RIESZ-MARTIN REPRESENTATION FOR POSITIVE SUPER-POLYHARMONIC FUNCTIONS IN A RIEMANNIAN MANIFOLD

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Let u be a super-biharmonic function, that is, $\Delta^2 u \geq 0$, on the unit disc D in the complex plane, satisfying certain conditions. Then it has been shown that u has a representation analogous to the Poisson-Jensen representation for subharmonic functions on D. In the same vein, it is shown here that a function u on any Green domain Ω in a Riemannian manifold satisfying the conditions $(-\Delta)^i u \geq 0$ for $0 \leq i \leq m$ has a representation analogous to the Riesz-Martin representation for positive superharmonic functions on Ω.

1. Introduction

Let u be a locally Lebesgue integrable function defined on the unit disc D in the complex plane. u is called a super-biharmonic function if $\Delta^2 u \geq 0$ in the sense of distributions. Abkar and Hedenmalm [1] consider a super-biharmonic function u on D, satisfying two conditions which regulate the growth of u near the boundary ∂D. These conditions are used to split u into its biharmonic Green potential part and its biharmonic part. Using this decomposition, they show that u can be represented by three measures, one on D and two on the boundary ∂D. This comes out as a generalization of the Riesz-Poisson integrals to the super-biharmonic functions on D. However, an extension of this representation in the case of the unit ball in $\mathbb{R}^n$, $n > 2$ (or to the case of $\Delta^m u \geq 0$ with suitable restrictions on u in the unit disc itself) seems complicated.

In this paper, we consider a set of two other conditions on a function u satisfying $\Delta^2 u \geq 0$, namely, $u \geq 0$ and $\Delta u \leq 0$. These conditions are more appropriate as a generalization of the positive superharmonic functions. For, suppose u is a locally Lebesgue integrable function on a bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, such that $u \geq 0$, $\Delta u \leq 0$, and $\Delta^2 u \geq 0$. Then u can be represented by three positive measures, one on Ω and two on the Martin boundary of Ω. Interestingly, the method of proof is general enough to be used in the case of $(-\Delta)^i u \geq 0$, $0 \leq i \leq m$, for any integer $m \geq 2$, and any domain Ω in $\mathbb{R}^n$ on which the Green function is defined (in particular on any bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$); actually, it goes through in the case of a Riemannian manifold also. Accordingly, we prove this result in the context of a Riemannian manifold.

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2. Preliminaries

Let R be an oriented Riemannian manifold of dimension ≥ 2, with local coordinates $x = (x^1,\ldots,x^n)$ and a C^∞-metric tensor g_{ij} such that $g_{ij}x^ix^j$ is positive definite. Denote the volume element by $dx = \sqrt{\det(g_{ij})}dx^1,\ldots,dx^n$. Let Δ be the Laplace-Beltrami operator which, acting on a C^2-function f, gives $\Delta f = \text{div}\, grad\, f$. However, we will assume that Δ is taken in the sense of distributions. Thus, a locally dx-integrable function f on an open set ω in R is said to be superharmonic (resp., harmonic) if $\Delta f \leq 0$ (resp., $\Delta f = 0$) on ω; a positive superharmonic function u on ω is called a potential if and only if the greatest harmonic minorant of u on ω is 0, (i.e., if h is harmonic on ω and $h \leq u$, then h should be negative).

For each open set ω in R, let $H(\omega)$ denote the class of C^2-functions u on ω such that $\Delta u = 0$. If ω is a domain, $H(\omega)$ has the Harnack property, namely, if h_n is an increasing sequence in $H(\omega)$ and if $h = \sup h_n$, then $h \in H(\omega)$ or $h \equiv \infty$. We can also solve the Dirichlet problem on any parametric ball. This means that the set of harmonic functions $H(\omega)$ satisfies the axioms 1, 2, 3 of Brelot [7, pages 13-14]. Consequently, we can use the results and the terminology of the Brelot axiomatic potential theory in the context of the Riemannian manifold R.

