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Research Article

On Weak Statistical Convergence

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The main object of this paper is to introduce a new concept of weak statistically Cauchy sequence in a normed space. It is shown that in a reflexive space, weak statistically Cauchy sequences are the same as weakly statistically convergent sequences. Finally, weak statistical convergence has been discussed in l_p spaces.

1. Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was formally introduced by Steinhaus [2] and Fast [3] and later was reintroduced by Schoenberg [4]. Although statistical convergence was introduced over nearly the last fifty years, it has become an active area of research in recent years. This concept has been applied in various areas such as number theory [5], measure theory [6], trigonometric series [1], summability theory [7], locally convex spaces [8], in the study of strong integral summability [9], turnpike theory [10–12], and Banach spaces [13].

If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \le n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K (see [14, chapter 11]) is given by $\delta(K) = \lim_{n\to\infty} n^{-1}|K_n|$. K is said to be statistically dense [15] if $\delta(K) = 1$. The set $\{k \in \mathbb{N} : k \neq m^2, m = 1,2,\dots\}$ is statistically dense, while the set $\{3k : k = 1,2,\dots\}$ is not. A subsequence of a sequence is called statistically dense [15] if the set of all indices of its elements is statistically dense. A sequence (x_k) of (real or complex) numbers is said to be statistically convergent to some number L, if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \ge \epsilon\}$ has natural density zero; in this case, we write $\text{st-lim}_k x_k = L$.

For real-valued sequences, statistically convergent sequences often satisfy statistical analogs of the usual attributes of convergent sequences. For instance, statistically convergent sequences are statistically bounded; a sequence is statistically convergent if and only if it is statistically Cauchy, and there are statistical analogs of the lim sup, lim inf, and so forth, see [3, 16–19].

We recall (see [16]) that if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that $x = (x_k)$ satisfies P for "almost all k," and we abbreviate this by "a.a. k."

The following concept is due to Fridy [16]. A sequence (x_k) is said to be statistically Cauchy if for each $\epsilon > 0$ there exists a number $N(= N(\epsilon))$ such that $|x_k - x_N| < \epsilon$, for a.a. k, that is, $\delta(\{k \in \mathbb{N} : |x_k - x_N| \geq \epsilon\}) = 0$.

Fridy [16] proved that a number sequence is statistically convergent if and only if it is statistically Cauchy. It was shown by Kolk [20] that this result remains true in case the entries of the sequences come from a Banach space instead of being scalars.

A number sequence $x = (x_k)$ is statistically bounded [18] if there is a number B such that $\delta\{k : |x_k| > B\} = 0$, that is, $|x_k| \leq B$, for a.a. k.

The concept of statistical limit superior and inferior was introduced by Fridy and Orhan [18] as follows. For a real number sequence x, the statistical limit superior of x is given by

$$\text{st-}\limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \varnothing, \\ -\infty, & \text{if } B_x = \varnothing. \end{cases} \tag{1.1}$$

Also, the statistical limit inferior of x is given by

$$\text{st-}\liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \varnothing, \\ +\infty, & \text{if } A_x = \varnothing, \end{cases} \tag{1.2}$$

where

$$\begin{aligned} B_x &= \{b \in \mathbb{R} : \delta\{k : x_k > b\} \neq 0\}, \\ A_x &= \{a \in \mathbb{R} : \delta\{k : x_k < a\} \neq 0\}. \end{aligned} \tag{1.3}$$

Maddox [8] extended the concept of statistical convergence to sequences with values in arbitrary locally convex Hausdorff topological vector spaces. The statistical convergence in Banach spaces was studied by Kolk [20].

Quite recently, Connor et al. [13] have introduced a new concept of weak statistical convergence and have characterized Banach spaces with separable duals via weak statistical convergence. Pehlivan and Karaev [21] have also used the idea of weak statistical convergence in strengthening a result of Gokhberg and Krein on compact operators. Following Connor et al. [13], we define norm and weak statistical convergence as follows.

Definition 1.1. Let X be a normed linear space, let (x_k) be an X-valued sequence, and $x \in X$.

(i) The sequence (x_k) is norm statistically convergent to x provided that $\delta(\{k : \|x_k - x\| > \epsilon\}) = 0$ for all $\epsilon > 0$. In this case, we write st-$\lim x_k = x$.

