REDUCED P.P.-RINGS WITHOUT IDENTITY

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We give some necessary and sufficient conditions for p.p.-rings without identity to be reduced. Our results strengthen and extend the results of Fraser and Nicholson as well as some recent results we obtained on reduced p.p.-rings with identity.

1. Introduction

Throughout this paper, the ring R is not necessarily with an identity. We denote the set of all idempotents of R by $E(R)$. Also, for a subset $X \subseteq R$, we denote the right (resp., left) annihilator of X in R by $\operatorname{ann}_r(X)$ (resp., $\operatorname{ann}_\ell(X)$).

Now, according to Fraser and Nicholson in [5], we call a ring R a *left p.p.-ring*, in brevity, l.p.p.-ring, if for all $x \in R$, there exists an idempotent e such that $\operatorname{ann}_\ell(x) = \operatorname{ann}_\ell(e)$ and $ex = x$. Dually, we may define a *right p.p.-ring*. Naturally, we call a ring R a p.p.-*ring* if it is both an l.p.p.-ring and an r.p.p.-ring. Clearly, if the ring R has an identity, the above (left; right) p.p.-rings coincide with the (left; right) p.p.-rings discussed in [6]. It can be observed that the class of p.p.-rings contains the classes of regular (von Neumann) rings, hereditary rings, Baer rings, and semi-hereditary rings as its proper subclasses. In the literature, p.p.-rings have already been studied by many authors (see [1, 2, 5–9, 11]). It is noteworthy that the definition of p.p.-rings has been extended to semigroups; in particular, Fountain [4] has introduced the concept of *abundant semigroups* which are both l.p.p.- and r.p.p.- semigroups. Similar to p.p.-rings, the class of abundant semigroups contains the class of regular semigroups as its proper subclass. An r.p.p.-semigroup in which every idempotent is central is called a C-r.p.p.-*semigroup*. In 1977 Fountain [3] first proved that a C-r.p.p.-monoid can be expressed as a strong semilattice of left cancellative monoids. This shows that a C-r.p.p.-monoid does not contain any nonzero nilpotent element and hence it is a reduced semigroup.

On the other hand, Cornish and Stewart [2] called a ring R *reduced* if it contains no nonzero nilpotent elements. Obviously, the left annihilator $\operatorname{ann}_\ell(X)$ of X in a reduced ring R is always a two-sided ideal of R. Moreover, if R is a reduced ring, then $ef = 0$ if and

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only if $fe = 0$ for any nonzero idempotents $e, f \in R$. By using the concept of a reduced ring, Fraser and Nicholson [5] obtained an analogous result of Fountain's [4] that a ring R is a reduced p.p.-ring if and only if R is a (left; right) p.p.-ring in which every idempotent is central.

In view of the above result, one would naturally ask whether we can give some necessary and sufficient conditions for a p.p.-ring to be reduced. In this aspect, the authors [6] have recently given some characterization theorems for reduced p.p.-rings with identity. In this paper, we further investigate this question for p.p.-rings without identity by using the properties of abundant semigroups. Some new characterization theorems for a p.p.-ring without identity to be reduced are obtained. Our results strengthen and extend the results obtained by Fraser and Nicholson [5] and those by the authors in [6]. The concept of perpetual ideals in p.p.-rings is also introduced.

2. Preliminaries

We first cite some known results of abundant semigroups which will be used in the sequel.

Let $\mathcal{L}^*$ and $\mathcal{R}^*$ be the left and right Green starred relations on an abundant semigroup S, as described by Fountain in [3, 4]. Then we have the following lemma.

Lemma 2.1 [4]. *Let S be a semigroup. Then, for any elements $a, b \in S$, $a\mathcal{L}^* b$ $[a\mathcal{R}^* b]$ if and only if $ax = ay \Leftrightarrow bx = by$ $[xa = ya \Leftrightarrow xb = yb]$, for all $x, y \in S^1$, where S^1 is the semigroup S adjoined with an identity 1.*

As an easy but useful consequence, we have the following corollary.

