{r+3}{r+2} \right) P_{r+1}^p - P_{r+2}^p - \left(\frac{r+2}{r+1} \right) P_r^p + P_{r+1}^p + \left(\frac{r+3}{r+1} \right) P_r^p - \left(\frac{r+3}{r+2} \right) P_{r+1}^p + P_{r+2}^p \right] \\
&\quad - \left(\frac{r+2}{r+1} \right) P_r^p P_{r+2}^p \\
&= P_{r+1}^p \left[\frac{1}{r+1} P_r^p + P_{r+1}^p \right] - \left(\frac{r+2}{r+1} \right) P_r^p P_{r+2}^p \\
&= \frac{1}{r+1} \left[P_r^p P_{r+1}^p + (r+1) (P_{r+1}^p)^2 - (r+2) P_r^p P_{r+2}^p \right] \\
&= \frac{P_r^p P_{r+1}^p}{r+1} \left[1 + (r+1) \left(\frac{P_{r+1}}{P_r} \right)^p - (r+2) \left(\frac{P_{r+2}}{P_{r+1}} \right)^p \right] \geq 0,
\end{aligned} \tag{46}$$

which is [7, Theorem 1, condition (1)] without any monotonicity condition on the $\{p_n\}$. Thus g is monotone decreasing in r .

Since $\{p_n\}$ is nondecreasing, $P_r \leq (r+1)p_r$; that is, $p_r/P_r \geq (r+1)^{-1}$. Thus

$$\frac{P_{r+1}}{P_r} = 1 + \frac{p_{r+1}}{P_r} \geq 1 + \frac{p_r}{P_r} \geq \frac{r+2}{r+1}, \tag{47}$$

and $P_{r+1}/P_r(r+2) \geq P_r(r+1)$.

Using (44), since $p > 1$,

$$\begin{aligned}
\lim |g(r)| &= \lim \left| \frac{(r+1)P_r^p}{P_{r+1}^p [(r+2)P_r^p - (r+1)P_{r+1}^p]} \right| \\
&= \lim \left| \frac{r+1}{P_{r+1}^p [(r+2) - (r+1)(P_{r+1}/P_r)^p]} \right| \\
&= \lim \frac{(r+1)}{P_{r+1}^p [(r+1)(P_{r+1}/P_r)^p - (r+2)]}.
\end{aligned} \tag{48}$$

Using the fact that $(1+x)^p \geq 1+px$ for $p > 1, x > -1$,

$$\begin{aligned}
\lim |g(x)| &\leq \lim \frac{(r+1)}{P_{r+1}^p [(r+1)(1+pp_{r+1}/P_r) - (r+2)]} \\
&= \lim \frac{r+1}{P_r [p(r+1)p_{r+1}/P_r - 1]} \\
&\leq \lim \frac{r+1}{P_r(p-1)} = 0.
\end{aligned} \tag{49}$$

Therefore $\lim g(r) = 0$ and g is positive for all r . From (44), since $\Delta(P_r^p/(r+1)) < 0$, $L^p = f(0)$. \square

COROLLARY 9. *Suppose (\bar{N}, p) is a weighted mean method with $p_n = (n+1)^\alpha$, $\alpha \geq 1$. Then $L^p = f(0)$.*

Proof. To show that (42) is satisfied, it is sufficient to show that

$$(r+1) \left(\frac{P_{r+1}}{P_r} \right)^p \tag{50}$$

is convex. The function $r = 1$ is trivially convex. Since $p > 1$, it will be enough to show that P_{r+1}/P_r is convex.

Define

$$n(r) = 1 + \frac{p_{r+1}}{P_r} = 1 + \frac{(r+2)^\alpha}{\sum_{k=0}^r (k+1)^\alpha}. \tag{51}$$

Then

$$\begin{aligned} \Delta^2 n(r) &= \frac{(r+2)^\alpha}{\sum_{k=0}^r (k+1)^\alpha} - \frac{2(r+3)^\alpha}{\sum_{k=0}^{r+1} (k+1)^\alpha} + \frac{(r+4)^\alpha}{\sum_{k=0}^{r+2} (k+1)^\alpha} \\ &= \frac{n(r)}{(\sum_{k=0}^r (k+1)^\alpha)(\sum_{k=0}^{r+1} (k+1)^\alpha)(\sum_{k=0}^{r+2} (k+1)^\alpha)}, \end{aligned} \tag{52}$$