(ii) The sequence (x_k) is weak statistically convergent to x provided that, for any f in the continuous dual X^* of X, the sequence $(f(x_k - x))$ is statistically convergent to 0. In this case, we write w-st-lim $x_k = x$ and x is called the weak statistical limit of (x_k).

By an application of Hahn-Banach theorem, it is easy to see that the weak statistical limit of a weakly statistically convergent sequence is unique.

In this paper, we show that weak statistical convergence is a generalization of the usual notion of weak convergence and that in finite dimensional normed spaces the concepts of norm and weak statistical convergence coincide. After introducing a new concept of weak statistically Cauchy sequence, it is established that every weak statistically Cauchy sequence in a normed space is statistically bounded and this fact has been used to show that in a reflexive space weak statistically Cauchy sequences and weak statistically convergent sequences are the same. As a final result, we see how weak statistical convergence looks like in l_p spaces.

The following well-known lemmas are required for establishing the results of this paper.

Lemma 1.2 [4]. *If* st-lim $x_k = l$ *and* $g(x)$, *defined for all real* x, *is continuous at* $x = l$, *then* st-lim $g(x_k) = g(l)$.

Lemma 1.3 [19]. *A number sequence* (x_k) *is statistically convergent to* l *if and only if there exists such a set* $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ *that* $\delta(K) = 1$ *and* $\lim_{n\to\infty} x_{k_n} = l$.

Lemma 1.4 [19]. *If* st-lim $x_k = l$ *and* st-lim $y_k = m$ *and* α *is a real number, then*

(i) st-lim $(x_k + y_k) = l + m$,

(ii) st-lim $(\alpha x_k) = \alpha l$.

Lemma 1.5 [22]. *A number sequence* (x_k) *is statistically bounded if and only if there exists such a set* $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ *that* $\delta(K) = 1$ *and* (x_{k_n}) *is bounded.*

Lemma 1.6 [23]. *Let* $x_k \le y_k$, *for a.a.* k. *If* st-lim x_k *and* st-lim y_k *exist, then* st-lim $x_k \le$ st-lim y_k.

Lemma 1.7 [15]. *A statistically dense subsequence of a statistically convergent sequence is statistically convergent.*

By $l_p (1 \le p < \infty)$, *we denote the space of absolutely* p-*summable scalar sequences and it is a normed linear space with the norm defined by* $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$, *where* $x = (x_k) \in l_p$. *By* c_{00}, *we denote the space of scalar sequences* $x = (x_k)$, *each of which has only finitely many nonzero terms. Clearly,* $c_{00} \subset l_p$ $(1 \le p < \infty)$.

2. Main results

Our first result shows that weak statistical convergence is a generalization of the usual notion of weak convergence.

Theorem 2.1. *Let* (x_k) *be a weakly convergent sequence in a normed space* X, *and* w-lim $x_k = x$. *Then* (x_k) *is weakly statistically convergent to* x. *The converse is not generally true.*

Proof. If w-lim $x_k = x$, then $(f(x_k))$ is convergent to $f(x)$, for all $f \in X^*$ which implies that w-st-lim $x_k = x$. □

To show that the converse is not true, we give the following example.

Example 2.2. Let $(x_k) \in l_p$ $(1 < p < \infty)$ be defined by

$$x_j^{(k)} = \begin{cases} m, & \text{if } j \le k,\ k = m^2, \\ \dfrac{1}{k}, & \text{if } j \le k,\ k \ne m^2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For $k \ne m^2$ and arbitrary $f \in l_p^*$, there is unique $y \in l_q$ such that

$$\begin{aligned} |f(x_k)| &= \left| \sum_{j=1}^{\infty} x_j^{(k)} y_j \right| \\ &\le \left(\sum_{j=1}^{\infty} \left| x_j^{(k)} \right|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^q \right)^{1/q}, \quad \text{by Hölder's inequality} \\ &\le \left(\sum_{j=1}^{k} \frac{1}{k^p} \right)^{1/p} M^{1/q} \text{ for some positive constant } M \\ &= \left(\frac{M}{k} \right)^{1/q} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{2.2}$$

Hence, st-lim $f(x_k) = 0$, by Lemma 1.3, which in turn implies that w-st-lim $x_k = 0$.

For $k = m^2$, consider the functional f_1 defined on l_p by $f_1(x) = x_1$, where $x = (x_k) \in l_p$. Clearly, $f_1(x_k) = x_1^{(k)} = \sqrt{k} \to \infty$, as $k \to \infty$. Hence, (x_k) is not weakly convergent.

