Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 10836, 5 pages
doi:10.1155/2007/10836

Research Article

A Note on ⊕-Cofinitely Supplemented Modules

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Received 9 January 2007; Accepted 2 May 2007

Recommended by Francois Goichot

Let R be a ring and M a right R-module. In this note, we show that a quotient of an ⊕-cofinitely supplemented module is not in general ⊕-cofinitely supplemented and prove that if a module M is an ⊕-cofinitely supplemented multiplication module with $\text{Rad}(M) \ll M$, then M can be written as an irredundant sum of local direct summand of M. An extension of the result of Calisici and Pancar [1], here it is shown that an arbitrary module is cofinitely semiperfect if and only if it is an (amply) cofinitely supplemented by supplements which have projective covers.

1. Introduction and preliminaries

Throughout this paper, R is an associative ring with identity and all modules are unital right R-modules. We use $N \leq M$ to indicate that N is a submodule of M.

Let M be a module and $S \leq M$. S is called *small* in M (notation $S \ll M$) if $M \neq S+T$ for any proper submodule T of M. Let N and L be submodules of M, N is called a *supplement* of L in M if $N+L=M$ and N is minimal with respect to this property. Equivalently, $M=N+L$ and $N \cap L \ll N$. A submodule N of a module M is said to be *cofinite* in M if the factor module M/N is finitely generated. A module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M, and M is called *amply cofinitely supplemented* if for every cofinite submodule N of M such that $M=N+K$, there is a supplement L of N such that $L \leq K$. A module is called an ⊕*-supplemented module* if every submodule of M has a supplement that is a direct summand of M. As a proper generalization of ⊕-supplemented modules, the notion of ⊕-cofinitely supplemented modules was introduced by Calisici and Pancar [2]. A module M is called an ⊕*-cofinitely supplemented module* if every cofinite submodule of M has a supplement that is a direct summand of

M. In Section 2, we show that a quotient of an $\oplus$-cofinitely supplemented module is not in general $\oplus$-cofinitely supplemented. It is proven that if a module M is an $\oplus$-cofinitely supplemented multiplication module with $\mathrm{Rad}(M) \ll M$, then M can be written as an irredundant sum of local direct summand of M. Calisici and Pancar [1] defined the concept of cofnitely semiperfect modules and proved that for a projective module M, M is cofinitely semiperfect if and only if M is amply cofinitely supplemented. In Section 3, we will show that an arbitrary module is cofinitely semiperfect if and only if it is (amply) cofinitely supplemented by supplements which have projective covers. This extends [1, Theorem 2.3].

2. $\oplus$-cofinitely supplemented modules

Let M be a module. M is called an $\oplus$-cofinitely supplemented module if every cofinite submodule of M has a supplement that is a direct summand of M.

We start with the following.

Example 2.1. Let R be a commutative local ring which is not a valuation ring and let $n \geq 2$. By [3, Theorem 2], there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than n elements. By [4, Corollary 1], $R^{(n)}$ is $\oplus$-cofinitely supplemented. However, M is not $\oplus$-cofinitely supplemented by [4, Proposition 2].

Example 2.2. Let R be a commutative local ring which is not a valuation ring. Let a and b be elements of R, neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a) \cap (b) = 0$ and $am = bm = 0$, where m is the maximal ideal of R. Let F be a free module with generators x_1, x_2, and x_3. Let K be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus, $M = (Rx_1 \oplus Rx_2 \oplus Rx_3)/R(ax_1 - bx_2) = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}$. Following by [5, Example 2.3], $Rx_1 \oplus Rx_2 \oplus Rx_3$ is $\oplus$-cofinitely supplemented, but M is not $\oplus$-cofinitely supplemented.

The above two examples show that a factor module of an $\oplus$-cofinitely supplemented module is not in general $\oplus$-cofinitely supplemented.

To deal with a special case of factor modules of $\oplus$-cofinitely supplemented modules, we need the following lemma.

Lemma 2.3 [5, Lemma 2.4]. *Let M be a module and U a fully invariant submodule of M. If $M = M_1 \oplus M_2$, then $U = (U \cap M_1) \oplus (U \cap M_2)$.*

Proposition 2.4. *Let M be a module and U a fully invariant submodule of M. If M is an $\oplus$-cofinitely supplemented module, then M/U is an $\oplus$-cofinitely supplemented module. If, moreover, U is a cofinite direct summand of M, then U is also an $\oplus$-cofinitely supplemented module.*

Proof. Suppose that M is an $\oplus$-cofinitely supplemented module and L/U is a cofinite submodule of M/U. Thus $M/L \cong (M/U)/(L/U)$ is finitely generated, and hence L is a cofinite submodule of M. Since M is an $\oplus$-cofinitely supplemented module, there exist submodules N and N' of M such that $M = N \oplus N'$, $M = N + L$, and $N \cap L \ll N$. It is easy to see that $(N + U)/U$ is a supplement of L/U. Now apply Lemma 2.3 to get that

$U = (U \cap N) \oplus (U \cap N')$. Thus we have $(N + U) \cap (N' + U) = U$ and $((N + U)/U) \oplus ((N' + U)/U) = M/U$, and hence $(N + U)/U$ is a direct summand of M/U. So M/U is an $\oplus$-cofinitely supplemented module.

