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*Research Article*

# Schur Algebras over $C^*$-Algebras

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Let $\mathcal{A}$ be a $C^*$-algebra with identity 1, and let $s(\mathcal{A})$ denote the set of all states on $\mathcal{A}$. For $p,q,r \in [1,\infty)$, denote by $\mathcal{S}^r(\mathcal{A})$ the set of all infinite matrices $A = [a_{jk}]_{j,k=1}^{\infty}$ over $\mathcal{A}$ such that the matrix $(\varphi[A^{[2]}])^{[r]} := [(\varphi(a_{jk}^* a_{jk}))^r]_{j,k=1}^{\infty}$ defines a bounded linear operator from $\ell^p$ to $\ell^q$ for all $\varphi \in s(\mathcal{A})$. Then $\mathcal{S}^r(\mathcal{A})$ is a Banach algebra with the Schur product operation and norm $\|A\| = \sup\{\|(\varphi[A^{[2]}])^r\|^{1/(2r)} : \varphi \in s(\mathcal{A})\}$. Analogs of Schatten's theorems on dualities among the compact operators, the trace-class operators, and all the bounded operators on a Hilbert space are proved.



## 1. Introduction

The *Schur product* operation has been studied by many authors since the seminal work of Schur [1]. The idea has been used in the studies of completely positive maps, analytic function theory, matrix analysis, operator theory, and operator algebras, and so forth. Recently there has been work done on the Schur product of matrices over Banach algebras (see [2–4]) and operators on Hilbert spaces [4]. Here we consider another direction: matrices with entries from a fixed $C^*$-algebra $\mathcal{A}$. The Gelfand-Naimark-Segal construction gives a representation of the elements in $\mathcal{A}$ as bounded linear operators on a Hilbert space. Rather than having the whole matrix considered as an operator on the direct sum of the underlying Hilbert space [4], we use states on the $C^*$-algebra to convert the matrix into a nonnegative numerical matrix and consider it as a bounded linear transformation from $\ell^p$ to $\ell^q$ $(1 \le p,\, q < \infty)$. Using the norm of the nonnegative matrix as an operator from $\ell^p$ to $\ell^q$ to define a norm on a certain set of matrices over $\mathcal{A}$, we show that the set of all matrices that define bounded operators is a Banach algebra. We then consider analogs of Schatten's theorems on these new Banach algebras, as in [3, 5].

## 2. Notation and preliminaries

For $1 \le p, q < \infty$, the space of all $p$th power absolutely summable sequences of complex numbers is denoted by $\ell^p$, and the space of all bounded linear transformations (or operators) from $\ell^p$ to $\ell^q$ is denoted by $\mathcal{B}(\ell^p, \ell^q)$. Elements in $\mathcal{B}(\ell^p, \ell^q)$ will be represented as matrices with respect to the standard bases for $\ell^p$ and $\ell^q$.

For a given matrix $A = [a_{jk}]$ (over the complex field $\mathbb{C}$ or a Banach algebra), and a positive integer $n \in \mathbb{N}$, $A_{n\lrcorner}$ denotes the matrix whose $(j,k)$-entry is $a_{jk}$ for $1 \le j, k \le n$, and is 0 otherwise. We will also use the same notation for the $n \times n$ matrix obtained by erasing all $k$th rows and $k$th columns, for $k > n$, from $A$. We include the following lemmas concerning the norm of operators in $\mathcal{B}(\ell^p, \ell^q)$ for the convenience of reference.

Lemma 2.1. *Let $[\alpha_{jk}]$ and $[\beta_{jk}]$ be matrices over $\mathbb{C}$ such that $|\alpha_{jk}| \le \beta_{jk}$ for all $j$ and $k$. Suppose that $[\beta_{jk}] \in \mathcal{B}(\ell^p, \ell^q)$. Then $[\alpha_{jk}] \in \mathcal{B}(\ell^p, \ell^q)$ and $\|[\alpha_{jk}]\| \le \|[\beta_{jk}]\|$.*

Lemma 2.2. *Let $\Lambda = [\alpha_{jk}]$ be a complex matrix. Then*

(1) *$\Lambda$ defines a bounded operator from $\ell^p$ to $\ell^q$ if and only if $\Lambda x$ exists as a sequence for every $x \in \ell^p$, and the sequence of norms $\{\|\Lambda_{\nu\lrcorner}\|_{p,q}\}_{\nu=1}^{\infty}$ is bounded;*

(2) *if (1) holds, $\|\Lambda_{\nu\lrcorner}\|_{p,q} \nearrow \|\Lambda\|_{p,q}$ as $\nu \to \infty$.*

Let $\mathcal{A}$ be a $C^*$-algebra with identity 1, and let $s(\mathcal{A})$ be the set of all *states* on $\mathcal{A}$ (i.e., the set of all positive linear functionals of norm 1 (or taking the value 1 at the identity), see [6, 7, page 256, Theorem 4.3.2]). Then $\|a\| = \sup_{\varphi\in s(\mathcal{A})} \varphi(a) = \max_{\varphi\in s(\mathcal{A})} \varphi(a)$ for all selfadjoint (positive, in particular) $a \in \mathcal{A}$ [6, 7, page 261, Theorem 4.3.4]. By convention $|x| = \sqrt{x^* x}$ for every $x \in \mathcal{A}$.