Our next result shows that in finite dimensional normed spaces the norm statistical convergence and weak statistical convergence coincide.

Theorem 2.3. *In a normed space X,*

(i) *norm statistical convergence implies weak statistical convergence with the same limit,*
(ii) *the converse of (i) is not generally true,*
(iii) *if dim $X < \infty$, the weak statistical convergence implies norm statistical convergence.*

Proof. The proof of (i) is straightforward.

To prove (ii), let (e_k) be an orthonormal sequence in a Hilbert space H. Every $f \in H^*$ has a Riesz representation $f(x) = \langle x, z \rangle$. Hence, $f(e_k) = \langle e_k, z \rangle$. By Bessel's inequality $\sum_{k=1}^{\infty} |\langle e_k, z \rangle|^2 \le \|z\|^2$. This implies that $f(e_k) = \langle e_k, z \rangle$ is convergent, and hence statistically convergent, to zero. Since $f \in H^*$ was arbitrary, w-st-lim $e_k = 0$. Let, if possible, (e_k) be norm statistically convergent. Then (e_k) is statistically Cauchy and so for each $\epsilon > 0$ there exists a positive integer $N = (N(\epsilon))$ such that $\|e_k - e_N\| < \epsilon$, for a.a. k, that is, $\delta(\{k : \|e_k - e_N\| \ge \epsilon\}) = 0$, which is absurd because $\|e_k - e_N\| = \sqrt{2}$ $(k \ne N)$.

As another example, let (x_k) in l_p $(1 < p < \infty)$ be defined by

$$x_j^{(k)} = \begin{cases} m, & \text{if } j \leq k,\ k = m^2, \\ 0, & \text{if } j > k,\ k = m^2, \\ 1, & \text{if } j = k,\ k \neq m^2, \\ 0, & \text{if } j \neq k,\ k \neq m^2. \end{cases} \tag{2.3}$$

It is easy to see that (x_k) is weakly statistically null sequence but it is not norm statistically null sequence.

(iii) Suppose $\{e_1, e_2, \ldots, e_m\}$ is any basis for X and that w-st-lim $x_k = x$. Then $x_k = \sum_{i=1}^{m} \alpha_i^{(k)} e_i$ $(k = 1, 2, \ldots)$ and $x = \sum_{i=1}^{m} \alpha_i e_i$ for scalars $\alpha_i^{(k)}$ and α_i. Consider the linear functionals $f_j \in X^*$ $(1 \leq j \leq m)$ defined by $f_j(e_j) = 1$, $f_j(e_k) = 0$ $(j \neq k)$. Since w-st-lim $x_k = x$, it follows that, for $j = 1, 2, \ldots, m$, st-lim $f_j(x_k) = f_j(x)$, which, by the definition of f_j, implies that st-lim $\alpha_j^{(k)} = \alpha_j$, and so for a given $\epsilon > 0$, $|\alpha_j^{(k)} - \alpha_j| < \epsilon / Km$, for a.a. k, where $K = \max_j \|e_j\|$. Hence,

$$||x_k - x|| = \left\| \sum_{j=1}^{m} \left(\alpha_j^{(k)} - \alpha_j \right) e_j \right\| \leq K \sum_{j=1}^{m} |\alpha_j^{(k)} - \alpha_j| < \epsilon, \quad \text{for a.a. } k, \tag{2.4}$$

which implies that st-lim $x_k = x$. □

Remark 2.4. Does there exist an infinite dimensional space in which the concepts of weak and norm statistical convergence coincide? This is an open problem.

We now introduce a new concept of weak statistically Cauchy sequence in a normed space.

Definition 2.5. A sequence (x_k) in a normed space X is said to be weak statistically Cauchy if $(f(x_k))$ is statistically Cauchy for every $f \in X^*$.

Obviously, every weakly statistically convergent sequence in a normed space is weak statistically Cauchy, but the converse need not be true.

Example 2.6. Consider the normed linear space c_{00} with $\|\cdot\|_p$, $1 < p < \infty$. Let $(x_k) \in c_{00}$ be defined by

$$x_j^{(k)} = \begin{cases} j, & \text{if } j \leq k,\ k = m^2, \\ \dfrac{1}{j}, & \text{if } j \leq k,\ k \neq m^2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

Using standard techniques, it is easy to see that this sequence is weak statistically Cauchy but not weakly statistically convergent.

