

AN IDENTITY RELATED TO JORDAN'S INEQUALITY

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The main purpose of this note is to establish an identity which states that the function $\sin x/x$ is a power series of $(\pi^2 - 4x^2)$ with positive coefficients for all $x \neq 0$. This enable us to obtain a much stronger Jordan's inequality than that obtained before.

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1. Introduction

The well-known Jordan's inequality states that

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right], \quad (1.1)$$

with equality holds if and only if $x = \pi/2$ (see [5]). It plays an important role in many areas of pure and applied mathematics. The inequality (1.1) is first extended to the following:

$$\frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right], \quad (1.2)$$

and then, it is further extended to the following:

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right], \quad (1.3)$$

with equality holds if and only if $x = \pi/2$ (see [2, 4, 6]). The inequality (1.3) is slightly stronger than the inequality (1.2) and is sharp in the sense that $1/\pi^3$ cannot be replaced by a larger constant. More recently, the monotone form of L'Hopital's rule (see [1, Lemma 5.1]) has been successfully used by Zhu [9, 10], Wu and Debnath [7, 8] in the sharpening

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Jordan's inequality. For example, it has been shown that if $0 < x \leq \pi/2$, then

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \quad (1.4)$$

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5}(\pi^2 - 4x^2)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^5}(\pi^2 - 4x^2)^2 \quad (1.5)$$

hold with equality if and only if $x = \pi/2$. Furthermore, the constants $1/\pi^3$ and $(\pi - 2)/\pi^3$ in (1.4) as well as the constants $(12 - \pi^2)/(16\pi^5)$ and $(\pi - 3)/\pi^5$ in (1.5) are the best. Also, in the process of sharpening Jordan's inequality, one can use the same method as did in [7] to introduce a parameter θ ($0 < \theta \leq \pi$) to replace the value $\pi/2$. Unfortunately, the preceding method will become cumbersome to execute in the further generalization of Jordan's inequality.

In this note we establish an identity which states that the function $\sin x/x$ is a power series of $(\pi^2 - 4x^2)$ with positive coefficients for all $x \neq 0$. This enables us to obtain a much better inequality than (1.4) or (1.5) if $0 < x \leq \pi/2$.

2. Main result

The main result relating to Jordan's inequality is contained in the following.

THEOREM 2.1. *For any $x > 0$, the following identity*

$$\frac{\sin x}{x} = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{(-1)^k R_k}{k! \pi^{2k}} (\pi^2 - 4x^2)^k \quad (2.1)$$

holds, where

$$R_k = \sum_{n=k}^{\infty} \frac{(-1)^n n!}{(2n+1)!(n-k)!} \left(\frac{\pi}{2}\right)^{2n} \quad (2.2)$$

satisfying $(-1)^k R_k > 0$ ($k = 1, 2, 3, \dots$).

Proof. Let $x = \sqrt{\pi^2 - t}/2$ and $t = \pi^2 - 4x^2$. It follows from the Taylor expansion for $\sin x$ that

$$\frac{\sin \sqrt{(\pi^2 - t)}/2}{\sqrt{(\pi^2 - t)}/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi^2 - t}{4}\right)^n \quad (-\infty < t < \pi^2). \quad (2.3)$$

Since

$$(\pi^2 - t)^n = \pi^{2n} + \sum_{k=1}^n \frac{(-1)^k n!}{k!(n-k)!} \pi^{2(n-k)} t^k \quad (n = 1, 2, 3, \dots), \quad (2.4)$$

we see that

$$\begin{aligned}
 \frac{\sin\sqrt{(\pi^2-t)}/2}{\sqrt{(\pi^2-t)}/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!4^n} \sum_{k=1}^n \frac{(-1)^k n!}{k!(n-k)!} \pi^{2(n-k)} t^k \\
 &= \frac{2}{\pi} + \sum_{k=1}^{\infty} \left\{ \sum_{n=k}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{n!}{(n-k)!} \left(\frac{\pi}{2}\right)^{2n} \right\} \frac{(-1)^k}{k! \pi^{2k}} t^k \\
 &= \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{(-1)^k R_k}{k! \pi^{2k}} t^k \quad (-\infty < t < \pi^2)
 \end{aligned} \tag{2.5}$$

which yields (2.1) with R_k given by (2.2).

Let $R_k = \sum_{n=k}^{\infty} (-1)^n c_{n,k}$ with

$$c_{n,k} = \frac{1}{(n-k)!} \frac{1}{n+1} \frac{\pi^2/4}{n+2} \frac{\pi^2/4}{n+3} \cdots \frac{\pi^2/4}{n+n} \frac{\pi^2/4}{2n+1}. \tag{2.6}$$

Then for each given k ($k = 1, 2, 3, \dots$),

$$c_{n,k} > c_{n+1,k}, \quad (n \geq k), \quad \lim_{n \rightarrow \infty} c_{n,k} = 0. \tag{2.7}$$

Hence the alternating series $\sum_{n=k}^{\infty} (-1)^n c_{n,k}$ converges, and its sum

$$R_k = (-1)^k \sum_{j=0}^{\infty} (-1)^j c_{k+j,k} \tag{2.8}$$

satisfies $(-1)^k R_k > 0$ and $|R_k| < c_{k,k}$ for each given k ($k = 1, 2, 3, \dots$). This completes the proof of the theorem. \square

Next we give a formula of calculating R_k . Let

$$d_k(x) = \sum_{n=k}^{\infty} \frac{(-1)^n n!}{(2n+1)!(n-k)!} x^{2n}. \tag{2.9}$$