A domain Ω in R is called a Green domain if the Green function $G(x,y)$ is well defined on Ω. On a Green domain Ω in R, we can construct the Martin compactification $\overline{\Omega}$ of Ω as in [8, pages 111–115]. Some of the important points to remember here are the following: fix a point y_0 in a Green domain Ω. If $G(x,y)$ is the Green function on Ω, write $k_y(x) = k(x,y) = G(x,y)/G(x,y_0)$ with the convention $k(y_0,y_0) = 1$. Then there exists only one (metrizable) compactification $\overline{\Omega}$ up to homeomorphism such that

(i) Ω is dense open in the compact space $\overline{\Omega}$;

(ii) $k_y(x)$, $y \in \Omega$, extends as a continuous function of x on $\overline{\Omega}$;

(iii) the family of these extended continuous functions on $\overline{\Omega}$ separates the points $x \in \Delta = \overline{\Omega}\backslash\Omega$.

$\overline{\Omega}$ is called the Martin compactification of Ω and $\Delta = \overline{\Omega}\backslash\Omega$ is called the Martin boundary. A positive harmonic function $u > 0$ is called *minimal* if and only if for any harmonic function v, $0 \leq v \leq u$, we should have $v = \alpha u$ for a constant α, $0 \leq \alpha \leq 1$. It can be proved that every minimal harmonic function $u(y)$ on Ω is of the form $u(y_0)k(x,y)$ for some $x \in \Delta$, and the points $x \in \Delta$ corresponding to these minimal harmonic functions are called the minimal points of Δ, and the set of minimal points of Δ is denoted by Δ_1, called the *minimal boundary*.

With these remarks, we can state the Martin representation theorem: for any harmonic function $u \geq 0$ on Ω, there exists a unique Radon measure $\mu \geq 0$ on Δ with support in the minimal boundary $\Delta_1 \subset \Delta$ such that $u(y) = \int_{\Delta_1} k(x,y)d\mu(x)$.

In the particular case of $R = \mathbb{R}^n$, $n \geq 2$, and $\Omega = B(0,1)$ the unit ball, taking the fixed point y_0 as the centre 0, we have the following: the Martin boundary $\Delta = \overline{\Omega}\backslash\Omega$ is homeomorphic to the unit sphere S and $k(x,y)$ is the Poisson kernel; also $\Delta_1 = \Delta = S$. Then the Martin representation gives the familiar result (see, e.g., Axler et al. [4, page 105]): if u is positive and harmonic on B, then there exists a unique positive Borel measure on S such that $u(x) = \int_S p(x,y)d\mu(y)$, where $p(x,y)$, $x \in B$, $y \in S$, is the Poisson kernel.

3. Riesz-Martin representation for positive super-biharmonic functions

Let Ω be a Green domain in a Riemannian manifold R, with the Green function $G(x,y)$ which is a symmetric function and for fixed y, $G_y(x) = G(x,y)$ is a potential on Ω; we have also $\Delta G_y(x) = -\delta_y(x)$, after a normalization.

Definition 3.1. A Green domain Ω in R called a biharmonic Green domain if for a pair of points x and y in Ω, $G^2(x,y) = \int_\Omega G(x,z)G(z,y)dz$ is finite. Then $G^2(x,y)$ is called the biharmonic Green function of Ω.

The above definition is given in Sario [10] when $\Omega = R$, a hyperbolic manifold. On an arbitrary hyperbolic Riemannian manifold R, the biharmonic Green function may or may not exist. It is shown in [2, Theorem 3.2] that the biharmonic Green function $G^2(x,y)$ can be defined on a hyperbolic Riemannian manifold R if and only if there exist two positive potentials p and q on R such that $\Delta q = -p$.

Consequently, any relatively compact domain Ω in a Riemannian manifold R is a biharmonic Green domain, whether R is hyperbolic or parabolic. Note that if Ω is a biharmonic Green domain in R, then $u(x) = G^2(x,y)$ is a potential on Ω, for fixed y; and $\Delta u(x) = \Delta_x G_y^2(x) = -G_y(x)$ so that $\Delta^2 u(x) = \delta_y(x)$.

Given a Radon measure $\mu \geq 0$ on Ω, if we set $p(x) = \int_\Omega G(x,y)d\mu(y)$, then we know that $p \equiv \infty$ or $p(x)$ is a potential such that $\Delta p = -\mu$. Let now $q(x) = \int_\Omega G^2(x,y)d\mu(y)$ be finite at some point $x_0 \in \Omega$. Then,

$$\infty > \int_\Omega \left(\int_\Omega G(x_0,z)G(z,y)dz \right) d\mu(y) = \int_\Omega G(x_0,z) \left[\int_\Omega G(z,y)d\mu(y) \right] dz. \tag{3.1}$$

Hence $p(z) = \int_\Omega G(z,y)d\mu(y) \not\equiv \infty$, so that $p(z)$ is a potential on Ω, and $q(x) = \int_\Omega G(x,z)p(z)dz$, which shows that $q(x)$ is a potential on Ω and $\Delta q(x) = -p(x) = -\int_\Omega G(x,y)d\mu(y)$.

