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Research Article

Cesàro Statistical Core of Complex Number Sequences

Abdullah M. Alotaibi

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We establish some core inequalities for complex bounded sequences using the concept of statistical $(C,1)$ summability.

1. Introduction and preliminaries

The concept of statistical convergence was first introduced by Fast [1] and further studied by Šalát [2], Fridy [3], and many others, and for double sequences it was introduced and studied by Mursaleen and Edely [4] and Móricz [5] separately in the same year. Many concepts related to the statistical convergence have been introduced and studied so far, for example, statistical limit point, statistical cluster point, statistical limit superior, statistical limit inferior, and statistical core. Recently, Móricz [6] defined the concept of statistical $(C,1)$ summability and studied some tauberian theorems. In this paper, we introduce $(C,1)$-analogues of the above-mentioned concepts and mainly study C_1-statistical core of complex sequences and establish some results on C_1-statistical core.

Let $\mathbb{N}$ be the set of positive integers and $K \subseteq \mathbb{N}$. Let $K_n := \{k \in K : k \leq n\}$. Then the natural density of K is defined by $\delta(K) := \lim_n (1/n)|K_n|$, where the vertical bars denote cardinality of the enclosed set. A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$ the set $K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero; in this case, we write $st\text{-}\lim x = L$. By the symbol st, we denote the set of all statistically convergent sequences.

Define the (first) arithmetic means σ_n of a sequence (x_k) by setting

$$\sigma_n := \frac{1}{n+1}\sum_{k=0}^{n} x_k, \quad n = 0,1,2,\dots. \tag{1.1}$$

We say that $x = (x_k)$ is statistically summable $(C,1)$ to L if the sequence $\sigma = (\sigma_n)$ is statistically convergent to L, that is, $st\text{-}\lim\sigma = L$. We denote by $C_1(st)$ the set of all sequences which are statistically summable $(C,1)$.

Let $\mathbb{C}$ be the set of all complex numbers. In [7], it was shown that for every bounded complex sequence x,

$$K\text{-core}(x) := \bigcap_{z\in\mathbb{C}} K_x^*(z), \tag{1.2}$$

where

$$K_x^*(z) := \{\omega \in \mathbb{C} : |\omega - z| \leq \limsup_k |x_k - z|\}. \tag{1.3}$$

Note that Knopp's core (or K-core) of a real-bounded sequence x is defined to be the closed interval $[\liminf x, \limsup x]$, and the statistical core (or st-core) as $[st\text{-}\liminf x, st\text{-}\limsup x]$.

Recently, Fridy and Orhan [8] proved that

$$st\text{-core}(x) := \bigcap_{z\in\mathbb{C}} S_x^*(z), \tag{1.4}$$

where

$$S_x^*(z) := \{\omega \in \mathbb{C} : |\omega - z| \leq st\text{-}\limsup_k |x_k - z|\} \tag{1.5}$$

if x is a statistically bounded sequence.

For different types of cores and core theorems, we refer to [9–15], for single sequences, and [16–20], for double sequences.

2. Lemmas

In this section, we quote some results on matrix classes which are already known in the literature and these results will be used frequently in the text of the paper.

Let X and Y be any two sequence spaces and $A = (a_{nk})_{n,k=1}^{\infty}$ an infinite matrix. If for each $x \in X$ the sequence $Ax = (A_n(x)) \in Y$, we say that the matrix A maps X into Y, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k, \quad n = 1,2,\ldots, \tag{2.1}$$

provided that the series on the right converges for each n. We denote by (X,Y) the class of all matrices A which map X into Y, and by $(X,Y)_{\text{reg}}$ we mean $A \in (X,Y)$ such that the limit is preserved.

Let c and l_∞ denote the spaces of convergent and bounded sequences $x = (x_k)$, respectively. A matrix A is called regular, that is, $A \in (c,c)_{\text{reg}}$ if $A \in (c,c)$ and $\lim Ax = \lim x$ for all $x \in c$.

The following is the well-known characterization of regular matrices due to Silverman and Toplitz.

LEMMA 2.1. *A matrix* $A = (a_{nk})_{n,k}^{\infty}$ *is regular if and only if*

(i) $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$;

(ii) $\lim_n a_{nk} = 0$ *for each* k;

(iii) $\lim_n \sum_k a_{nk} = 1$.

The following result will also be very useful (see Simons [21]).

LEMMA 2.2. *Let* $\|A\| < \infty$ *and* $\limsup_n a_{nk} = 0$. *Then there exists* $y \in l_\infty$ *such that* $\|y\| \leq 1$ *and*

$$\limsup_n \sum_{k=1}^{\infty} a_{nk} y_k = \limsup_n \sum_{k=1}^{\infty} |a_{nk}|. \tag{2.2}$$

The following lemma is due to Kolk [22].