Then

$$\begin{aligned}
 d_1(x) &= \frac{x}{2} \left(\frac{\sin x}{x}\right)', \quad d_2(x) = \frac{x^3}{2^2} \left(\frac{1}{x} \left(\frac{\sin x}{x}\right)'\right)', \dots, \\
 d_k(x) &= \frac{x^{2k-1}}{2^k} \left(\frac{1}{x} \left(\frac{1}{x} \left(\dots \left(\frac{1}{x} \left(\frac{\sin x}{x}\right)'\right)'\dots\right)'\right)'\right)'
 \end{aligned} \tag{2.10}$$

with k th derivative, and

$$d_{k+1}(x) = -k d_k(x) + \frac{x}{2} d_k'(x) \quad (k = 1, 2, 3, \dots). \tag{2.11}$$

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Hence $R_k = d_k(\pi/2)$ with

$$\begin{aligned} R_1 &= d_1\left(\frac{\pi}{2}\right) = -\frac{1}{\pi}, & R_2 &= d_2\left(\frac{\pi}{2}\right) = \frac{12 - \pi^2}{8\pi}, \\ R_3 &= d_3\left(\frac{\pi}{2}\right) = \frac{-3(10 - \pi^2)}{8\pi}, & R_4 &= d_4\left(\frac{\pi}{2}\right) = \frac{\pi^4 - 180\pi^2 + 1680}{128\pi}. \end{aligned} \quad (2.12)$$

Also the above established identity (2.1) gives

$$\begin{aligned} \frac{\sin x}{x} &= \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5}(\pi^2 - 4x^2)^2 \\ &\quad + \frac{10 - \pi^2}{16\pi^7}(\pi^2 - 4x^2)^3 + \frac{\pi^4 - 180\pi^2 + 1680}{3072\pi^9}(\pi^2 - 4x^2)^4 \\ &\quad + \sum_{k=5}^{\infty} \frac{(-1)^k R_k}{k! \pi^{2k}} (\pi^2 - 4x^2)^k \end{aligned} \quad (2.13)$$

for all $x > 0$. Since $(-1)^k R_k > 0$ ($k = 1, 2, 3, \dots$), we have the following corollary.

COROLLARY 2.2. *If $0 < x \leq \pi/2$, then*

$$\begin{aligned} \frac{\sin x}{x} &\geq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5}(\pi^2 - 4x^2)^2 \\ &\quad + \frac{10 - \pi^2}{16\pi^7}(\pi^2 - 4x^2)^3 + \frac{\pi^4 - 180\pi^2 + 1680}{3072\pi^9}(\pi^2 - 4x^2)^4 \end{aligned} \quad (2.14)$$

holds with equality if and only if $x = \pi/2$. Furthermore, the constants $1/\pi^3$, $(12 - \pi^2)/(16\pi^5)$, $(10 - \pi^2)/(16\pi^7)$, and $(\pi^4 - 180\pi^2 + 1680)/(3072\pi^9)$ in (2.14) are the best.

The above established inequality (2.14) is much stronger than the left-hand side of inequality (1.5). Also one can add more positive terms to the right-hand side of inequality (2.14) to get higher accuracy.

Finally, it should be pointed out that, in order to give the right-hand side of inequality (1.4) or (1.5), the following Taylor expansion for $x/\sin x$ will play an important role as the above established identity (2.1) or (2.13).

Taylor expansion of $x/\sin x$.

$$\begin{aligned} \frac{x}{\sin x} &= \sum_{n=0}^{\infty} (-1)^{n+1} B_{2n} \frac{2^{2n} - 2}{(2n)!} x^{2n} \\ &= 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \dots \quad (|x| < \pi), \end{aligned} \quad (2.15)$$

where B_{2n} are the Bernoulli numbers satisfying $(-1)^{n+1} B_{2n} > 0$ ($n = 1, 2, 3, \dots$).

Recall that the Bernoulli numbers B_n and the functions $B_n(x)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad (|t| < 2\pi). \quad (2.16)$$

It is familiar that they have the following properties (see [3, Section I-13]):

$$\begin{aligned}
 B_0(x) = B_0 = 1, \quad B_1 = \frac{-1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = \frac{-1}{30}, \quad B_6 = \frac{1}{42}; \\
 B_{2n+1} = 0, \quad (-1)^{n+1}B_{2n} > 0, \quad B_n(0) = B_n \quad (n \geq 1); \\
 B_n(1-x) = (-1)^n B_n(x) \quad (n \geq 1); \\
 B_{2n}(x) = \frac{2(-1)^{n+1}(2n)!}{(2\pi)^{2n}} \sum_{r=1}^{\infty} \frac{1}{r^{2n}} \cos(2\pi r x) \quad (n \geq 1, 0 \leq x \leq 1),
 \end{aligned}
 \tag{2.17}$$

from which, we have

$$B_{2n-1}\left(\frac{1}{2}\right) = 0, \quad 2^{2n}B_{2n}\left(\frac{1}{2}\right) = (2 - 2^{2n})B_{2n} \quad (n \geq 1).
 \tag{2.18}$$

Therefore, for $|x| < \pi$,

$$\begin{aligned}
 \frac{x}{\sin x} &= \frac{2xie^{xi}}{e^{2xi} - 1} = \sum_{n=0}^{\infty} B_n\left(\frac{1}{2}\right) \frac{2^n i^n}{n!} x^n \\
 &= \sum_{n=0}^{\infty} B_{2n}\left(\frac{1}{2}\right) \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^{n+1} B_{2n} \frac{2^{2n} - 2}{(2n)!} x^{2n}.
 \end{aligned}
 \tag{2.19}$$

From (2.15), we have the following type of strengthened right-hand Jordan’s inequality:

$$\frac{\sin x}{x} \leq \left(1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6\right)^{-1} \quad (|x| < \pi).
 \tag{2.20}$$

Also one can add more positive terms to the right-hand side of inequality (2.20) to get higher accuracy.

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