Corollary 2.2 [4]. *Let S be a semigroup. Then $a\mathcal{L}^* e$ $[a\mathcal{R}^* e]$ if and only if $a = ae$ $[a = ea]$ and $ax = ay \Rightarrow ex = ey$ $[xa = ya \Rightarrow xe = ye]$, for all idempotents $e \in S$ and $x, y \in S^1$.*

In view of the above corollary, we call an element $a \in S$ a *right abundant element* if there exists an idempotent $e \in E(S)$ such that $e\mathcal{L}^* a$. Dually, we call an element $a \in S$ a *left abundant element* if there exists an idempotent $f \in E(S)$ such that $f\mathcal{R}^* a$. An element is called *abundant* if it is both left and right abundant. A semigroup S is called *abundant* if every element of S is abundant.

The following lemma was due to Lawson [10].

Lemma 2.3 [10]. *Let S be an abundant semigroup and $e^2 = e \in S$. Then eSe is an abundant subsemigroup of S.*

Lemma 2.4. *The following statements hold on a ring R.*

(1) *If R has an identity, then R is a p.p.-ring if and only if the multiplicative semigroup $(R, \bullet)$ of R is an abundant semigroup.*

(2) *If R is a p.p.-ring, then the multiplicative semigroup $(R, \bullet)$ is an abundant semigroup.*

Proof. (1) This part is trivial. (2) Assume that R is a p.p.-ring. Let $a \in R$ and $u, v \in (R, \bullet)^1$ with $ua = va$. Then because R is a p.p.-ring, there exists $e \in E(R)$ such that $\operatorname{ann}_\ell(a) = \operatorname{ann}_\ell(e)$ and $ea = a$. We consider the following two cases.

(i) $u, v \in R$. Then $(u-v)a = 0$ and so $(u-v)e = 0$, that is, $ue = ve$.

(ii) One of u and v does not lie in R; say $v \notin (R, \bullet)$. Then, in this case, $v = 1$ and hence we have $ua = a$. Since $ea = a$, we have $(u-e)a = 0$ and thereby, $(u-e) \in \mathrm{ann}_\ell\, a = \mathrm{ann}_\ell\, e$, that is, $ue = e$.

Thus $a\mathcal{R}^* e$ and hence $(R, \bullet)$ is a left abundant semigroup. Dually, we can similarly prove that $(R, \bullet)$ is a right abundant semigroup. Therefore $(R, \bullet)$ is an abundant semigroup. □

It is well known that the Green starred relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are congruences on an abundant semigroup S. Now, following Fountain [4], we define $\mathcal{H}^*$ as the intersection of the congruences $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroup S. Then, we call a semigroup S a *superabundant semigroup* if each $\mathcal{H}^*$-class of S contains an idempotent in S. The structure of superabundant semigroups has recently been studied by Ren and Shum in [12]. Of course, a superabundant semigroup is always abundant.

By a *band* we mean a semigroup B whose elements are idempotents. In addition, if a band B is commutative, then we call B a *semilattice*.

As in [6], we now denote the 2×2 upper triangular matrix rings over $\mathbb{Z}$ and $\mathbb{Z}_p$ by $\mathrm{UTM}_2(\mathbb{Z})$ and $\mathrm{UTM}_2(\mathbb{Z}_p)$, respectively.

The following result for reduced p.p.-rings with identity was proved in [6].

Lemma 2.5 [6]. *Let R be a p.p.-ring with an identity. Then R is reduced if and only if R has no subrings which are isomorphic either to $UTM_2(\mathbb{Z})$ or to $UTM_2(\mathbb{Z}_p)$, where p is a prime.*

3. Characterization theorems

By using the results cited in the above section, we now establish some new characterization theorems for reduced p.p.-rings possibly without an identity.