LEMMA 2.3. $A \in (c, st \bigcap l_\infty)_{\text{reg}}$, *that is,* $Ax \in st \bigcap l_\infty$ *for all* $x \in c$ *with* $st\text{-}\lim Ax = \lim x$, *if and only if*

(i) $\|A\| < \infty$,

(ii) $st\text{-}\lim_n a_{nk} = 0$ *for each* k,

(iii) $st\text{-}\lim_n \sum_k a_{nk} = 1$.

Analogously, we can easily prove the following.

LEMMA 2.4. $A \in (c, C_1(st) \bigcap l_\infty)_{\text{reg}}$ *if and only if*

(i) $\|A\| < \infty$,

(ii) $st\text{-}\lim_n b_{nk} = 0$ *for each* k, *where* $b_{nk} = (1/n)\sum_{i=1}^{n} a_{ik}$,

(iii) $st\text{-}\lim_n \sum_k b_{nk} = 1$.

3. C_1-statistical core

In this section, we define $(C,1)$ analogous of some notions related to statistical convergence (see Fridy [3], Fridy and Orhan [23]).

Definition 3.1. (i) A sequence $x = (x_k)$ is said to be lower C_1-statistically bounded if there exists a constant M such that $\delta(\{k : \sigma_k < M\}) = 0$, or equivalently we write $\delta_{c_1}(\{k : x_k < M\}) = 0$. (ii) A sequence $x = (x_k)$ is said to be upper C_1-statistically bounded if there exists a constant N such that $\delta(\{k : \sigma_k > N\}) = 0$, or equivalently we write $\delta_{c_1}(\{k : x_k > N\}) = 0$. (iii) If $x = (x_k)$ is both lower and upper C_1-statistically bounded, we say that $x = (x_k)$ is C_1-statistically bounded $C_1(st)$-bdd, for short).

We denote the set of all $C_1(st)$-bdd sequences by $C_1(st_\infty)$.

Definition 3.2. For any $M, N \in \mathbb{R}$, let

$$\begin{aligned} K_x &= \{M : \delta(\{k : \sigma_k < M\}) = 0\}, \\ L_x &= \{N : \delta(\{k : \sigma_k > N\}) = 0\}, \end{aligned} \tag{3.1}$$

then

$$\begin{aligned} C_1(st)\text{-superior of } x &= \inf L_x, \\ C_1(st)\text{-inferior of } x &= \sup K_x. \end{aligned} \tag{3.2}$$

Remark 3.3. It is clear that every bounded sequence is statistically bounded and every statistically bounded sequence is $C_1(st)$-bdd, but not conversely in general. For example, define $x = (x_k)$ by

$$x = (x_k) = \begin{cases} 2k-1 & \text{if } k \text{ is an odd square,} \\ 2 & \text{if } k \text{ is an even square,} \\ 1 & \text{if } k \text{ is an odd non square,} \\ 0 & \text{if } k \text{ is an even non square.} \end{cases} \tag{3.3}$$

Then, $\sup x_k = \infty$, but $L_x = [2,\infty)$, $K_x = (-\infty,0]$, $C_1(st)\text{-}\sup x_k = 2$, and $C_1(st)\text{-}\inf x_k = 0$.

Now let us introduce the following notation: a subsequence $y = \{y_k\}_{k=1}^{\infty} = \{x_{k(j)}\}_{k=1}^{\infty}$.

Definition 3.4. The number λ is a C_1-statistical limit point of the number sequence $x = (x_k)$ provided that there is a nonthin subsequence of x that is $(C,1)$-summable to λ.

Definition 3.5. The number β is a C_1-statistical cluster point of the number sequence $x = (x_k)$ provided that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |\sigma_k - \beta| < \varepsilon\}$ does not have density zero.

Now, let us introduce the following notation: if x is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that x_k satisfies P for "almost all k," and we abbreviate this by "a.a.k."

We define the $C_1(st)$-core of a sequence $x = (x_k)$ of complex numbers as follows.

Definition 3.6. For any complex sequence x, let $C(x)$ denote the collection of all closed half-planes that contain σ_k for almost all k. Then the C_1-statistical core of x is defined by

$$C_1(st)\text{-core}(x) = \bigcap_{G \in C(x)} G. \tag{3.4}$$

Note that in defining $C_1(st)$-core (x), we have simply replaced x_k by its $(C,1)$-mean in the definition of st-core (x) in the same manner as Moricz has defined statistical summability $(C,1)$ (or C_1-statistically convergence). Hence, it follows that

$$C_1(st)\text{-core}(x) \subseteq st\text{-core}(x) \subseteq K\text{-core}(x). \tag{3.5}$$

It is easy to see that for a statistically bounded real sequence x,

$$C_1(st)\text{-core}(x) = [st\text{-}\liminf \sigma,\ st\text{-}\limsup \sigma], \tag{3.6}$$

where $\sigma = (\sigma_k)$.

