

Research Article

A Finite-Interval Uniqueness Theorem for Bilateral Laplace Transforms

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Two or more bilateral Laplace transforms with a complex argument “ s ” may be equal in a finite vertical interval when, in fact, the transforms correspond to different functions. In this article, we prove that the existence of a bilateral Laplace transform in any finite horizontal interval uniquely determines the corresponding function. The result appears to be new as we could not find it in the literature. The novelty of the result is that the interval need not contain zero, the function need not be nonnegative and need not be integrable. The result has a potential to be useful in the context of fitting probability distributions to data using Laplace transforms or moment generating functions.

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1. Introduction

The bilateral Laplace transform of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx. \quad (1.1)$$

The transform is named in honor of the French mathematician and astronomer Pierre-Simon Laplace, who used the transform in his work on probability theory. The domain of $L_f(s)$ is the set of all complex numbers $s = r + it$, $r, t \in \mathbb{R}$, for which the integral exists. If $r = 0$ and the function f is integrable, then the bilateral Laplace transform exists for all $t \in \mathbb{R}$. In this case, the bilateral Laplace transform is called the Fourier transform of f . It may be the case that the bilateral Laplace transform does not exist for any $r \neq 0$, even if the

function f is integrable. For example, if

$$f(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \quad (1.2)$$

then the bilateral Laplace transform does not exist for any $r \neq 0$.

If the domain of f is $[0, \infty)$, the bilateral Laplace transform becomes the unilateral Laplace transform. For integrable functions, the unilateral Laplace transform exists for all $r \geq 0$ and all $t \in \mathbb{R}$, that is, in the entire right half of the complex plane. In standard text books such as [1], the better-known unilateral Laplace transform is almost always what is meant by “the Laplace transform”. However, in some more recent text books (e.g., [2]), the term “Laplace transform” refers to the bilateral Laplace transform. The use of the term “Laplace transform” to refer to the bilateral Laplace transform is becoming more common. As indicated above, the (bilateral) Laplace transform includes the Fourier transform. If f is a probability density function, then the complex conjugate of the Fourier transform is called the characteristic function [3]; and if $t = 0$; the function $M_f(s) = L_f(-s)$ is called the moment generating function [4].

The Laplace transform or moment generating function, has many theoretical and practical applications in engineering, physics, probability and statistics. The applications include pricing options [5]; finding moments of a distribution [6]; finding limiting distributions [4]; proving laws of large numbers [7–10]; estimation [11]; and solving differential equations [1]. A property of the Laplace transform which makes it a useful tool is that it uniquely determines the corresponding function f . Under suitable conditions on the growth of the function f , for example if f is of an exponential order ($|f(x)| \leq Me^{ax}$) and does not have any singularity in the right half-plane ($r = \text{Re}(s) \geq 0$), then the function f can be recovered from the Laplace transform by using the inversion formula, which is given by

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{sx} L_f(s) ds \quad (1.3)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} L_f(it) dt, \quad \text{by using the substitution } s = it. \quad (1.4)$$

The inversion formula implies the uniqueness theorem for Fourier transforms [12, page 170] or characteristic functions [3, page 143]. It is important to note here that the uniqueness theorem or the inversion formula, given in (1.4), implies that for two Laplace transforms to correspond to the same function, the two transforms must be equal for all $t \in \mathbb{R}$. That is, the transforms being equal in a finite interval does not guarantee that they correspond to the same function. For example, the Laplace transforms (characteristic functions), given by

$$L_f(it) = \begin{cases} 1 - |t|, & t \in [-1, 1], \\ 0, & |t| > 1, \end{cases}$$

$$L_g(it) = \begin{cases} 1 - |t|, & 0 \leq |t| \leq 1/2, \\ (1/3)(2 - |t|), & 1/2 < |t| \leq 2, \\ 0, & |t| > 2, \end{cases} \quad (1.5)$$

are equal on the interval $[-1/2, 1/2]$, but the transforms correspond to different probability density functions:

$$\begin{aligned} f(x) &= \frac{1 - \cos(x)}{\pi x^2}, \quad x \neq 0, \\ g(x) &= \frac{3 - 2\cos(x/2) - \cos(2x)}{3\pi x^2}, \quad x \neq 0, \end{aligned} \quad (1.6)$$

respectively. In some applications, the need for two transforms to be equal for all $t \in \mathbb{R}$, for them to correspond to the same function, is not a major impediment. However, in other applications involving complicated transforms or fitting probability distributions to data, the requirement may be a serious drawback. The author is currently developing statistical procedures for fitting probability distributions to data using Laplace transforms. The purpose of this article is to show that while on the vertical axis equality must occur for all $t \in \mathbb{R}$, for two transforms to correspond to the same function, on the horizontal axis, the existence of the Laplace in any (finite) interval in the complex plane uniquely determines the corresponding function f . We begin with the case where the argument s is real and the function f is nonnegative and integrable.