Theorem 3.1. *The following statements are equivalent for a p.p.-ring R:*

(1) *R is reduced;*

(2) *$E(R)$ is central in R;*

(3) *$(E(R), \bullet)$ is a semilattice;*

(4) *$(E(R), \bullet)$ is a band;*

(5) *for all $e, f \in E(R)$, $ef = 0$, if and only if $fe = 0$.*

Proof. We only need to prove that (1)⇒(2) and (2)⇒(1). This is because that (2)⇒(3)⇒(4) ⇒(5)⇒(2) is easy to see.

For (1)⇒(2), we suppose that the ring R is reduced. Then $ex = exe$ since $(ex - exe)^2 = 0$, for all $e \in E(R)$ and $x \in R$. Similarly, $xe = exe$. Thus $ex = xe$, for all $e \in E(R)$ and $x \in R$. That is, (2) holds.

For (2)⇒(1), we suppose that every element of $E(R)$ is central in R. In order to prove (1), it suffices to prove that for all $x \in R$, $x^2 = 0$ implies that $x = 0$. Since $(R, \bullet)$ is an abundant semigroup, there exists $e \in E(R)$ such that $e\mathcal{L}^* x$, and hence $xe = x$ by Corollary 2.2. Now, by Lemma 2.1, we see that $x^2 = 0$ implies that $ex = 0$, hence by (2), $xe = x = 0$. □

Theorem 3.2. *Let R be a p.p.-ring. Then R is a reduced ring if and only if $(R, \bullet)$ is a superabundant semigroup.*

Proof

Necessity. If R is reduced, then $E(R)$ is central in R. By a result of Fountain in [3], $(R,\bullet)$ is a strong semilattice of cancellative monoids and by a result of Fountain in [4], $(R,\bullet)$ is a superabundant semigroup.

Sufficiency. Suppose that $(R,\bullet)$ is a superabundant semigroup. To verify that R is reduced, we need only to prove that $x^2 = 0$ implies that $x = 0$ for all $x \in R$. Because $(R,\bullet)$ is a superabundant semigroup, there exists a unique idempotent $e \in E(R)$ such that $x\mathcal{H}^*e$. Hence, we have $ex = x$ by Corollary 2.2. Now, by Lemma 2.1, $x^2 = 0$ implies that $ex = 0$, and so we have $x = ex = 0$, as required. □

We now introduce a new definition.

Definition 3.3. A ring R is said to be *locally reduced* if the subring eRe of R is reduced for every $e \in E(R) \setminus \{1\}$.

Clearly, a locally reduced ring is not necessarily reduced. For example, if we consider the matrix ring

$$M_2(\mathbb{Z}) =: \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z} \right\}, \tag{3.1}$$

then $M_2(\mathbb{Z})$ is not reduced because $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. On the other hand, if the nontrivial idempotents in $E(R) \setminus \{1\}$ are $e_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 0 \end{smallmatrix}\right)$, then

$$\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & ax \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

Now, since

$$\begin{pmatrix} a & ax \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} a^2 & a^2x \\ 0 & 0 \end{pmatrix} = 0 \Longleftrightarrow a^2 = 0 \Longleftrightarrow a = 0, \tag{3.3}$$

we can see that every subring eRe of R is a reduced ring, and hence the ring R itself is locally reduced.

However, if the p.p.-ring R does not have an identity, then a locally reduced ring and a reduced ring are the same, as the following theorem shows.

We now give the following interesting theorem.

Theorem 3.4. *Let R be a p.p.-ring without identity. Then R is a reduced ring if and only if R is a locally reduced ring.*

Proof. Obviously, a reduced ring is locally reduced. To prove the converse part, we suppose that the ring R is locally reduced, that is, eRe is a reduced ring for all $e \in E(R)$. Then, we can let $e, f \in E(R)$ such that $ef = 0$. Denote $g = e - fe$. It is now easy to check from the fact $ef = 0$ that $g \in E(R)$ and $fg = gf = 0$. This leads to $u = f + g \in E(R)$. By using $ef = 0$ and $fg = 0$ again, we have $eu = e = ue$ and $fu = f = uf$. That is, $e, f \in uRu$. On the other hand, by Lemma 2.3, $(uRu,\bullet)$ is an abundant semigroup. Since the ring uRu is reduced, $fe = 0$. Thus, by Theorem 3.1, R is a reduced ring. □

The following corollary is a generalized version of the main result in [6].