Let $\overline{\Omega}$ be the Martin compactification of Ω, $\Delta = \overline{\Omega} \setminus \Omega$ the Martin boundary, and Δ_1 the minimal boundary $\subset \Delta$. Let $k(x,y)$ be the Martin kernel, $(x,y) \in \overline{\Omega} \times \Omega$.

Notation 3.2. (1) Let π_2 denote the set of positive Radon measures μ on Ω such that $q(x) = \int_\Omega G^2(x,y)d\mu(y)$ is a potential on Ω.

(2) Let $\wedge_0$ denote the set of positive Radon measures ν on Δ, with supp $\nu \subset \Delta_1$.

(3) Let $\wedge_1$ denote the positive Radon measures $\nu \in \wedge_0$ such that $u(x) = \int_\Omega G(x,y)[\int_{\Delta_1} k(X,y)d\nu(X)]dy$ is a potential on Ω. In that case, $\Delta u(x) = -\int_{\Delta_1} k(X,x)d\nu(X)$ which is harmonic, so that $u(x)$ is also a biharmonic function on Ω. (Remark that $\wedge_1$ can be empty as in the case of $\Omega = \mathbb{R}^n$.) If $\nu \in \wedge_1$, we will write $k_1(X,x) = \int_\Omega G(x,y)k(X,y)dy$ for $X \in \Delta_1$, and $x \in \Omega$, so that $u(x) = \int_\Omega G(x,y)[\int_{\Delta_1} k(X,y)d\nu(X)]dy$ can be more elegantly represented as $u(x) = \int_{\Delta_1} k_1(X,x)d\nu(X)$.

LEMMA 3.3. *Let $\mu \geq 0$ be a Radon measure on an open set ω in a Riemannian manifold R, hyperbolic or parabolic. Then there exists a superharmonic function s on ω with μ as the associated measure in a local Riesz representation.*

Proof. The statement means that for every point $x_0 \in \omega$, there is a neighborhood δ, $x_0 \in \delta \subset \overline{\delta} \subset \omega$, with the Green function $G^\delta(x,y)$ such that $s(x) = \int_\delta G^\delta(x,y)d\mu(y) +$ (a harmonic function $h(x)$) in δ.

For the construction of s in $\mathbb{R}^n$, we refer to Brelot [6]. A similar method, with the use of an approximation property given in Bagby and Blanchet [5, Theorem 3.10], proves the result in a Riemannian manifold. (For a more general discussion of this result, see [3, Section 2].) □

By Definition 3.1, a Green domain Ω in a Riemannian manifold R (whether hyperbolic or parabolic) is a biharmonic Green domain if and only if $G^2(x,y) \not\equiv \infty$ on Ω. Note that $u(x) = G_y^2(x) > 0$, $\Delta u(x) = G_y(x) < 0$, and $\Delta^2 u(x) = \delta_y(x) \geq 0$ on Ω. Hence on a biharmonic Green domain Ω, functions v of the type $v > 0$, $\Delta v \leq 0$, and $\Delta^2 v \geq 0$ exist. The following theorem gives an integral representation for such functions.

Theorem 3.4. *Let Ω be a biharmonic Green domain in a Riemannian manifold R (whether R is hyperbolic or parabolic) and let v be a locally dx-integrable function on Ω. Then the following are equivalent.*

(a) $v \geq 0$, $\Delta v \leq 0$, *and* $\Delta^2 v \geq 0$ *on* Ω.

(b) $v(x) = \int_\Omega G^2(x,y)d\mu(y) + \int_{\Delta_1} k_1(X,x)d\nu_1(X) + \int_{\Delta_1} k(X,x)d\nu_0(X)$ *a.e. on* Ω, *where* $(\mu,\nu_1,\nu_0) \in \pi_2 \times \wedge_1 \times \wedge_0$ *is uniquely determined.*

Proof. (b)⇒(a). Let

$$u(x) = \int_\Omega G^2(x,y)d\mu(y) + \int_{\Delta_1} k_1(X,x)d\nu_1(X) + \int_{\Delta_1} k(X,x)d\nu_0(X), \tag{3.2}$$

where $\mu \geq 0$ is a Radon measure on Ω, and ν_0, ν_1 are positive Radon measures on Δ_1. Then $u(x) = v(x)$ a.e. on Ω by the assumption. Hence $u \not\equiv \infty$.