2. The finite interval uniqueness theorem for Laplace transforms

LEMMA 2.1. *Let $f(x)$ be a nonnegative and integrable function. Then $f(x) = 0$, a.e., if and only if $\int_{-\infty}^{\infty} f(x)dx = 0$.*

The proof of the lemma is straightforward and hence omitted.

THEOREM 2.2. *Let f and g be nonnegative and integrable functions, $s \in \mathbb{R}$ and $L_f(s)$ and $L_g(s)$ be the Laplace transforms of f and g , respectively. Suppose that there is an open interval (a, b) , not necessarily containing zero, such that both $L_f(s)$ and $L_g(s)$ exist in (a, b) and $L_f(s) = L_g(s)$ for all $s \in (a, b)$. Then $f(x) = g(x)$, a.e.*

Proof. Let $s' = s - (a + b)/2$. Then

$$\int_{-\infty}^{\infty} e^{-sx} f(x) dx = \int_{-\infty}^{\infty} e^{-s'x} u(x) dx, \quad s' \in (-(b-a)/2, (b-a)/2), \quad (2.1)$$

where $u(x) = e^{-(a+b)x/2} f(x)$. Similarly,

$$\int_{-\infty}^{\infty} e^{-sx} g(x) dx = \int_{-\infty}^{\infty} e^{-s'x} v(x) dx, \quad s' \in (-(b-a)/2, (b-a)/2), \quad (2.2)$$

where $v(x) = e^{-(a+b)x/2} g(x)$. The reparameterization $s' = s - (a + b)/2$ ensures that s' belongs to an open interval which contains zero.

If $L_u(0) = L_v(0) = 0$, then $u(x) = e^{-(a+b)x/2}f(x) = 0$, a.e. and $v(x) = e^{-(a+b)x/2}g(x) = 0$, a.e., by Lemma 2.1. This implies that $f(x) = g(x) = 0$, a.e. That is, the null Laplace transform $L_f(s) \equiv 0$, is uniquely identified with a null function f such that $f(x) = 0$, a.e.

Now suppose that $L_u(0) = L_v(0) = \sigma^2 > 0$, then the functions $u^*(x) = u(x)/\sigma^2$ and $v^*(x) = v(x)/\sigma^2$ are probability density functions. Since

$$L_{u^*}(s') = L_u(s')/\sigma^2 = L_v(s')/\sigma^2 = L_{v^*}(s') \tag{2.3}$$

for all $s' \in (- (b - a)/2, (b - a)/2)$, it follows from Curtiss' Theorem [13] that $u^*(x) = v^*(x)$, a.e. $\Rightarrow u(x) = v(x)$, a.e. $\Rightarrow f(x) = g(x)$, a.e. This completes the proof of Theorem 2.2. \square

The essence of Theorem 2.2 is that the original interval need not contain the origin. That is, Curtiss' theorem may be reformulated to include moment generating functions for which there is no open interval $(-\delta, \delta)$, $\delta > 0$ containing zero, in which the moment generating function exists for all $t \in (-\delta, \delta)$. Thus, by Theorem 2.2, the function

$$L_f(s) = \frac{1}{2} \left(e^{-2\sqrt{s}} + \frac{1}{1-s} \right), \quad s \in [0, 1), \tag{2.4}$$

is the bilateral Laplace transform of the unique distribution with density

$$f(x) = \begin{cases} \frac{1}{2\sqrt{\pi(-x)^3}} \exp(1/x), & x < 0, \\ e^{-x}/2, & x \geq 0. \end{cases} \tag{2.5}$$

To fit a model such as the one given in (2.5) to a given dataset, it would be more convenient to use the Laplace transform or moment generating function, which has a closed form, rather than the corresponding distribution function which does not have a closed form. Thus, the uniqueness theorem presented above has a potential to be useful in statistical inference. For example, it can be used to develop procedures for parameter estimation and (parametric) tests for goodness of fit of distributions, based on Laplace transforms that exist in a specified interval. The research is currently being pursued by this author.

