STRONG IMPLICATIVE HYPER K-IDEALS

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A condition for a strong hyper K-ideal to be a strong implicative hyper K-ideal is given. Homomorphic images and inverse images of strong implicative hyper K-ideals are considered.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. The hyperstructure theory (called also multialgebras) is introduced in 1934 by Marty [6] at the 8th congress of Scandinavian Mathematicians. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, Japan, and Iran. Hyperstructures have many applications to several sectors of both pure and applied sciences. Recently in [2] Borzoei et al. and Borzoei and Zahedi [3] constructed the hyper K-algebras, and studied (positive implicative) hyper K-ideals in hyper K-algebras. In [4] Jun and Roh studied strong hyper K-ideals in hyper K-algebras. Also Jun et al. [5] introduced the notion of s-implicative hyper K-ideal of a hyper K-algebra, and investigated related properties. As a continuation of [5], in this paper, we find a condition for a strong hyper K-ideal to be a strong implicative hyper K-ideal. We also consider homomorphic images and inverse images of strong implicative hyper K-ideals.

2. Preliminaries

We include some elementary aspects of hyper K-algebras that are necessary for this paper, and for more details we refer to [2, 7]. Let H be a nonempty set endowed with a hyper operation "$\circ$," that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$.

By a hyper I-algebra we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

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(H1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(H3) $x < x$,
(H4) $x < y$ and $y < x$ imply $x = y$,
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper I-algebra $(H, \circ, 0)$ satisfies an additional condition:
(H5) $0 < x$ for all $x \in H$,
then $(H, \circ, 0)$ is called a hyper K-algebra (see [2]).

In a hyper I-algebra H, the following hold (see [2, Proposition 3.4]):
(a1) $(A \circ B) \circ C = (A \circ C) \circ B$,
(a2) $x \circ (x \circ y) < y$,
(a3) $x \circ y < z \Leftrightarrow x \circ z < y$,
(a4) $A \circ B < C \Leftrightarrow A \circ C < B$,
(a5) $(x \circ z) \circ (x \circ y) < y \circ z$,
(a6) $(A \circ C) \circ (B \circ C) < A \circ B$,
(a7) $A \circ (A \circ B) < B$,
(a8) $A < A$,
(a9) $A \subseteq B$ implies $A < B$,
for all $x, y, z \in H$ and for all nonempty subsets A, B, and C of H.

A nonempty subset I of a hyper K-algebra H is called a *weak hyper K-ideal* of H (see [2]) if it satisfies
(I1) $0 \in I$,
(I2) (for all $x, y \in H$) $(x \circ y \subseteq I,\ y \in I \Rightarrow x \in I)$.

A nonempty subset I of a hyper K-algebra H is called a hyper K-ideal of H (see [2]) if it satisfies (I1) and

$$(\forall x, y \in H) \quad (x \circ y < I,\ y \in I \Longrightarrow x \in I). \tag{2.1}$$

Note that every hyper K-ideal is a weak hyper K-ideal (see [1]). A nonempty subset I of a hyper K-algebra H is called a *strong hyper K-ideal* of H (see [4]) if it satisfies (I1) and

$$(\forall x, y \in H) \quad ((x \circ y) \cap I \neq \varnothing,\ y \in I \Longrightarrow x \in I). \tag{2.2}$$

Note that every strong hyper K-ideal is a hyper K-ideal (see [4]).

A nonempty subset I of a hyper K-algebra H is called a *weak implicative hyper K-ideal* of H (see [1]) if it satisfies (I1) and

$$(\forall x, y, z \in H) \quad ((x \circ z) \circ (y \circ x) \subseteq I,\ z \in I \Longrightarrow x \in I). \tag{2.3}$$

A nonempty subset I of a hyper K-algebra H is called an *implicative hyper K-ideal* of H (see [1]) if it satisfies (I1) and

$$(\forall x, y, z \in H) \quad ((x \circ z) \circ (y \circ x) < I,\ z \in I \Longrightarrow x \in I). \tag{2.4}$$

Note that every implicative hyper K-ideal is a weak implicative hyper K-ideal (see [1]).

Table 3.1

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{b\}$
c	$\{c\}$	$\{a,b\}$	$\{0,a\}$	$\{0,b\}$

3. Strong implicative hyper K-ideals

In what follows let H denote a hyper K-algebra unless otherwise specified. In [5], Jun et al. introduced the notion of s-implicative hyper K-ideal of H as follows.

Defintion 3.1 [5]. A nonempty subset I of H is called an *s-implicative hyper K-ideal* of H if it satisfies (I1) and

$$(\forall x,y,z \in H) \quad (((x \circ z) \circ (y \circ x)) \cap I \neq \varnothing,\ z \in I \Longrightarrow x \in I). \tag{3.1}$$

In this paper it is called a *strong implicative hyper K-ideal* of H.

Example 3.2. Let $H = \{0,a,b,c\}$ be a hyper K-algebra with the Cayley table (Table 3.1). Then $I := \{0,a,c\}$ is a strong implicative hyper K-ideal of H. But $I := \{0,a\}$ is not a strong implicative hyper K-ideal of H since $((c \circ a) \circ (0 \circ c)) \cap I = \{a\}$ and $a \in I$ but $c \notin I$.