Now suppose that U is a cofinite direct summand of M. Then there is a submodule U' such that $M = U \oplus U'$ and U' is finitely generated. Let V a cofinite submodule of U. Note that $M/V = (U \oplus U')/V \cong U/V \oplus U'$ is finitely generated so that V is a cofinite submodule of M. Since M is an $\oplus$-cofinitely supplemented module, there are submodules K and K' of M such that $M = K \oplus K'$, $M = V + K$, and $V \cap K \ll K$. Thus $U = V + (U \cap K)$. But $U = (U \cap K) \oplus (U \cap K')$, and hence $U \cap K$ is a direct summand of U. Moreover, $V \cap (U \cap K) = V \cap K \ll K$. Then $V \cap (U \cap K) \ll (U \cap K)$. Therefore, $U \cap K$ is a supplement of V in U and it is a direct summand of U. Thus, U is an $\oplus$-cofinitely supplemented module. □

Remark 2.5. Let M be an $\oplus$-cofinitely supplemented module. It is clear that $M/\operatorname{Rad}(M)$ and $M/\operatorname{Soc}(M)$ are also $\oplus$-cofinitely supplemented modules.

Lemma 2.6. *Let M be an $\oplus$-cofinitely supplemented module. If M contains a maximal submodule, then M contains a local direct summand.*

Proof. Let L be a maximal submodule of M. Since M is an $\oplus$-cofinitely supplemented module, there exists a direct summand K of M such that K is a supplement of L in M. Then for any proper submodule X of K, X is contained in L since L is a maximal submodule and $L + X$ is a proper submodule of M by minimality of K. Hence $X \leq L \cap K$ and $X \ll K$. Thus K is a hollow module, and the lemma is proven. □

Theorem 2.7. *Let M be an $\oplus$-cofinitely supplemented multiplication module with* $\operatorname{Rad}(M) \ll M$, *then M can be written as an irredundant sum of local direct summands of M.*

Proof. Since $\operatorname{Rad}(M) \ll M$, M contains a maximal submodule, and hence M contains a local direct summand. Let N be the sum of all local direct summands of M. If N is a proper submodule of M, then there is a maximal submodule L of M such that $N \leq L$ by [6, Theorem 2.5]. Let P be a direct summand of M such that P is a supplement of L in M. Note that P is a local module, and hence it is contained in N, so $M = L + P \leq L + N = L$. This is a contradiction. Hence we have $N = M$. Now let $M = \Sigma_{i \in I} L_i$, where each L_i is a local direct summand of M. Then

$$\frac{M}{\operatorname{Rad}(M)} = \Sigma_{i \in I} \left[\frac{L_i + \operatorname{Rad}(M)}{\operatorname{Rad}(M)} \right] \tag{2.1}$$

and each

$$\frac{Li + \operatorname{Rad}(M)}{\operatorname{Rad}(M)} \cong \frac{L_i}{Li \cap \operatorname{Rad}(M)} \tag{2.2}$$

is simple. Hence

$$\frac{M}{\operatorname{Rad}(M)} = \bigoplus_{k \in K} \left[\frac{L_k + \operatorname{Rad}(M)}{\operatorname{Rad}(M)} \right] \tag{2.3}$$

for some subset $K \subseteq I$. Thus $M = \Sigma_{k\in K} L_k$ since $\mathrm{Rad}(M) \ll M$. It is clear that the sum $\Sigma_{k\in K} L_k$ is irredundant. □

Remark 2.8. The module $M = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}$ in Example 2.2 is not an $\oplus$-cofinitely supplemented module. On the other hand, M can be written as follows: $M = (R\overline{x_1} + R\overline{x_2}) \oplus R(\overline{x_1} - \overline{x_3})$; $M = (R\overline{x_1} + R\overline{x_2}) \oplus R(\overline{x_2} - \overline{x_3})$; and $M = R(\overline{x_1} - \overline{x_3}) + R(\overline{x_2} - \overline{x_3}) + R\overline{x_3}$. Therefore, M is an irredundant sum of local direct summands of M.

3. Cofinitely semiperfect modules

Let M be a module. M is called semiperfect if every factor module of M has a projective cover. Calisici and Pancar [1] introduced the notion of cofinitely semiperfect modules as a proper generalization of semiperfect modules. M is called a cofinitely semiperfect module if every finitely generated factor module of M has a projective cover. The following result generalizes [1, Theorem 2.3].

Theorem 3.1. *The following statements are equivalent for a module M:*

(1) *M is a cofinitely semiperfect module;*
(2) *M is amply cofinitely supplemented by supplements which have projective covers;*
(3) *M is cofinitely supplemented by supplements which have projective covers.*

Proof. "(1)⇒ (2)" Let $M = N + L$ with N cofinite. Let P be a projective cover of M/N with epimorphism f. Since P is projective and M/N is isomorphic to L/D, where D is the intersection of N and L, the epimorphism f lifts to a homomorphism g from P to L. Next, we show that $\mathrm{Im}g$ is a supplement of N in M and P is a projective cover of $\mathrm{Im}g$. Since $\mathrm{Ker}f \ll P$ and $g(\mathrm{Ker}f) = \mathrm{Im}g \cap D$, $\mathrm{Im}g \cap D \ll \mathrm{Im}g$. Since f is an epimorphism, $\mathrm{Im}g + D = L$. Thus, $\mathrm{Im}g$ is a supplement of D in L. It is easy to verify that $\mathrm{Im}g$ is a supplement of N in M. Since $\mathrm{Ker}g \leq \mathrm{Ker}f$, $\mathrm{Ker}g \ll P$ and hence P is a projective cover of $\mathrm{Im}g$ which is clearly contained in L.

"(2)⇒ (3)" is clear.

"(3)⇒ (1)" Let N be a cofinite submodule of M and L is a supplement of N in M. Since if D denotes the intersection of N and L, L is a cover of L/D, then any projective cover of L is a projective cover of L/D, which is isomorphic to M/N. This completes the proof. □

Acknowledgment

This research was supported by a Doctoral Initial Fund of Lanzhou University of Technology.

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