Lemma 2.3 (Minkowski's inequality). *Let $a, b \in \mathcal{A}$, and $\varphi \in s(\mathcal{A})$. Then*

$$[\varphi(|a+b|^2)]^{1/2} \le [\varphi(|a|^2)]^{1/2} + [\varphi(|b|^2)]^{1/2}. \tag{2.1}$$

This is just the triangle inequality for the seminorm induced by the semi-inner product defined by $\langle a, b\rangle_\varphi = \varphi(b^* a)$ for all $a, b \in \mathcal{A}$.

Lemma 2.4. *Let $x, y \in \mathcal{A}$ and $\varphi \in s(\mathcal{A})$. Then*

$$\varphi(|xy|^2) \le \|x\|^2 \varphi(|y|^2). \tag{2.2}$$

The is also a well-known standard result.

## 3. The Schur algebras

For a matrix $A = [a_{jk}]$ with entries from $\mathcal{A}$ (a $C^*$-algebra with identity 1) and $r \in [1, \infty)$, the *absolute Schur $r$th power* of $A$ is the matrix

$$A^{[r]} = [\,|a_{jk}|^r\,] = [(a_{jk}^* a_{jk})^{r/2}]. \tag{3.1}$$

(It is the matrix with $(j,k)$-entry $(a_{jk}^{*}a_{jk})^{r/2}$.) In particular, if $\Lambda = [\lambda_{jk}]$ is a complex or real matrix, then $\Lambda^{[r]} = [|\lambda_{jk}|^{r}]$. Note that since $\mathscr{A}$ contains an identity, each matrix $\Lambda = [\lambda_{jk}]$ over $\mathbb{R}$ or $\mathbb{C}$ can be treated as one over $\mathscr{A}$ via the identification $\lambda_{jk} \leftrightarrow (\lambda_{jk} \cdot 1)$. Let $\mathscr{S}^{r}(\mathscr{A})$ be the set of all matrices $A = [a_{jk}]$ with $a_{jk} \in \mathscr{A}$ such that

$$\begin{aligned} \varphi[A^{[2]}]^{[r]} &:= [\varphi(|a_{jk}|^{2})]^{[r]} := [(\varphi(|a_{jk}|^{2}))^{r}] \\ &= [(\varphi(a_{jk}^{*}a_{jk}))^{r}] \in \mathscr{B}(\ell^{p},\ell^{q}) \quad \forall \varphi \in s(\mathscr{A}). \end{aligned} \tag{3.2}$$

That is, for all $\varphi \in s(\mathscr{A})$, the matrix $\varphi[A^{[2]}]^{[r]}$, with $(\varphi(a_{jk}^{*}a_{jk}))^{r}$ as its $(j,k)$-entry, defines a bounded linear operator from $\ell^{p}$ to $\ell^{q}$. This is also equivalent to saying that $\varphi[A^{[2]}]$ is in $\mathscr{S}^{r}$ (the Schur algebra of matrices over $\mathbb{C}$ with Schur $r$th power defining bounded linear operator from $\ell^{p}$ to $\ell^{q}$, see [2]) for all $\varphi \in s(\mathscr{A})$. For each $A = [a_{jk}] \in \mathscr{S}^{r}(\mathscr{A})$, define

$$\|A\| := \sup_{\varphi \in s(\mathscr{A})} \|(\varphi[A^{[2]}])^{[r]}\|_{p,q}^{1/(2r)} = \sup_{\varphi \in s(\mathscr{A})} \|[(\varphi(a_{jk}^{*}a_{jk}))^{r}]\|_{p,q}^{1/(2r)}, \tag{3.3}$$

where $\|\cdot\|_{p,q}$ denotes the norm on $\mathscr{B}(\ell^{p},\ell^{q})$. We will prove in Theorem 3.2 that this indeed defines a norm on $\mathscr{S}^{r}(\mathscr{A})$. In the sequel, we will suppress the subscripts $_{p,q}$ in $\|\cdot\|_{p,q}$, and use $\|\cdot\|$ to denote both the norm on $\mathscr{S}^{r}(\mathscr{A})$ and the norm on $\mathscr{B}(\ell^{p},\ell^{q})$, letting the context determine which one is intended.