The next result shows that if the space is reflexive, then every weak statistically Cauchy sequence is weakly statistically convergent.

Theorem 2.7. *If the normed space is reflexive, then every weak statistically Cauchy sequence is weakly statistically convergent.*

To prove this result, we need the following Lemma.

Lemma 2.8. *Every weak statistically Cauchy sequence in a normed space is statistically bounded.*

Proof. Let (x_k) be a weak statistically Cauchy sequence in a normed space X. Then $(f(x_k))$ is a statistically Cauchy sequence for all $f \in X^*$ and hence is statistically bounded. So by Lemma 1.5, for any $f \in X^*$, there exists a set $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and $(f(x_{k_n}))$ is bounded. Consider the canonical mapping $C : X \to X^{**}$ defined by $C(x) = g_x$ for all $x \in X$, where $g_x \in X^{**}$ is defined by $g_x(f) = f(x)$ for all $f \in X^*$. Also $\|g_x\| = \|x\|$. Now for any $f \in X^*$, $\sup_n |g_{x_{k_n}}(f)| = \sup_n |f(x_{k_n})| < \infty$. Since X^* is a Banach space, by Banach Steinhaus theorem $\sup_n \|g_{x_{k_n}}\| < \infty$ and hence $\sup_n \|x_{k_n}\| < \infty$. Again, by Lemma 1.5, it follows that (x_k) is statistically bounded. □

Corollary 2.9. *Every weakly statistically convergent sequence in a normed space is statistically bounded.*

The following example shows that the converse of Lemma 2.8 is not true in general.

Example 2.10. Let (x_k) in $\mathbb{R}$ be defined by

$$x_k = \begin{cases} k, & \text{if } k \text{ is a square,} \\ 0, & \text{if } k \text{ is an even nonsquare,} \\ 1, & \text{if } k \text{ is an odd nonsquare.} \end{cases} \tag{2.6}$$

Then (x_k) is statistically bounded, but not statistically convergent and hence not weakly statistically convergent.

Proof of Theorem 2.7. Suppose (x_k) is a weak statistically Cauchy sequence in X, that is, $(f(x_k))$ is statistically Cauchy for all $f \in X^*$. Consider the canonical mapping $C : X \to X^{**}$ as defined in Lemma 2.8. $(Cx_k(f))$ is statistically Cauchy and hence statistically convergent sequence of scalars for every $f \in X^*$. Define $y(f) = \text{st-lim}_{k\to\infty} Cx_k(f)$. The linearity of y follows by Lemma 1.4. Moreover, by Lemma 2.8, (x_k) is statistically bounded, so there exists some positive number M such that $\|x_k\| \le M$, for a.a. k. Hence for any $f \in X^*$, $|Cx_k(f)| = |f(x_k)| \le M\|f\|$, for a.a. k, and hence by Lemma 1.6, st-lim $|Cx_k(f)| \le M\|f\|$. This implies $|y(f)| \le M\|f\|$, and hence $y \in X^{**}$. Since X is reflexive, there exists $x \in X$ such that $y = Cx$. Hence for any $f \in X^*$, st-lim $f(x_k) = y(f) = Cx(f) = f(x)$ which shows that w-st-lim $x_k = x$. □

Proposition 2.11. *If* w-st-lim $x_k = x$ *in a normed space* X*, then* $\|x\| \le \text{st-lim inf}\, \|x_k\|$.

Proof. For each $f \in X^*$,

$$\begin{aligned} |f(x)| &= \text{st-lim}\, |f(x_k)|, \quad \text{using Lemma 1.2} \\ &= \text{st-lim inf}\, |f(x_k)| \le \|f\| \text{st-lim inf}\, \|x_k\|. \end{aligned} \tag{2.7}$$

Taking supremum over all $f \in X^*$ with $\|f\| = 1$, we get $\|x\| \le \text{st-lim inf}\, \|x_k\|$. □

We know that every subsequence of a weakly convergent sequence is again weakly convergent, but this is not true in case of weak statistical convergence.

Example 2.12. Let (x_k) in $\mathbb{R}$ be defined by

$$x_k = \begin{cases} k, & \text{if } k = m^2, \\ \dfrac{1}{k}, & \text{otherwise}. \end{cases} \tag{2.8}$$

Then (x_k) is statistically convergent and hence weakly statistically convergent, but its subsequence $\{k^2 : k = 1,2,\dots\}$ being statistically divergent is not weakly statistically convergent.