Theorem 2.2 may be extended to the class of all integrable functions. The following lemma makes the proof of the result flow more smoothly.

LEMMA 2.3. *Let $h(x)$ be an integrable function and suppose that there is an interval (a, b) such that $\int_{-\infty}^{\infty} e^{-sx}h(x)dx = 0$ for all $s \in (a, b)$. Then $h(x) = 0$, a.e.*

Proof. Suppose that $\int_{-\infty}^{\infty} e^{-sx}h(x)dx = 0$ for all $s \in (a, b)$.

Let

$$\begin{aligned} h^+(x) &= \max(0, h(x)), \\ h^-(x) &= -\min(0, h(x)). \end{aligned} \tag{2.6}$$

Then, $\int_{-\infty}^{\infty} e^{-sx}h(x)dx = 0 \Rightarrow \int_{-\infty}^{\infty} e^{-sx}(h^+(x) - h^-(x))dx = 0$. It follows from Theorem 2.2, that $h^+(x) = h^-(x)$, a.e. Whence $h(x) = 0$, a.e. \square

THEOREM 2.4. *Let f and g be integrable functions and suppose that there is an interval (a, b) such that $\int_{-\infty}^{\infty} -e^{-sx}(f(x) - g(x))dx = 0$ for all $s \in (a, b)$. Then $f(x) = g(x)$, a.e.*

Proof. The proof of Theorem 2.4 follows readily from Lemma 2.3 with $h = f - g$.

From Theorem 2.4 one can state the corresponding uniqueness theorem for Laplace transforms with a complex argument “ s ” and for which function f may not be integrable. This is the main result of this paper. \square

THEOREM 2.5. *Let f and g be measurable functions, $r, t \in \mathbb{R}$, $i = \sqrt{-1}$, $s = r + it$, $L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$ and $L_g(s) = \int_{-\infty}^{\infty} e^{-sx} g(x)dx$. Suppose that there is an interval $(a, b) \subset \mathbb{R}$ such that for any fixed t , $L_f(s) = L_g(s)$ and finite for all $r \in (a, b)$. Then $f(x) = g(x)$ a.e.*

Proof. The transforms $L_f(s)$ and $L_g(s)$ can be expressed as

$$\begin{aligned} L_f(s) &= \int_{-\infty}^{\infty} e^{-rx} \cos(tx) f(x) dx - i \int_{-\infty}^{\infty} e^{-rx} \sin(tx) f(x) dx, \\ L_g(s) &= \int_{-\infty}^{\infty} e^{-rx} \cos(tx) g(x) dx - i \int_{-\infty}^{\infty} e^{-rx} \sin(tx) g(x) dx. \end{aligned} \tag{2.7}$$

If $L_f(s) = L_g(s)$ for all $r \in (a, b)$, then from Theorem 2.4 we have, $\cos(tx)(f(x) - g(x)) = 0$, a.e., which implies that $f(x) - g(x) = 0$ a.e., since $\cos(tx) \neq 0$, a.e. The existence of the integral $\int_{-\infty}^{\infty} e^{-rx} \cos(tx) f(x) dx$ and the reparameterization $r' = r - (a + b)/2$ implies that the function $e^{-(a+b)x/2} \cos(tx) f(x)$ is integrable. This completes the proof of the theorem. \square

Example 2.6. The function,

$$f(x) = \begin{cases} e^x, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{2.8}$$

is not integrable but its Laplace transform, $1/(s - 1)$, exists for all $r = \text{Re}(s) > 1$. From Theorem 2.5, the function f is uniquely determined by the values of the Laplace transform in any finite horizontal line segment $(a, b) \subset (1, \infty)$, such as the line from 1 to 2 or from $3 + 4i$ to $5 + 4i$ in the complex plane.

3. Conclusion

In this article we have presented a proof of the fact that the existence of the bilateral Laplace transform in any finite horizontal interval uniquely determines the corresponding function. The result seems to be new and to have a potential to increase the practical utility of Laplace transforms especially in probability and statistics. For example, the result can be used to develop parametric tests for goodness of fit of probability distributions using Laplace transforms or one-sided moment generating functions that exist only on one side of the origin.

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