Lemma 3.1. *Let* $A = [a_{jk}] \in \mathscr{S}^{r}(\mathscr{A})$.
(1) $\|A\| < \infty$.
(2) *For each* $(\mu,\nu) \in \mathbb{N} \times \mathbb{N}$, $\|a_{\mu\nu}\| \le \|A\|$.
(3) *For each subset* $S$ *of* $\mathbb{N} \times \mathbb{N}$, *denote by* $A(S)$ *the matrix obtained from* $A$ *by replacing by* 0 *for all* $(j,k)$*-entries with* $(j,k) \notin S$. *Then* $\|A(S)\| \le \|A\|$.
(4) $\|A_{\nu\lrcorner}\| \nearrow \|A\|$ *as* $\nu \to \infty$.

*Proof.* (1) Let $\mathscr{A}^{\#}$ be the dual space of $\mathscr{A}$. By [6, 7, Corollary 4.3.7, page 260], each $f \in \mathscr{A}^{\#}$ is a linear combination of at most four states, that is, for some $\alpha_{\nu} = \alpha_{\nu}(f) \in \mathbb{C}$, $\varphi_{\nu} = \varphi_{\nu}(f) \in s(\mathscr{A})$, $\nu = 1,2,3,4$, $f = \sum_{\nu=1}^{4}\alpha_{\nu}\varphi_{\nu}$. For each $A = [a_{jk}] \in \mathscr{S}^{r}(\mathscr{A})$, $[\varphi_{\nu}(|a_{jk}|^{2})] \in \mathscr{S}^{r}$, for $\nu = 1,2,3,4$. Since $\mathscr{S}^{r}$ is a Banach algebra under the Schur multiplication and usual addition [2],

$$f[A^{[2]}] = \sum_{\nu=1}^{4} \alpha_{\nu}\varphi_{\nu}[A^{[2]}] \in \mathscr{S}^{r}. \tag{3.4}$$

Thus $\mathscr{T}_{A} : f \mapsto f[A^{[2]}] = \sum_{\nu=1}^{4}\alpha_{\nu}(\varphi_{\nu}[A^{[2]}])$ defines a linear transformation from $\mathscr{A}^{\#}$ to $\mathscr{S}^{r}$. Since both the domain $\mathscr{A}^{\#}$ and the codomain $\mathscr{S}^{r}$ of $\mathscr{T}_{A}$ are Banach spaces, and $\mathscr{T}_{A}$ is linear, it suffices to show that the graph $\mathscr{G}_{\mathscr{T}_{A}}$ of $\mathscr{T}_{A}$ is closed in $\mathscr{A}^{\#} \oplus_{\infty} \mathscr{S}^{r}$ to conclude that $\mathscr{T}_{A}$ is bounded. To that end, let $\{f_{n}\}_{n=1}^{\infty}$ be a sequence in $\mathscr{A}^{\#}$ such that $f_{n} \to f$ in $\mathscr{A}^{\#}$ and $\mathscr{T}_{A}(f_{n}) \to \Lambda = [\lambda_{jk}]$ in $\mathscr{S}^{r}$. Then, for each $(j,k) \in \mathbb{N} \times \mathbb{N}$, $f_{n}(|a_{jk}|^{2}) \to f(|a_{jk}|^{2})$. From the convergence of $\mathscr{T}_{A}(f_{n})$ to $\Lambda$ in $\mathscr{S}^{r}$, we also have $f_{n}(|a_{jk}|^{2}) \to \lambda_{jk}$. Therefore,

$\lambda_{jk} = f(|a_{jk}|^2)$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. It then follows that $\mathcal{T}_A(f) = f[A^{[2]}] = [f(|a_{jk}|^2)] = [\lambda_{jk}] = \Lambda$. Therefore $\mathcal{T}_A$ has a closed graph, and hence bounded. This implies that

$$\begin{aligned} \|A\| &= \sup_{\varphi\in s(\mathcal{A})} \|(\varphi[A^{[2]}])^{[r]}\|^{1/(2r)} = \sup_{\varphi\in s(\mathcal{A})} \|\mathcal{T}_A(\varphi)\|_{\mathcal{S}^r}^{1/2} \\ &\le \sup_{\varphi\in s(\mathcal{A})} \|\mathcal{T}_A\|^{1/2} \|\varphi\|^{1/2} \le \|\mathcal{T}_A\|^{1/2} < \infty. \end{aligned} \tag{3.5}$$

(2) By [6, 7, Theorem 4.3.4, page 261],

$$\begin{aligned} \|a_{\mu\nu}\|^2 &= \|a_{\mu\nu}^* a_{\mu\nu}\| = \sup_{\varphi\in s(\mathcal{A})} \varphi(a_{\mu\nu}^* a_{\mu\nu}) \\ &\le \sup_{\varphi\in s(\mathcal{A})} \|[(\varphi(a_{jk}^* a_{jk}))^r]\|^{1/r} = \|A\|^2. \end{aligned} \tag{3.6}$$

(3) follows directly from Lemma 2.1.