Corollary 3.5. *Let R be a p.p.-ring without identity. Then R is a reduced ring if and only if R has no subrings which are isomorphic either to* $UTM_2(\mathbb{Z})$ *or to* $UTM_2(\mathbb{Z}_p)$*, where p is a prime.*

4. Perpetual ideals and reduced p.p.-rings

In this section, we define the (left; right) perpetual ideal and we characterize the reduced p.p.-rings by using the (right; left) perpetual ideals.

Definition 4.1. Let R be a ring and I a right ideal of R. Then, I is called a *right perpetual ideal* of R if for all $x \in I$, the set $\{r \in R : \mathrm{ann}_\ell(x) \subseteq \mathrm{ann}_\ell(r)$ *and* $ux = x \Rightarrow ur = r$ for all $u \in R\}$ is contained in I.

Evidently, every ring R has two right perpetual ideals $\{0\}$ and R. Dually, we can also define left perpetual ideals. By a *perpetual ideal* of R, we mean an ideal of R which is a both left and right perpetual ideal. It is easy to see that the intersection of left (right) perpetual ideals of R is still a left (right) perpetual ideal of R. Consequently, there exists the smallest (left; right) perpetual ideal of R containing X for all $X \subseteq R$. We usually call this smallest (left; right) perpetual ideal of R the *(left; right) perpetual ideal generated by* X and denote it by $(L^*(X); R^*(X))\ J^*(X)$. If $X = a$, then we write $L^*(X) = L^*(a)$, $R^*(X) = R^*(a)$ and $J^*(X) = J^*(a)$. Also, we simply call the (left; right) perpetual ideal generated by a the *principal (left; right) perpetual ideal generated by a.*

The following proposition describes the construction of $R^*(a)$ by $a \in R$.

Lemma 4.2. *Let R be a ring with identity. Then* $R^*(x) = \mathrm{ann}_r(\mathrm{ann}_\ell(x))$*, for any* $x \in R$.

Proof. We can easily check that $\mathrm{ann}_r(\mathrm{ann}_\ell(x))$ is a right ideal of R. Now let $u \in R$ and $a \in \mathrm{ann}_r(\mathrm{ann}_\ell(x))$ such that $\mathrm{ann}_\ell(a) \subseteq \mathrm{ann}_\ell(u)$. Then, since $(\mathrm{ann}_\ell(x))a = 0$, we have $(\mathrm{ann}_\ell(x))u = 0$, that is, $u \in \mathrm{ann}_r(\mathrm{ann}_\ell(x))$. Now, we have

$$ux = x \Longleftrightarrow (u-1)x = 0 \Longleftrightarrow u - 1 \in \mathrm{ann}_\ell(x). \tag{4.1}$$

Therefore, $\mathrm{ann}_r(\mathrm{ann}_\ell(x))$ is indeed a right perpetual ideal of R. Now let I be a right perpetual ideal of R containing x. Then $\{u \in R : \mathrm{ann}_\ell(x) \subseteq \mathrm{ann}_\ell(u)\} \subseteq I$. Since $\mathrm{ann}_\ell(x) \subseteq \mathrm{ann}_\ell(u)$ if and only if $(\mathrm{ann}_\ell(x))u = 0$ if and only if $u \in \mathrm{ann}_r(\mathrm{ann}_\ell(x))$, we can easily observe that $\mathrm{ann}_r(\mathrm{ann}_\ell(x)) \subseteq I$. Thus $\mathrm{ann}_r(\mathrm{ann}_\ell(x))$ is the smallest right perpetual ideal of R containing x, whence $R^*(x) = \mathrm{ann}_r(\mathrm{ann}_\ell(x))$. □

By using the above result, we obtain the following lemma.