Fridy and Orhan [8] proved the following lemma.

Lemma 3.7. *Let x be statistically bounded sequence; for each $z \in \mathbb{C}$ let*

$$B_x(z) := \{\omega \in \mathbb{C} : |\omega - z| \leq st\text{-}\limsup_k |x_k - z|\}, \tag{3.7}$$

then

$$st\text{-core}(x) = \bigcap_{z\in\mathbb{C}} B_x(z). \tag{3.8}$$

The following lemma is an analog to the previous lemma.

LEMMA 3.8. *Let x be C_1-statistically bounded sequence; for each $z \in \mathbb{C}$ let*

$$M_x(z) := \left\{\omega \in \mathbb{C} : |\omega - z| \le st\text{-}\limsup_k \left|\sigma_k - z\right|\right\}, \tag{3.9}$$

then

$$C_1(st)\text{-core}(x) = \bigcap_{z\in\mathbb{C}} M_x(z). \tag{3.10}$$

Proof. Let ω be a C_1-statistical limit point of x, then

$$|\omega - z| \le st\text{-}\limsup_k \left|\sigma_k - z\right|. \tag{3.11}$$

Thus, $\cap_{z\in\mathbb{C}} M_x(z)$ contains all of C_1-statistical limit points of x. Since $C_1(st)$-core (x) is the smallest closed half-planes that contains C_1-statistical limit points of x, we have

$$C_1(st)\text{-core} \subseteq \bigcap_{z\in\mathbb{C}} M_x(z). \tag{3.12}$$

Converse set inclusion follows analogously as in the lemma of Fridy and Orhan [8]. □

Note that the present form of $C_1(st)$-core (x) given in the above lemma is not valid if x is not C_1-statistically bounded. For example, consider $x_k := 2k - 1$ for all k, then $C_1(st)$-core $(x) = \varnothing$ and $st - \limsup|\sigma_k - z| = \infty$ for any $z \in \mathbb{C}$. Thus we have

$$C_1(st)\text{-core}(x) := \begin{cases} \bigcap_{z\in\mathbb{C}} M_x(z) & \text{if } x \text{ is } C_1\ (st)\text{-bounded}, \\ \mathbb{C} & \text{otherwise}. \end{cases} \tag{3.13}$$

4. Some core theorems

Schoenberg [24] proved that if $x = (x_k) \in l_\infty$, then

$$st\text{-}\lim x_k = L \Longrightarrow st\text{-}\lim \sigma_n = L, \tag{4.1}$$

that is, $st \subseteq C_1(st)$. Thus we have

$$C_1(st)\text{-core}(x) \subseteq st\text{-core}(x). \tag{4.2}$$

Hence, we can sharpen [8, Theorem 1], as follows.

THEOREM 4.1. *If the matrix A satisfies $\|A\| < \infty$, then*

$$K\text{-core}(Ax) \subseteq C_1(st)\text{-core}(x) \tag{4.3}$$

for every $x \in l_\infty$ *if and only if*

(i) A *is regular and* $\lim_n \sum_{k\in E}|a_{nk}| = 0$ *whenever* $\delta(E) = 0$ *for* $E \subseteq \mathbb{N}$;

(ii) $\lim_n \sum_k |a_{nk}| = 1$.

Proof. Necessity. Let K-core $(Ax) \subseteq C_1(st)$-core (x). Then by [8, Theorem 1], necessity of (i) and (ii) follows immediately since $C_1(st)$-core $(x) \subseteq st$-core (x). □

Sufficiency. Assume that (i) and (ii) hold and $\omega \in K$-core (Ax). Then for $z \in \mathbb{C}$, we have

$$|\omega - z| \le \limsup_n \left| \sum_k a_{nk}(z - x_k) \right|. \tag{4.4}$$

Let $r := C_1(st)\text{-}\limsup |x_k - z|$ and $E := \{k : |z - \sigma_k| > r + \epsilon\}$. Then for any $\epsilon > 0$, we have $\delta(E) = 0$. We can write

$$\left| \sum_k a_{nk}(z - x_k) \right| \le C_1(st) - \sup_k |z - x_k| \sum_{k\in E} |a_{nk}| + (r+\epsilon) \sum_{k\notin E} |a_{nk}|. \tag{4.5}$$

Applying conditions (i) and (ii), we obtain

$$\limsup_n \left| \sum_k a_{nk}(z - x_k) \right| \le C_1(st)\text{-}\limsup_k |x_k - z|. \tag{4.6}$$

From (4.4), we get $|\omega - z| \le C_1(st)\text{-}\limsup_k |x_k - z|$ since ϵ was arbitrary. Hence $\omega \in C_1(st)\text{-core}(x)$ by Lemma 3.8. This completes the proof.