(4) Let $\nu \in \mathbb{N}$ and $\varphi \in s(\mathcal{A})$. By Lemmas 2.1 and 2.2, and since $(\varphi[A^{[2]}])^{[r]} \in \mathcal{B}(\ell^p, \ell^q)$, we have

$$\begin{aligned} \left\|(\varphi[(A_{\nu\lrcorner})^{[2]}])^{[r]}\right\| &= \left\|[(\varphi[A^{[2]}])^{[r]}]_{\nu\lrcorner}\right\| \nearrow \left\|[(\varphi[A^{[2]}])^{[r]}]\right\| \\ &= \left\|(\varphi[(A)^{[2]}])^{[r]}\right\|. \end{aligned} \tag{3.7}$$

Taking suprema, as $\varphi$ runs over the set $s(\mathcal{A})$, on both sides of this inequality, we have, after taking the $(2r)$th roots, $\|A_{\nu\lrcorner}\| \nearrow \|A\|$. ☐

Theorem 3.2. *Let $\mathcal{S}^r(\mathcal{A})$ be as defined above. Then $\|\cdot\|$ as defined in (3.3) is a norm on $\mathcal{S}^r(\mathcal{A})$. Equipped with this norm, $\mathcal{S}^r(\mathcal{A})$ is a Banach algebra under the Schur product and usual addition and scalar multiplication.*

*Proof.* By Lemma 3.1(1), the function $\|\cdot\|$ as defined in (3.3) satisfies $\|A\| < \infty$ for all $A \in \mathcal{S}^r(\mathcal{A})$. To see that $\|\cdot\|$ is indeed a norm, we first note that for each $\varphi \in s(\mathcal{A})$ and each $\alpha \in \mathbb{C}$,

$$\begin{aligned} \|(\varphi[(\alpha A)^{[2]}])^{[r]}\|^{1/(2r)} &= \|[\{\varphi((\overline{\alpha} a_{jk}^*)(\alpha a_{jk}))\}^r]\|^{1/(2r)} \\ &= |\alpha| \|[(\varphi(a_{jk}^* a_{jk}))^r]\|^{1/(2r)}. \end{aligned} \tag{3.8}$$

Taking suprema on both end expressions in the above equality, we have $\|\alpha A\| = |\alpha| \|A\|$. For the triangle inequality, let $A = [a_{jk}]$, $B = [b_{jk}] \in \mathcal{S}^r(\mathcal{A})$. For each $\varphi \in s(\mathcal{A})$, since

$[\varphi[A^{[2]}]]^{[1/2]} = [(\varphi(a_{jk}^* a_{jk}))^{1/2}]$ and $[\varphi[B^{[2]}]]^{[1/2]} = [(\varphi(b_{jk}^* b_{jk}))^{1/2}]$ are in $\mathcal{S}^{2r}$, we have

$$\begin{aligned} [\varphi[A^{[2]}]]^{[1/2]} + [\varphi[B^{[2]}]]^{[1/2]} &= [\{\varphi(a_{jk}^* a_{jk})\}^{1/2} + \{\varphi(b_{jk}^* b_{jk})\}^{1/2}] \in \mathcal{S}^{2r}, \\ ||(\varphi[(A+B)^{[2]}])^{[r]}||^{1/(2r)} &= ||[\{\varphi((a_{jk}+b_{jk})^*(a_{jk}+b_{jk}))\}^r]||^{1/(2r)} \\ &\le ||[\{(\{\varphi(a_{jk}^* a_{jk})\}^{1/2} + \{\varphi(b_{jk}^* b_{jk})\}^{1/2})^2\}^r]||^{1/(2r)} \\ &\qquad \text{(by Lemmas 2.1 and 2.3)} \\ &= ||[(\{\varphi(a_{jk}^* a_{jk})\}^{1/2} + \{\varphi(b_{jk}^* b_{jk})\}^{1/2})^{2r}]||^{1/(2r)} \\ &\le ||[\{\varphi(a_{jk}^* a_{jk})\}^r]||^{1/(2r)} + ||[\{\varphi(b_{jk}^* b_{jk})\}^r]||^{1/(2r)} \\ &\qquad \text{(by the triangle inequality for the norm on } \mathcal{S}^{2r}\text{)} \\ &\le \|A\| + \|B\|. \end{aligned} \tag{3.9}$$

Since this is true for all $\varphi \in s(\mathcal{A})$, we have

$$\|A+B\| = \sup_{\varphi \in s(\mathcal{A})} ||(\varphi[(A+B)^{[2]}])^r||^{1/(2r)} \le \|A\| + \|B\|. \tag{3.10}$$

Thus $\|\cdot\|$ is a norm on $\mathcal{S}^r(\mathcal{A})$.