(i) Let $u_1(x) = \int_\Omega G^2(x,y)d\mu(y)$. Then $u_1 \geq 0$ is a potential on Ω, such that $\Delta u_1(x) = -\int G(x,y)d\mu(y)$ and $\Delta^2 u_1 = \mu$.

(ii) Let

$$u_2(x) = \int_{\Delta_1} k_1(X,x)d\nu_1(X) = \int_\Omega G(x,y)\left[\int_{\Delta_1} k(X,y)d\nu_1(X)\right]dy. \tag{3.3}$$

Then $u_2 \geq 0$ is a potential on Ω, such that $\Delta u_2(x) = -\int_{\Delta_1} k(X,x)d\nu_1(X) = -h_1(x)$, where $h_1(x)$ is a positive harmonic function on Ω, so that $\Delta u_2 \leq 0$ and $\Delta^2 u_2 \equiv 0$.

(iii) Let $u_3(x) = \int_{\Delta_1} k(X,x)d\nu_0(X)$.

Then $u_3 \geq 0$ is harmonic on Ω, so that $\Delta u_3 \equiv 0$ and $\Delta^2 u_3 \equiv 0$.

Consequently, $u = u_1 + u_2 + u_3 \geq 0$ on Ω such that $\Delta u \leq 0$ and $\Delta^2 u \geq 0$ on Ω. Since $u = v$ a.e., the statement (a) is proved.

(a)⇒(b). Since $\Delta^2 v \geq 0$, $\Delta^2 v = \mu$, where μ is a positive Radon measure on Ω. Since $\Delta(\Delta v) = \mu$, Δv is a subharmonic function on Ω. Since $\Delta v \leq 0$ by hypothesis, $-\Delta v$ is a positive superharmonic function on Ω. Hence by the Riesz representation theorem,

$$-\Delta v(x) = \int_\Omega G(x,y)d\mu(y) + h(x), \tag{3.4}$$

where $h(x)$ is a positive harmonic function on Ω.

Let us choose (using the lemma above) two superharmonic functions $q(x)$ and $H(x)$ on Ω such that

$$\Delta q(x) = -\int_{\Omega} G(x,y)\,d\mu(y), \qquad \Delta H(x) = -h(x). \tag{3.5}$$

Then from (3.4),

$$v(x) = q(x) + H(x) + (\text{a harmonic function } h_1) \quad \text{on } \Omega. \tag{3.6}$$

Since $v \geq 0$ on Ω, $q(x) \geq -H(x) - h_1(x)$; that is, $q(x)$ has a subharmonic minorant on Ω. Hence $q(x)$ has the greatest harmonic minorant $h_2(x)$ on Ω, and by the Riesz representation theorem,

$$\begin{aligned} q(x) &= \int_{\Omega} G(x,y)\big(-\Delta q(y)\big)dy + h_2(x) \quad \text{on } \Omega \\ &= \int_{\Omega} G(x,z)\Big[\int_{\Omega} G(z,y)\,d\mu(y)\Big]dz + h_2(x) \\ &= \int_{\Omega} G^2(x,y)\,d\mu(y) + h_2(x). \end{aligned} \tag{3.7}$$

Similarly, dealing with the superharmonic function $H(x)$ and its greatest harmonic minorant $h_3(x)$ on Ω, we can write

$$\begin{aligned} H(x) &= \int_{\Omega} G(x,y)\big(-\Delta H(y)\big)dy + h_3(x) \quad \text{on } \Omega \\ &= \int_{\Omega} G(x,y)h(y)\,dy + h_3(x) \\ &= \int_{\Omega} G(x,y)\Big(\int_{\Delta_1} k(X,y)\,dv_1(X)\Big)dy + h_3(x), \end{aligned} \tag{3.8}$$

by using the Martin representation for the positive harmonic function h on Ω. Note that $v_1 \in \wedge_1$ and is uniquely determined. Consequently,

$$H(x) = \int_{\Delta_1} k_1(X,x)\,dv_1(X) + h_3(x). \tag{3.9}$$

Now, using (3.6), (3.7), and (3.9), we write

$$v(x) = \int_{\Omega} G^2(x,y)\,d\mu(y) + \int_{\Delta_1} k_1(X,x)\,dv_1(X) + h_0(x), \tag{3.10}$$

where $h_0 = h_1 + h_2 + h_3$ is harmonic on Ω.