The next result tells which subsequences of a weakly statistically convergent sequence are weakly statistically convergent.

THEOREM 2.13. (i) *Every statistically dense subsequence of a weakly statistically convergent sequence is weakly statistically convergent.*

(ii) *The converse of (i) is not true, in general.*

Proof. (i) follows from Lemma 1.7.

The converse of (i) is not true and follows from the following example. □

Example 2.14. Let (x_k) in $\mathbb{R}$ be defined by

$$x_k = \begin{cases} 1, & \text{if } k = m^2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.9}$$

then (x_k) is statistically convergent, and hence weakly statistically convergent, to 0. Its subsequence $\{1,1,\dots\}$ is weakly statistically convergent but not statistically dense.

3. Weak statistical convergence in l_p $(1 < p < \infty)$

In this section, we see that how weak statistical convergence "looks like" in l_p space.

THEOREM 3.1. *In the space l_p $(1 < p < \infty)$, we have w-st-$\lim x_k = x$ if and only if*

(i) *the sequence $(\|x_k\|)$ is statistically bounded;*

(ii) *for every fixed j, we have st-$\lim x_j^{(k)} = x_j$; here $x_k = (x_j^{(k)})$ and $x = (x_j)$.*

The proof is completely analogous to the classical theorem (see [24, page 236]) once we establish the following lemma.

LEMMA 3.2. *In a normed space X, we have w-st-$\lim x_k = x$ if and only if*

(i) *the sequence $(\|x_k\|)$ is statistically bounded;*

(ii) *for every element f of a total subset $M \subset X^*$, we have st-$\lim f(x_k) = f(x)$.*

Proof. In the case of weak statistical convergence, (i) follows from Corollary 2.9 and (ii) is trivial.

Conversely, suppose that (i) and (ii) hold. Consider any $h \in X^*$ and we will show that st-$\lim h(x_k) = h(x)$. This will be done in two steps. First, it will be shown that this is true for all $h \in \operatorname{span} M$ and then for $h \in \overline{\operatorname{span} M}$.

To prove the first conclusion, let $g \in \operatorname{span} M$. Then $g = \sum_{i=1}^{n} \alpha_i f_i$ for $f_1, f_2, \dots, f_n \in M$ and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$. By hypothesis (ii), st-$\lim f_i(x_k) = f_i(x)$ for all i, $1 \le i \le n$ and hence st-$\lim g(x_k) = g(x)$, by Lemma 1.4. Thus the first conclusion is established.

For the second conclusion, suppose $h \in \overline{\text{span}\, M}$. By hypothesis (i), there exists a constant $c > 0$ such that $\|x_k\| < c$, for a.a. k, and therefore, for any $f \in M \subset X^*$, we have $|f(x_k)| < c\|f\|$, for a.a. k, which by Lemma 1.6 gives that st-lim$|f(x_k)| < c\|f\|$. Again using Lemma 1.2, we have $|f(x)| < c\|f\|$ which implies $\|x\| < c$. Since $h \in \overline{\text{span}\, M}$, for a given $\epsilon > 0$, there exists $g_j \in \text{span}\, M$ $(j = 1, 2, \dots)$ such that $\|h - g_j\| < \epsilon/3c$ for all $j > n_0$. Consider

$$\begin{aligned} |h(x_k) - h(x)| &\le \|h - g_j\| \, \|x_k\| + |g_j(x_k) - g_j(x)| + \|g_j - h\| \|x\| \\ &< \frac{\epsilon}{3c} c + |g_j(x_k) - g_j(x)| + \frac{\epsilon}{3c} c, \text{ for a.a. } k, \text{ provided } j > n_0. \end{aligned} \tag{3.1}$$

Since $g_j \in \text{span}\, M$, so by the first part of the proof, st-lim$\, g_j(x_k) = g_j(x)$, and hence $|g_j(x_k) - g_j(x)| < \epsilon/3$, for a.a. k. Hence $|h(x_k) - h(x)| < \epsilon$, for a.a. k, and so w-st-lim$\, x_k = x$. □

Proposition 3.3. *In a Hilbert space H,* w-st-lim$\, x_k = x$ *if and only if* st-lim$\langle x_k, y\rangle = \langle x, y\rangle$, *for all $y \in H$.*

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