To see submultiplicativity of $\|\cdot\|$, let $A = [a_{jk}], B = [b_{jk}] \in \mathcal{S}^r(\mathcal{A})$:

$$\begin{aligned} \|A \bullet B\|^{2r} &= \sup_{\varphi \in s(\mathcal{A})} ||[(\varphi(b_{jk}^* a_{jk}^* a_{jk} b_{jk}))^r]|| \\ &\le \sup_{\varphi \in s(\mathcal{A})} ||[\{\varphi(\|a_{jk}\|^2 (b_{jk}^* b_{jk}))\}^r]|| \\ &\qquad \text{(by Lemmas 2.1 and 2.4)} \\ &\le \sup_{\varphi \in s(\mathcal{A})} ||[\{\varphi(\|A\|^2 (b_{jk}^* b_{jk}))\}^r]|| \\ &= \|A\|^{2r} \Big( \sup_{\varphi \in s(\mathcal{A})} ||[(\varphi(b_{jk}^* b_{jk}))^r]|| \Big) \\ &\qquad \text{(by Lemmas 2.1 and 3.1)} \\ &\le \|A\|^{2r} \|B\|^{2r}. \end{aligned} \tag{3.11}$$

This submultiplicativity of the norm also shows that $\mathcal{S}^r(\mathcal{A})$ is closed under the Schur multiplication. We next show that $\mathcal{S}^r(\mathcal{A})$ is complete in this norm. To that end, let $\{A^{(n)} = [a_{jk}^{(n)}]\}$ be a Cauchy sequence in $\mathcal{S}^r(\mathcal{A})$. Then by Lemma 3.1, for each $(j,k) \in \mathbb{N} \times \mathbb{N}$, $\{a_{jk}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{A}$. Thus the completeness of $\mathcal{A}$ provides an $a_{jk} \in \mathcal{A}$ such that $a_{jk}^{(n)} \to a_{jk}$, in $\mathcal{A}$, as $n \to \infty$. For each $\varphi \in s(\mathcal{A})$, since the sequence $\{(\varphi[(A^{(n)})^{[2]}])^{[r]} = [(\varphi((a_{jk}^{(n)})^* a_{jk}^{(n)}))^r]\}_{n=1}^{\infty}$ of matrices is a Cauchy sequence in $\mathcal{B}(\ell^p, \ell^q)$, there is a matrix $\Lambda^{(\varphi)} = [\lambda_{jk}^{(\varphi)}] \in \mathcal{B}(\ell^p, \ell^q)$ such that $(\varphi[(A^{(n)})^{[2]}])^{[r]} \to \Lambda^{(\varphi)}$ in $\mathcal{B}(\ell^p, \ell^q)$. Thus, for each $(j,k) \in \mathbb{N} \times \mathbb{N}$, $(\varphi(|a_{jk}^{(n)}|^2))^r \to \lambda_{jk}^{(\varphi)}$ as $n \to \infty$. But we also have, by the convergence $a_{jk}^{(n)} \to a_{jk}$,

$\varphi(|a_{jk}^{(n)}|^2) \to \varphi(|a_{jk}|^2)$ as $n \to \infty$. Thus $(\varphi(|a_{jk}|^2))^r = \lambda_{jk}^{(\varphi)}$ for each $(j,k) \in \mathbb{N} \times \mathbb{N}$ and each $\varphi \in s(\mathcal{A})$. With $A = [a_{jk}]$, we see that $(\varphi[A^{[2]}])^{[r]} = \Lambda^{(\varphi)} \in \mathcal{B}(\ell^p, \ell^q)$, for each $\varphi \in s(\mathcal{A})$. Therefore $A \in \mathcal{S}^r(\mathcal{A})$.