Now by hypothesis $v \geq 0$, so that

$$-h_0(x) \leq \int_{\Omega} G^2(x,y)\,d\mu(y) + \int_{\Delta_1} k_1(X,x)\,dv_1(X) \quad \text{on } \Omega. \tag{3.11}$$

Now the two terms on the right side are potentials on Ω and hence their sum also is a potential on Ω. This means that the harmonic function $-h_0$ is majorized by a potential on Ω, so that $-h_0 \leq 0$. Thus h_0 is a positive harmonic function Ω. Use the Martin representation to conclude that there exists a unique measure ν_0 on the Martin boundary with support in Δ_1, such that

$$h_0(x) = \int_{\Delta_1} k(X,x)\,d\nu_0(X). \tag{3.12}$$

Thus, from (3.10) and (3.12), we finally arrive at the representation for $v(x)$ on Ω:

$$v(x) = \int_{\Omega} G^2(x,y)\,d\mu(y) + \int_{\Delta_1} k_1(X,x)\,d\nu_1(X) + \int_{\Delta_1} k(X,x)\,d\nu_0(X), \tag{3.13}$$

where $(\mu,\nu_1,\nu_0) \in \pi_2 \times \wedge_1 \times \wedge_0$ is uniquely determined. □

4. Representation for positive super-polyharmonic functions

By induction, we can extend Theorem 3.4 to obtain the Riesz-Martin representation for positive super-polyharmonic functions.

Let Ω be a Green domain in a Riemannian manifold R, with $G(x,y)$ as the Green function of Ω. For an integer $m \geq 2$, we will denote

$$G^m(x,y) = \int G(x,z_{m-1})G(z_{m-1},z_{m-2})\cdots G(z_1,y)\,dz_1\cdots dz_{m-1} \tag{4.1}$$

and say that a positive Radon measure μ on Ω is in π_m if $u(x) = \int_{\Omega} G^m(x,y)\,d\mu(y) \not\equiv \infty$ on Ω, in which case $u(x)$ is a potential on Ω and $(-\Delta)^m u = \mu$; also $(-\Delta)^j u \geq 0$ for $0 \leq j \leq m$. When such a function $u(x)$ exists on Ω, we say that Ω is an m-harmonic Green domain in R, whether R is hyperbolic or parabolic.

Let $\overline{\Omega}$ be the Martin compactification of Ω and let $k(x,y)$ be the Martin kernel. For any i, $1 \leq i \leq m-1$, let $\wedge_i$ denote the set of positive Radon measures ν_i on $\Delta = \overline{\Omega}\backslash\Omega$ with support in the minimal boundary Δ_1, such that

$$v_i(x) = \int G(x,z_i)G(z_i,z_{i-1})\cdots G(z_2,z_1)\left[\int_{\Delta_1} k(X,z_1)\,d\nu(X)\right]dz_1\cdots dz_i \not\equiv \infty. \tag{4.2}$$

In that case, $v_i(x)$ is a potential on Ω, $(-\Delta)^i v_i \equiv 0$; also $(-\Delta)^j v_i \geq 0$ for $0 \leq j \leq i$. Let us write for $X \in \Delta_1$ and $x \in \Omega$,

$$k_i(X,y) = \int G(x,z_i)\cdots G(z_2,z_1)k(X,z_1)\,dz_1\cdots dz_i. \tag{4.3}$$

Then, if $\nu \in \wedge_i$, $v_i(x) = \int_{\Delta_1} k_i(X,x)\,d\nu(X)$ is well defined on Ω with the above properties.

As before, let $\wedge_0$ denote the set of positive Radon measures ν on Δ, with support in Δ_1.

Then, the proof of Theorem 3.4 can be extended by using the method of induction to arrive at the following result.

Theorem 4.1. *Let Ω be an m-harmonic Green domain in a Riemannian manifold R and let v be a locally dx-integrable function on Ω. Let $m \geq 1$ be an integer. Then the following* *are equivalent.*

(a) $(-\Delta)^i v \geq 0$ *on* Ω *for* $0 \leq i \leq m$.