Lemma 4.3. *Let* $(R, +, \bullet)$ *be a ring. Now denote the semigroup* $(R, \bullet)$ *by R. Then the following statements hold:*

(1) *if* $e \in E(R)$*, then* $R^*(e) = eR$*;*

(2) *if* $x \in R, e \in E(R)$ *and* $x\mathcal{R}^*e$*, then* $R^*(x) = eR$.

Proof. (1) We need only to show that eR is a right perpetual ideal of R. Obviously, eR is a right ideal of R. Now let $y \in R$ such that $\mathrm{ann}_\ell(x) \subseteq \mathrm{ann}_\ell(y)$ and $u(ex) = ex \Rightarrow uy = y$

for all $u \in R$. Then, since $e(ex) = ex$, we have $ey = y \in eR$. This means that eR is a right perpetual ideal of R. Thus $R^*(e) = eR$.

(2) By Lemma 2.1, $e \in R^*(x)$ and $eR \subseteq R^*(x)$. On the other hand, since $ex = x$, we have $x \in eR$. But eR is a right perpetual ideal of R (by (1)); by the minimality of $R^*(x)$, we obtain that $R^*(x) \subseteq eR$. Thus $R^*(x) = eR$. □

Finally, we obtain the main result of this section.

Theorem 4.4. *The following statements are equivalent for a p.p.-ring R without identity:*
(1) *R is reduced;*
(2) *R is a locally reduced ring;*
(3) *every left perpetual ideal of R is a right perpetual ideal of R;*
(4) *every right perpetual ideal of R is a left perpetual ideal of R;*
(5) *for all* $a \in R$, $L^*(a)$ *is a right perpetual ideal of R;*
(6) *for all* $a \in R$, $R^*(a)$ *is a left perpetual ideal of R;*
(7) *for all* $a \in R$, $L^*(a) = R^*(a)$.

Proof. We have already proved that (1) and (2) are equivalent. We only to need prove that (1)⇒(3)⇒(5)⇒(1)⇒(7)⇒(1) because (1)⇒(4)⇒(6)⇒(1)⇒(7)⇒(1) can be similarly proved.

(1)⇒(3). Assume that the ring R is reduced. Then, by Theorem 3.1, $E(R)$ is central in R. Let I be a left perpetual ideal of R. Then $I = \bigcup_{a\in I} L^*(a)$. By Theorem 3.2, $(R,\bullet)$ is a superabundant semigroup and hence we have $e \in E(R)$ such that $a\mathcal{H}^*e$. Moreover, by Lemmas 2.1 and 4.3, $L^*(a) = Re = eR = R^*(a)$. Thus $I = \bigcup_{a\in I} Re = \bigcup_{a\in I} eR$ is a right ideal of R. By observing that $I = \bigcup_{a\in I} L^*(a) = \bigcup_{a\in I} R^*(a)$, we can easily see that I is a right perpetual ideal of R.

(3)⇒(5). This part is trivial.

(5)⇒(1). Assume that (5) holds. Then for all $e \in E(R)$, $L^*(e) = Re$ is a right ideal of R. But $e \in Re$, so we have $eR \subseteq Re$. If $f \in E(R)$, then $ef = xe = efe$, where $x \in R$, hence $ef = (ef)^2$. This means that $E(R)$ is a band. By Theorem 3.1, R is reduced.

(1)⇒(7). Assume that R is reduced. Then, by (6), $R^*(a)$ is a left perpetual ideal of R, for all $a \in R$. But $a \in R^*(a)$, we have $L^*(a) \subseteq R^*(a)$. Dually, we have $R^*(a) \subseteq L^*(a)$. Thus, $R^*(a) = L^*(a)$.

(7)⇒(1). If (7) holds, then by Lemma 4.3 we have $Re = L^*(e) = R^*(e) = eR$, for all $e \in E(R)$. By the proof of (5)⇒(1), we can prove that R is indeed reduced. □

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