To see that $A^{(n)} \to A$ in $\mathcal{S}^r(\mathcal{A})$, let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$||A^{(n)} - A^{(m)}|| < \frac{\epsilon}{2} \qquad \forall n, m \geq N. \tag{3.12}$$

Let $\nu \in \mathbb{N}$ be arbitrarily fixed. Since for each $(j,k) \in \mathbb{N} \times \mathbb{N}$, $\|a_{jk}^{(n)} - a_{jk}\| \to 0$ as $n \to \infty$, we have, by Lemma 2.1,

$$||A_{\nu\lrcorner}^{(n)} - A_{\nu\lrcorner}|| = ||(A^{(n)} - A)_{\nu\lrcorner}|| \leq ||[||a_{jk}^{(n)} - a_{jk}||]_{\nu\lrcorner}||_{\mathcal{B}(\ell^p,\ell^q)}^{1/(2r)} \longrightarrow 0 \tag{3.13}$$

as $n \to \infty$. Thus there is an $n_\nu > N$ such that

$$||(A^{(n)})_{\nu\lrcorner} - A_{\nu\lrcorner}|| < \frac{\epsilon}{3} \quad \forall n \geq n_\nu. \tag{3.14}$$

For $n \geq N$, we have

$$\begin{aligned} ||(A^{(n)} - A)_{\nu\lrcorner}|| &= ||(A^{(n)})_{\nu\lrcorner} - A_{\nu\lrcorner}|| \\ &\leq ||(A^{(n)})_{\nu\lrcorner} - (A^{(n_\nu)})_{\nu\lrcorner}|| + ||(A^{(n_\nu)})_{\nu\lrcorner} - A_{\nu\lrcorner}|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{3} = \frac{5\epsilon}{6}. \end{aligned} \tag{3.15}$$

That is, for each $n \geq N$, $\|(A^{(n)} - A)_{\nu\lrcorner}\| < 5\epsilon/6$ for all $\nu \in \mathbb{N}$. Thus after taking limit as $\nu \to \infty$, we have, by Lemma 3.1(4),

$$||A^{(n)} - A|| \leq \frac{5\epsilon}{6} < \epsilon \quad \forall n \geq N. \tag{3.16}$$

This completes the proof. □

## 4. The dual of a Schur algebra

In this section, we will prove a Schur algebra version of Schatten's theorem about the decomposition of the dual of the algebra of bounded operators on a Hilbert space as the direct sum of "singular functionals" and functionals given by the trace-class operators. Denote by $(\mathcal{AS})$ the space of all matrices $[\alpha_{jk}]$ with entries from the complex field $\mathbb{C}$ such that

$$|||[\alpha_{jk}]|||_{(\mathcal{AS})} := |||[\alpha_{jk}]||| := \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} |\alpha_{jk}| < \infty. \tag{4.1}$$

(This is just the Banach space $\ell^1(\mathbb{N}\times\mathbb{N})$.) Let $\mathcal{M}$ be the space of all matrices $\Phi = [\varphi_{jk}]$ over $\mathcal{A}^\#$, the dual space of $\mathcal{A}$, such that

$$\begin{aligned}\|\Phi\| &:= \|\Phi\|_{\mathcal{M}} := \big\|[\varphi_{jk}]\big\|_{\mathcal{M}} \\ &:= \sup\Bigg\{\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} |\varphi_{jk}(a_{jk})| : A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A}),\ \|A\| \le 1\Bigg\} < \infty.\end{aligned} \tag{4.2}$$

That is, $\Phi$ regarded as a map from $\mathcal{S}^r(\mathcal{A})$ to $(\mathcal{AS})$ given by

$$\Phi : [a_{jk}] \longmapsto [\varphi_{jk}(a_{jk})] =: \Phi[A] \tag{4.3}$$

is a bounded linear transformation from $\mathcal{S}^r(\mathcal{A})$ to $(\mathcal{AS})$. We also think of this as the *Schur product* of the two matrices, one over $\mathcal{A}^\#$ and one over $\mathcal{A}$, resulting in a matrix with entries from $\mathbb{C}$.

Lemma 4.1. *Let $\Phi = [\varphi_{jk}] \in \mathcal{M}$.*
(1) *For each $(\mu,\nu) \in \mathbb{N}\times\mathbb{N}$, $\|\varphi_{\mu\nu}\| \le \|A\|$.*
(2) *$\|\Phi_{\nu_\lrcorner}\| \nearrow \|\Phi\|$ as $\nu \to \infty$.*

*Proof.* (1) For each $a \in \mathcal{A}$, denote by $E_{\mu\nu}(a)$ the matrix whose $(\mu,\nu)$-entry is $a$ and all other entries are 0. Then $E_{\mu\nu}(a) \in \mathcal{S}^r(\mathcal{A})$ and $\|E_{\mu\nu}(a)\| = \|a\|$:

$$\begin{aligned}\|\varphi_{\mu\nu}\| &= \sup\{|\varphi_{\mu\nu}(a)| : a \in \mathcal{A},\ \|a\| \le 1\} \\ &\le \sup\{\|\Phi[E_{\mu\nu}(a)]\| : a \in \mathcal{A},\ \|E_{\mu\nu}(a)\| \le 1\} \le \|\Phi\|.\end{aligned} \tag{4.4}$$