where

$$\begin{aligned} n(r) &= (r+2)^\alpha \left(\sum_{k=0}^{r+1} (k+1)^\alpha \right) \left(\sum_{k=0}^{r+2} (k+1)^\alpha \right) - 2(r+3)^\alpha \left(\sum_{k=0}^r (k+1)^\alpha \right) \left(\sum_{k=0}^{r+2} (k+1)^\alpha \right) \\ &\quad + (r+4)^\alpha \left(\sum_{k=0}^r (k+1)^\alpha \right) \left(\sum_{k=0}^{r+1} (k+1)^\alpha \right) \\ &= (r+2)^\alpha \left[\sum_{k=0}^r (k+1)^\alpha + (r+2)^\alpha \right] \times \left[\sum_{k=0}^r (k+1)^\alpha + (r+2)^\alpha + (r+3)^\alpha \right] \\ &\quad - 2(r+3)^\alpha \left(\sum_{k=0}^r (k+1)^\alpha \right) \left(\sum_{k=0}^r (k+1)^\alpha + (r+2)^\alpha + (r+3)^\alpha \right) \\ &\quad + (r+4)^\alpha \left(\sum_{k=0}^r (k+1)^\alpha \right) \left(\sum_{k=0}^r (k+1)^\alpha + (r+2)^\alpha \right) \\ &= \left(\sum_{k=0}^r (k+1)^\alpha \right)^2 [(r+2)^\alpha - 2(r+3)^\alpha + (r+4)^\alpha] \end{aligned}$$

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$$\begin{aligned}
 &+ [(r+2)^{2\alpha} + (r+2)^{2\alpha} + ((r+2)(r+3))^\alpha \\
 &\quad - 2((r+2)(r+3))^\alpha - 2(r+3)^{2\alpha} + ((r+4)(r+2))^\alpha] \\
 &\quad \times \left(\sum_{k=0}^r (k+1)^\alpha \right) + (r+2)^{3\alpha} + (r+2)^{2\alpha}(r+3)^\alpha.
 \end{aligned} \tag{53}$$

Since $\alpha \geq 1$, $(r+2)^\alpha$ is convex, so the first quantity in brackets is positive. The second quantity in brackets can be written in the form

$$(r+2)^\alpha \{ (r+2)^\alpha - 2(r+3)^\alpha + (r+4)^\alpha \} + (r+2)^\alpha + (r+3)^\alpha ((r+2)^\alpha - 2), \tag{54}$$

which is positive. Therefore $n(r)$ is convex and (42) is satisfied. \square

COROLLARY 10. *Let $1 < p < \infty$, H the Hausdorff matrix generated by $\mu_n = a/(n+a)$, $a \geq 1$. Then $L^p = f(0)$.*

Proof. H is also a weighted mean matrix with $p_n = p_0 \Gamma(n+a)/\Gamma(a+1)\Gamma(n+1)$ and $P_n = p_0 \Gamma(n+a+1)/\Gamma(a+1)\Gamma(n+1)$. Substituting in (42), one obtains

$$1 + (r+1) \left(\frac{r+a+1}{r+1} \right)^p - (r+2) \left(\frac{r+a+2}{r+2} \right)^p, \tag{55}$$

and it is sufficient to prove that $(r+a+1)/(r+1)$ is convex, which it is. \square

The Cesàro matrix of order one, written $(C, 1)$, is a Hausdorff matrix with generating sequence $\mu_n = (n+1)^{-1}$.

COROLLARY 11. *For $(C, 1)$, $L^p = f(0)$.*

Proof. Use Corollary 10 with $a = 1$. \square

Remarks 12. (1) The condition that the $\{p_n\}$ be nondecreasing is not a necessary condition for $L^p = f(0)$. For example, take $p_n = 2/(n+1)(n+2)$. Then $\{p_n\}$ is monotone decreasing and satisfies (42) and

$$\Delta \left(\frac{P_r}{r+1} \right) < 0. \tag{56}$$

(2) Bennett [1] proved that $L^p = f(0)$ for the Hilbert matrix.

(3) In [2], Bennett has shown that $L^p = f(0)$ for each Hausdorff matrix $H \in B(\ell^p)$ with nonnegative entries.

(4) No results have been established for Nörlund matrices.

An interesting open question is the following. If $\{p_n\}$ is nondecreasing, must $\{p_n\}$ satisfy (42)?

References

- [1] G. Bennett, *Lower bounds for matrices*, Linear Algebra and Its Applications **82** (1986), 81–98.
- [2] ———, *Lower bounds for matrices. II*, Canadian Journal of Mathematics **44** (1992), no. 1, 54–74.
- [3] R. Lyons, *A lower bound on the Cesàro operator*, Proceedings of the American Mathematical Society **86** (1982), no. 4, 694.
- [4] H. C. Rhaly Jr., *Discrete generalized Cesàro operators*, Proceedings of the American Mathematical Society **34** (1986), 225–232.
- [5] ———, *p -Cesàro matrices*, Houston Journal of Mathematics **15** (1989), no. 1, 137–146.
- [6] ———, *Terraced matrices*, The Bulletin of the London Mathematical Society **21** (1989), no. 4, 399–406.
- [7] B. E. Rhoades, *Lower bounds for some matrices*, Linear and Multilinear Algebra **20** (1987), no. 4, 347–352.
- [8] ———, *Lower bounds for some matrices. II*, Linear and Multilinear Algebra **26** (1990), no. 1-2, 49–58.

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