(b) *There exist unique measures* $\mu \in \pi_m$ *and* $\nu_i \in \wedge_i$ *for* $0 \leq i \leq m-1$ *such that*

$$v(x) = \int_\Omega G^m(x,y)d\mu(y) + \sum_{i=0}^{m-1} \int_{\Delta_1} k_i(X,x)d\nu_i(X) \quad \text{a.e. on } \Omega. \tag{4.4}$$

5. Integral representations in a Riemann surface

We are not in a position to say that the above integral representation theorems in a Riemannian manifold R are automatically valid in a Riemann surface S. For, we have used the Laplace-Beltrami operator Δ on R to define polyharmonic-superharmonic functions on R and also to obtain some of their properties. But the Laplacian is not invariant under a parametric change in an abstract Riemann surface S. Hence there is a problem. We indicate in this section how to get over this difficulty.

Let S be a Riemann surface. Let $\mu \geq 0$ be a Radon measure defined on an open set ω in S. Then, using an approximation theorem of Pfluger [9, page 192], we can show that there exists a superhamonic function s on ω with associated measure μ in a local Riesz representation as explained in Lemma 3.3 (see [3, Theorem 2.3]). Let us symbolically denote this relation between s and μ by $Ls = -\mu$ on ω.

Let now $d\sigma$ denote the surface measure on S. Then, given any locally $d\sigma$-integrable function f on an open set ω, let λ be the signed measure on ω defined by $d\lambda = f\,d\sigma$. Construct as above two superharmonic functions s_1 and s_2 on ω, such that $Ls_1 = -\lambda^+$ and $Ls_2 = -\lambda^-$. Let us denote this relation between the δ-superharmonic function $s = s_1 - s_2$ and the locally $d\sigma$-integrable function f by $Ls = -f$.

We will say that $s = (s_m, s_{m-1}, \ldots, s_1)$ is a polyharmonic-superharmonic function of order m in an open set ω, if s_1 is superharmonic on ω and $Ls_i = -s_{i-1}$ for $2 \leq i \leq m$. We will say that $s \geq 0$ if each $s_i \geq 0$. If there exists a polyharmonic-superharmonic function $s = (s_m, s_{m-1}, \ldots, s_1) \geq 0$, $s_i \not\equiv 0$ for any i, on a domain Ω in S, we say that Ω is an m-harmonic Green domain in S.

Let now Ω be a Green domain in a Riemann surface S. As before, let $\overline{\Omega}$ be the Martin compactfication of Ω, let $\Delta = \overline{\Omega} \backslash \Omega$ be the Martin boundary, and let Δ_1 be the minimal boundary. Then, with the notations as in Section 4, we can prove the following.

Theorem 5.1. *Let* Ω *be an* m*-harmonic Green domain in a Riemann surface* S. *Let* $m \geq 1$ *be an integer. Then, the following are equivalent.*

(a) $s = (s_m, s_{m-1}, \ldots, s_1) \geq 0$ *is a polyharmonic-superharmonic function of order* m *in* Ω.

(b) *For any* j, $1 \leq j \leq m$, *there exist unique measures* $\mu \in \pi_j$ *and* $\nu_i \in \wedge_i$ *for* $0 \leq i \leq j-1$ *such that*

$$s_j(x) = \int_\Omega G^j(x,y)d\mu(y) + \sum_{i=0}^{j-1} \int_{\Delta_1} k_i(X,x)d\nu_i(X) \quad \text{a.e. on } \Omega. \tag{5.1}$$

(c) *The above property* (b) *is satisfied for* $j = m$.

Proof. (a)⇒(b). Fix j, $1 \le j \le m$. Then $(s_j, s_{j-1}, \dots, s_1)$ is a j-superharmonic function on Ω, since $(-L)s_{i+1} = s_i$ for $1 \le i \le j-1$ and s_1 is superharmonic. Moreover, since $(-L)s_{i+1} \ge 0$, each s_i is a positive superharmonic function. Write $s_1 = p_1 + h_1$ as the unique sum of a potential p_1 and a positive harmonic function h_1. Let $(-L)p_1^* = p_1$ and $(-L)h_1^* = h_1$. Then p_1^* and h_1^* are superharmonic on Ω and

$$(-L)s_2 = p_1 + h_1 = (-L)p_1^* + (-L)h_1^*. \tag{5.2}$$

That is, $s_2 = p_1^* + h_1^* +$ (a harmonic function) on Ω. Since $s_2 \ge 0$, p_1^* has a subharmonic minorant on Ω and hence $p_1^* =$ (a potential p_2) + (the greatest harmonic minorant of p_1^*, which may not necessarily be positive).