(2) For each $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A})$ and $\nu \in \mathbb{N}$, $\Phi[A_{\nu_\lrcorner}] = \Phi_{\nu_\lrcorner}[A] = (\Phi[A])_{\nu_\lrcorner}$. Since $\Phi[A] \in (\mathcal{AS})$,

$$\|\Phi_{\nu_\lrcorner}[A]\|_{(\mathcal{AS})} = \sum_{j,k=1}^{\nu} |\varphi_{jk}(a_{jk})| \le \sum_{j,k=1}^{\nu+1} |\varphi_{jk}(a_{jk})| = \|\Phi_{(\nu+1)_\lrcorner}[A]\|_{(\mathcal{AS})}. \tag{4.5}$$

Taking supremum over all $A \in \mathcal{A}$ with $\|A\| \le 1$, we have $\|\Phi_{\nu_\lrcorner}\| \le \|\Phi_{(\nu+1)_\lrcorner}\|$, showing the monotonicity of $\{\|\Phi_{\nu_\lrcorner}\|\}_{\nu\in\mathbb{N}}$.

To see the convergence, let $\epsilon > 0$. There is an $A = [a_{jk}] \in \mathcal{A}$ with $\|A\| \le 1$ such that $\sum_{j,k=1}^{\infty} |\varphi_{jk}(a_{jk})| > \|\Phi\| - \epsilon/2$. By the convergence of the series on the left-hand side of the preceding inequality, there is an $N$ such that

$$\sum_{j,k=1}^{N} |\varphi_{jk}(a_{jk})| > \sum_{j,k=1}^{\infty} |\varphi_{jk}(a_{jk})| - \frac{\epsilon}{2} > \|\Phi\| - \epsilon. \tag{4.6}$$

Since $\|\Phi_{N_\lrcorner}\| \ge \sum_{j,k=1}^{N} |\varphi_{jk}(a_{jk})| \ge \|\Phi\| - \epsilon$, we see that $\|\Phi_{\nu_\lrcorner}\| \nearrow \|\Phi\|$ as $\nu \to \infty$. □

Next we show that $\|\cdot\|_{\mathcal{M}}$ is indeed a norm on $\mathcal{M}$ and that $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a Banach space. Then an analog of a theorem of Schatten will also be proved.

Proposition 4.2. *The function $\|\cdot\|_{\mathcal{M}}$ on $\mathcal{M}$ as defined in (4.2) is a norm on the space $\mathcal{M}$ and $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ (with entry-wise addition and scalar multiplication) is a Banach space.*

*Proof.* Treat $\mathcal{M}$ as a subspace of the space $\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))$ of all bounded linear maps from $\mathcal{S}^r(\mathcal{A})$ to $(\mathcal{AS})$ as follows. For each $A=[a_{jk}]\in\mathcal{S}^r(\mathcal{A})$ and $\Phi=[\varphi_{jk}]\in\mathcal{M}$, define

$$\Phi : A \longmapsto \Phi[A] = \Phi\bullet A = [\varphi_{jk}(a_{jk})]. \tag{4.7}$$

Then $\Phi[A]\in(\mathcal{AS})$ and

$$\|\Phi\|_{\mathcal{M}} = \sup\{\|\Phi[A]\|_{(\mathcal{AS})} : A\in\mathcal{S}(\mathcal{A}),\ \|A\|\le 1\} \tag{4.8}$$

is just the norm on the space $\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))$ restricted to $\mathcal{M}$. Thus $\|\cdot\|_{\mathcal{M}}$ is a norm. It remains to prove that the space $\mathcal{M}$ is closed in $\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))$. To that end, suppose that $\{\Phi_n=[\varphi_{jk}^{(n)}]\}_{n=1}^{\infty}\subseteq\mathcal{M}$ is sequence such that $\Phi_n\to T$ for some $T\in\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))$. By Lemma 4.1(1), for each fixed $(j,k)$, the sequence $\{\varphi_{jk}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{A}^{\#}$, and thus by the completeness of $\mathcal{A}^{\#}$, there is a $\varphi_{jk}\in\mathcal{A}^{\#}$ such that $\varphi_{jk}^{(n)}\to\varphi_{jk}$ in $\mathcal{A}^{\#}$. Let $\Phi=[\varphi_{jk}]$. We show that $\Phi\in\mathcal{M}$ and $\Phi[A]=T(A)$ for all $A\in\mathcal{S}^r(\mathcal{A})$. Let $\epsilon>0$, $\nu\in\mathbb{N}$, and $A=[a_{jk}]\in\mathcal{S}^r(\mathcal{A})$ with $\|A\|\le 1$. Since $\{\Phi_n\}$ is a Cauchy sequence, there is an $N$ such that

$$\|\Phi_n-\Phi_m\|_{\mathcal{M}}<\epsilon \quad \forall n,m\ge N. \tag{4.9}$$

Thus

$$\begin{aligned}\|\Phi_n[A_{\nu\lrcorner}]-\Phi_m[A_{\nu\lrcorner}]\|_{(\mathcal{AS})} &\le \|\Phi_n[A]-\Phi_m[A]\|_{(\mathcal{AS})}\\ &\le \|\Phi_n-\Phi_m\|\|A\|<\epsilon \quad \forall n,m\ge N,\ \forall\nu\in\mathbb{N}.\end{aligned} \tag{4.10}$$