Theorem 4.2. *If* $\|A\| < \infty$, *then for every* $x \in l_\infty$,

$$C_1(st)\text{-core}(Ax) \subseteq K\text{-core}(x) \tag{4.7}$$

if and only if

(i) $C_1(st)\text{-}\lim_n \sum_{k\in E}|a_{nk}| = 1$, *whenever* $\mathbb{N} \setminus E$ *is finite, where* $E \subseteq \mathbb{N}$,

(ii) $A \in (c, C_1(st) \bigcap l_\infty)_{reg}$.

Proof. Necessity. Suppose that

$$C_1(st)\text{-core}(Ax) \subseteq K\text{-core}(x) \tag{4.8}$$

and x is convergent to l. Then

$$\{l\} = K\text{-core}(x) \supseteq C_1(st)\text{-core}(Ax). \tag{4.9}$$

Since $\|A\| < \infty$ implies $Ax \in l_\infty$ for $x \in l_\infty$, Ax has at least one statistical cluster point (see Mursaleen and Edely [13]) and hence the set of C_1-statistical cluster points is in $C_1(st)$-core (Ax). Therefore, $C_1(st)$-core $(Ax) \neq \varnothing$ and so $C_1(st)$-core $(Ax) = \{l\}$. Hence, $A \in (c, C_1(st) \bigcap l_\infty)_{\text{reg}}$, that is, condition (ii).

To prove condition (i), let us define $x = (x_k)$ by

$$x = \begin{cases} 1 & \text{if } k \in E, \\ 0 & \text{otherwise}, \end{cases} \tag{4.10}$$

where $E \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus E$ is finite, then

$$K\text{-core}(x) = \{1\}. \tag{4.11}$$

Since

$$\varnothing \neq C_1(st)\text{-core}(Ax) \subseteq K\text{-core}(x) = \{1\}, \tag{4.12}$$

we have $C_1(st)$-core $(Ax) = \{1\}$ and 1 is the only C_1-statistical cluster point of Ax. Using [25, Proposition 8], we have

$$C_1(st)\text{-}\lim Ax = 1. \tag{4.13}$$

Hence, by Lemma 2.2, we have

$$C_1 st\text{-}\lim_n \sum_{k\in E} |a_{nk}| = 1, \quad \text{whenever } \mathbb{N} \setminus E \text{ is finite.} \tag{4.14}$$

Sufficiency. Let conditions (i) and (ii) hold and $\omega \in C_1(st)$-core (Ax). Then for any $z \in \mathbb{C}$, we have

$$\begin{aligned} |\omega - z| &\leq C_1(st)\text{-}\limsup_n |z - A_n(x)| \\ &= C_1(st)\text{-}\limsup_n \left| z - \sum_k a_{nk} x_k \right| \\ &\leq C_1(st)\text{-}\limsup_n \left| \sum_k a_{nk}(z - x_k) \right| + C_1(st)\text{-}\limsup_n |z| \left| 1 - \sum_k a_{nk} \right|, \end{aligned} \tag{4.15}$$

and by (i), we have

$$|\omega - z| \leq C_1(st)\text{-}\limsup_n \left| \sum_k a_{nk}(z - x_k) \right|. \tag{4.16}$$

Let $r := \limsup_k |z - x_k|$ and $E := \{k : |z - x_k| > r + \epsilon\}$ for $\epsilon > 0$. Then $\delta(E) = 0$ as E is finite, and we have

$$\left| \sum_k a_{nk}(z - x_k) \right| \leq \sup_k |z - x_k| \sum_{k\in E} |a_{nk}| + (r + \epsilon) \sum_{k\notin E} |a_{nk}|. \tag{4.17}$$

Therefore, by conditions (i) and (ii), we obtain

$$C_1(st)\text{-}\limsup_n \left| \sum_k a_{nk}(z - x_k) \right| \leq r + \epsilon. \tag{4.18}$$

Hence, by (4.16), we have

$$|\omega - z| \leq r + \epsilon, \tag{4.19}$$

and since ϵ is arbitrary,

$$|\omega - z| \leq r = \limsup_{k} |z - x_k|, \tag{4.20}$$

that is $\omega \in K_x^*(z)$. Hence, $\omega \in K$-core (x), and so

$$C_1(st)\text{-core}(Ax) \subseteq K\text{-core}(x). \tag{4.21}$$

This completes the proof of the theorem. □

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Abdullah M. Alotaibi: School of Mathematics, King Abdulaziz University, P.O. Box 80203,
Jeddah 21589, Saudi Arabia
Email address: mathker11@hotmail.com