Theorem 3.3. *If $\{I_\lambda \mid \lambda \in \Lambda\}$ is a family of strong implicative hyper K-ideals of H, then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a strong implicative hyper K-ideal of H.*

Proof. Clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Let $x,y,z \in H$ be such that $((x \circ z) \circ (y \circ x)) \cap (\bigcap_{\lambda\in\Lambda} I_\lambda) \neq \varnothing$ and $z \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Then $((x \circ z) \circ (y \circ x)) \cap I_\lambda \neq \varnothing$ and $z \in I_\lambda$ for all $\lambda \in \Lambda$. By using (3.1), we have $x \in I_\lambda$ for all $\lambda \in \Lambda$, and hence $x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Hence $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a strong implicative hyper K-ideal of H. □

Lemma 3.4 [5]. *Every strong implicative hyper K-ideal of H is a strong hyper K-ideal.*

Proposition 3.5. *Let I be a strong hyper K-ideal of H, and $A,B \subseteq H$. If $(A \circ B) \cap I \neq \varnothing$ and $B \subseteq I$, then $A < I$.*

Proof. Since $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$ and $(A \circ B) \cap I \neq \varnothing$, there exists $t \in a \circ b$ for some $a \in A$, $b \in B$ such that $t \in I$. Hence $(a \circ b) \cap I \neq \varnothing$. Since I is a strong hyper K-ideal and $b \in B \subseteq I$, we have $a \in I$. Therefore we get $A < I$. □

Now we give a condition for a strong hyper K-ideal to be a strong implicative hyper K-ideal.

Theorem 3.6. *Let I be a strong hyper K-ideal of H such that*

$$(\forall x,y \in H) \quad (x \circ (y \circ x) < I \Longrightarrow\ x \in I). \tag{3.2}$$

Then I is a strong implicative hyper K-ideal of H.

Table 3.2

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{a,b\}$	$\{0,a\}$	$\{0,a\}$

Proof. Let $x,y,z \in H$ be such that $((x\circ z)\circ(y\circ x))\cap I \neq \emptyset$ and $z \in I$. Then we have $((x\circ(y\circ x))\circ z)\cap I \neq \emptyset$. It follows from Proposition 3.5 that $x\circ(y\circ x) < I$ so from (3.2) that $x \in I$. Therefore I is a strong implicative hyper K-ideal of H. □

Theorem 3.7. *Every strong implicative hyper K-ideal I of H satisfies (3.2).*

Proof. Let I be a strong implicative hyper K-ideal of H and let $x,y \in H$ be such that $x\circ(y\circ x) < I$. The inclusion $x\circ(y\circ x) \subseteq (x\circ 0)\circ(y\circ x)$ implies that $(x\circ 0)\circ(y\circ x) < I$. Hence $x \in I$ since $0 \in I$. □

Corollary 3.8. *Let I be a nonempty subset of H. Then I is a strong implicative hyper K-ideal of H if and only if I is a strong hyper K-ideal of H that satisfies the condition (3.2).*

Theorem 3.9. *Every strong implicative hyper K-ideal is a weak implicative hyper K-ideal.*

Proof. Straightforward. □

In general, a weak implicative hyper K-ideal may not be a strong implicative hyper K-ideal as seen in the following example.

Example 3.10. Let $H = \{0,a,b\}$ be a hyper K-algebra with the Cayley table (Table 3.2). Then $I := \{0,b\}$ is a weak implicative hyper K-ideal of H, but I is not a strong implicative hyper K-ideal since $((a\circ 0)\circ(a\circ a))\cap I \neq \emptyset$ and $0 \in I$ but $a \notin I$.

Our future work will focus on finding conditions for a weak implicative hyper K-ideal to be a strong implicative hyper K-ideal.

4. Homomorphisms of hyper K-algebras

Defintion 4.1 [7]. Let H_1 and H_2 be two hyper K-algebras. A mapping $f: H_1 \to H_2$ is said to be a *homomorphism* if it satisfies

(i) $f(0) = 0$,

(ii) (for all $x,y \in H_1$) $(f(x\circ y) = f(x)\circ f(y))$.

Moreover if f is $1-1$ (or onto), we say that f is a *monomorphism* (or *epimorphism*). And if f is both $1-1$ and onto, we say that f is an *isomorphism*.