Then $s_2 = p_2 + u_2$, where u_2 is superharmonic on Ω. Since $s_2 \ge 0$, $p_2 \ge -u_2$. Since p_2 is a potential and $-u_2$ is subharmonic, $-u_2 \le 0$. Hence $s_2 = p_2 + u_2$, where p_2 is a potential on Ω such that $(-L)p_2 = p_1$ and $u_2 \ge 0$ is superharmonic such that $(-L)u_2 = h_1$.

Thus proceeding, we can write

$$(s_j, \dots, s_2, s_1) = (p_j, \dots, p_2, p_1) + (u_j, \dots, u_2, h_1), \tag{5.3}$$

where $(-L)p_{i+1} = p_i$ for $1 \le i \le j-1$, and $p_1, \dots, p_j$ are all potentials; $(-L)u_{i+1} = u_i$ for $2 \le i \le j-1$ and $(-L)u_2 = h_1$.

Now take $(u_j, \dots, u_2, h_1)$ and proceed as before. Note now h_1 is positive harmonic, so that we can write

$$(u_j, \dots, u_2, h_1) = (q_j, \dots, q_2, h_1) + (f_j, \dots, f_3, h_2, 0), \tag{5.4}$$

where $(-L)q_{i+1} = q_i$ for $2 \le i \le j-1$, $(-L)q_2 = h_1$, and each q_i is a potential; $(-L)f_{i+1} = f_i \ge 0$ for $3 \le i \le j-1$, $(-L)f_3 = h_2$, and $(-L)h_2 = 0$, so that h_2 is positive harmonic.

Then take $(f_j, \dots, f_3, h_2, 0)$ and follow the same procedure, so that

$$(f_j, \dots, f_3, h_2, 0) = (r_j, \dots, r_3, h_2, 0) + (g_j, \dots, g_4, h_3, 0, 0), \tag{5.5}$$

where $(-L)r_{i+1} = r_i$ for $3 \le i \le j-1$, $(-L)r_3 = h_2$ and each r_i is a potential; $(-L)g_{i+1} = g_i \ge 0$ for $4 \le i \le j-1$, $(-L)g_4 = h_3$ and $(-L)h_3 = 0$, so that h_3 is harmonic ≥ 0.

Thus proceeding, we finally arrive at the decomposition

$$(s_j, \dots, s_1) = (p_j, \dots, p_1) + (q_j, \dots, q_2, h_1) + (r_j, \dots, r_3, h_2, 0) + \cdots + (h_j, 0, \dots, 0). \tag{5.6}$$

Let $(-L)p_1 = \mu$; let ν_i $(1 \le i \le j)$ be the positive Radon measure on Δ with support in Δ_1, associated with the positive harmonic function h_i in the Martin representation.

Then $s_j = p_j + q_j + r_j + \cdots + h_j$ has the integral representation

$$s_j(x) = \int_\Omega G^j(x, y)\, d\mu(y) + \sum_{i=0}^{j-1} \int_{\Delta_1} k_i(X, x)\, d\nu_i(X) \quad \text{a.e. on } \Omega. \tag{5.7}$$

(b)⇒(c). $j = m$ is a particular case of (b).

(c)⇒(a). By the assumption,

$$s_m(x) = \int_{\Omega} G^m(x,y)\,d\mu(y) + \sum_{i=0}^{m-1} \int_{\Delta_1} k_i(X,y)\,d\nu_i(X) \quad \text{a.e.} \tag{5.8}$$

Hence we can express s_m in the form $s_m(x) = p_m(x) + \sum_{j=0}^{m-1} q_j(x)$. We can calculate to find that $(-L)^i p_m$ is a potential for $1 \le i \le m-1$ and $(-L)^m p_m = \mu$, a positive Radon measure; and $(-L)^i q_j$ is a potential for $1 \le i \le j-1$ and $(-L)^j q_j = 0$.

Write now $(-L)s_m = s_{m-1}$, $(-L)s_{m-1} = s_{m-2}, \dots, (-L)s_2 = s_1$. We can see that each s_i $(1 \le i \le m)$ is a positive superharmonic function and $(-L)s_{i+1} = s_i$ for $1 \le i \le m-1$.

Hence $s = (s_m, s_{m-1}, \dots, s_1) \ge 0$ is a polyharmonic-superharmonic function of order m. □

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