Proposition 4.2. *Let $f: H_1 \to H_2$ be a homomorphism of hyper K-algebras. Then*

$$(\forall A,B \subseteq H_1) \quad (A < B \Longrightarrow f(A) < f(B)). \tag{4.1}$$

Proof. Let $A,B \subseteq H_1$ be such that $A < B$. Then $\exists a \in A$, $\exists b \in B$ such that $a < b$, that is, $0 \in a\circ b$. Hence $0 = f(0) \in f(a\circ b) = f(a)\circ f(b)$, which implies $f(A) < f(B)$. □

THEOREM 4.3. *Let $f : H_1 \to H_2$ be a homomorphism of hyper K-algebras. If I is a strong hyper K-ideal of H_2, then $f^{-1}(I)$ is a strong hyper K-ideal of H_1.*

Proof. Let I be a strong hyper K-ideal of H_2. Clearly $0 \in f^{-1}(I)$. Let $x, y \in H_1$ be such that $(x \circ y) \cap f^{-1}(I) \neq \varnothing$ and $y \in f^{-1}(I)$. Then we have

$$\varnothing \neq f\big((x \circ y) \cap f^{-1}(I)\big) \subseteq f(x \circ y) \cap f\big(f^{-1}(I)\big) \subseteq \big(f(x) \circ f(y)\big) \cap I, \tag{4.2}$$

and so $(f(x) \circ f(y)) \cap I \neq \varnothing$ and $f(y) \in f(f^{-1}(I)) \subseteq I$. Since I is a strong hyper K-ideal of H_2, we have $f(x) \in I$ and so $x \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a a strong hyper K-ideal of H_1. □

THEOREM 4.4. *Let $f : H_1 \to H_2$ be a homomorphism of hyper K-algebras. If I is a strong implicative hyper K-ideal of H_2, then $f^{-1}(I)$ is a strong implicative hyper K-ideal of H_1.*

Proof. The proof is similar to the proof of Theorem 4.3 by some modification. □

THEOREM 4.5. *Let $f : H_1 \to H_2$ be a homomorphism of hyper K-algebras. Then* $\ker f := \{x \in H_1 \mid f(x) = 0\}$ *is a strong hyper K-ideal of H_1.*

Proof. First we show that $\{0\}$ is a strong hyper K-ideal of H_2. To do this, let $x, y \in H_1$ be such that $(x \circ y) \cap \{0\} \neq \varnothing$ and $y \in \{0\}$. Then $y = 0$ and so $0 \in x \circ 0$ since $(x \circ 0) \cap \{0\} \neq \varnothing$. Thus we have $x < 0$. By (H4) and (H5), we get $x = 0 \in \{0\}$. This shows that $\{0\}$ is a strong hyper K-ideal of H_2. It follows from Theorem 4.3 that $\ker f = f^{-1}(\{0\})$ is a strong hyper K-ideal of H_1. □

In general, a strong hyper K-ideal $\{0\}$ may not be a strong implicative hyper K-ideal. For example, consider the hyper K-algebra H of Example 3.2. Clearly $\{0\}$ is a strong hyper K-ideal of H, while it is not a strong implicative hyper K-ideal of H since $((c \circ 0) \circ (b \circ c)) \cap \{0\} \neq \varnothing$ and $0 \in \{0\}$ but $c \notin \{0\}$.

THEOREM 4.6. *Let $f : H_1 \to H_2$ be a homomorphism of hyper K-algebras. If f is onto and I is a strong hyper K-ideal of H_1 which contains* $\ker f$, *then $f(I)$ is a strong hyper K-ideal of H_2.*

Proof. Let I be a strong hyper K-ideal of H_1. Clearly $0 \in f(I)$. Let $x, y \in H_2$ be such that $(x \circ y) \cap f(I) \neq \varnothing$ and $y \in f(I)$. Since $y \in f(I)$ and f is onto, there are $y_1 \in I$ and $x_1 \in H_1$ such that $y = f(y_1)$ and $x = f(x_1)$. Thus

$$\varnothing \neq (x \circ y) \cap f(I) = f(x_1 \circ y_1) \cap f(I), \tag{4.3}$$

and so there exists $a \in H_2$ such that $a \in f(x_1 \circ y_1)$ and $a \in f(I)$. It follows that there are $a_1 \in x_1 \circ y_1$ and $b_1 \in I$ such that $a = f(a_1)$ and $a = f(b_1)$ so that

$$0 \in a \circ a = f(a_1) \circ f(b_1) = f(a_1 \circ b_1), \tag{4.4}$$

which implies that $f(c) = 0$ for some $c \in a_1 \circ b_1$. Hence $c \in \ker f \subseteq I$ and so $(a_1 \circ b_1) \cap I \neq \varnothing$. Now since I is a strong hyper K-ideal of H_1 and $b_1 \in I$, we get $a_1 \in I$. Thus $(x_1 \circ y_1) \cap I \neq \varnothing$, which implies that $x_1 \in I$. Thereby $x = f(x_1) \in f(I)$, and so $f(I)$ is a strong hyper K-ideal of H_2. □

Theorem 4.7. *Let $f : H_1 \to H_2$ be a homomorphism of hyper K-algebras. If f is onto and I is a strong implicative hyper K-ideal of H_1 which contains* $\ker f$, *then $f(I)$ is a strong implicative hyper K-ideal of H_2.*

Proof. The proof is similar to the proof of Theorem 4.6. □

The following theorem is straightforward, and so we omit the proof.

Theorem 4.8. *Let $f : H_1 \to H_2$ be an epimorphism of hyper K-algebras. Then there is a one to one correspondence between the set of all strong (implicative) hyper K-ideals of H_1 containing* $\ker f$ *and the set of all strong (implicative) hyper K-ideals of H_2.*

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