It then follows that

$$\begin{aligned}&\sum_{j,k=1}^{\nu}|\varphi_{jk}^{(n)}(a_{jk})-\varphi_{jk}^{(m)}(a_{jk})|\\ &\quad=\|\Phi_n[A_{\nu\lrcorner}]-\Phi_m[A_{\nu\lrcorner}]\|_{(\mathcal{AS})}<\epsilon \quad \forall n,m\ge N,\ \forall\nu\in\mathbb{N}.\end{aligned} \tag{4.11}$$

Since the left-hand side is a finite sum, we may take the limits in the preceding inequality, as $m\to\infty$, to obtain

$$\begin{aligned}&\sum_{j,k=1}^{\nu}|\varphi_{jk}^{(n)}(a_{jk})-\varphi_{jk}(a_{jk})|\\ &\quad=\|\Phi_n[A_{\nu\lrcorner}]-\Phi[A_{\nu\lrcorner}]\|_{(\mathcal{AS})}\le\epsilon \quad \forall n\ge N,\ \forall\nu\in\mathbb{N}.\end{aligned} \tag{4.12}$$

Therefore, $\Phi_n[A_{\nu\lrcorner}]\to\Phi[A_{\nu\lrcorner}]$ as $n\to\infty$ for all $\nu\in\mathbb{N}$. Since we also have $\Phi_n[A_{\nu\lrcorner}]\to T(A_{\nu\lrcorner})$ as $n\to\infty$, $\Phi[A_{\nu\lrcorner}]=T(A_{\nu\lrcorner})$. Furthermore, for each $\nu\in\mathbb{N}$,

$$\begin{aligned}\|\Phi[A_{\nu\lrcorner}]\|_{(\mathcal{AS})} &\le \|\Phi[A_{\nu\lrcorner}]-\Phi_N[A_{\nu\lrcorner}]\|_{(\mathcal{AS})}+\|\Phi_N[A_{\nu\lrcorner}]\|_{(\mathcal{AS})}\\ &<\epsilon+\|\Phi_N[A]\|_{(\mathcal{AS})}\le\epsilon+\|\Phi_N\|.\end{aligned} \tag{4.13}$$

Taking supremum over all $\nu \in \mathbb{N}$, we have

$$\|\Phi[A]\|_{(\mathcal{AS})} = \sup_{\nu\in\mathbb{N}} \|(\Phi[A])_{\nu\lrcorner}\|_{(\mathcal{AS})} \leq \epsilon + \|\Phi_N\|, \tag{4.14}$$

and hence $\Phi[A] \in (\mathcal{AS})$. Taking suprema on both sides of inequality (4.12) over all $\nu \in \mathbb{N}$, we also have

$$\begin{aligned} \|\Phi_n[A] - \Phi[A]\|_{(\mathcal{AS})} &= \sup_{\nu\in\mathbb{N}} \|(\Phi_n[A] - \Phi[A])_{\nu\lrcorner}\|_{(\mathcal{AS})} \\ &= \sup_{\nu\in\mathbb{N}} \|\Phi_n[A_{\nu\lrcorner}] - \Phi[A_{\nu\lrcorner}]\|_{(\mathcal{AS})} \leq \epsilon \quad \forall n \geq N. \end{aligned} \tag{4.15}$$

Thus $\Phi_n[A] \to \Phi[A]$. But, since we also have by our assumption on $\{\Phi_n\}$ that $\Phi_n[A] \to T(A)$, therefore $T(A) = \Phi[A]$ for all $A \in \mathcal{S}^r(\mathcal{A})$. Thus, $\mathcal{M}$ is closed in $\mathcal{B}(\mathcal{S}^r(\mathcal{A}), (\mathcal{AS}))$, and hence complete. □

## 5. $\mathcal{M}$ as the dual of $\mathcal{K}^r(\mathcal{A})$

Let $\mathcal{K}^r(\mathcal{A})$ be the set of all $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A})$ such that $\|A - A_{\nu\lrcorner}\| \to 0$ as $\nu \to \infty$. We first identify the dual of $\mathcal{K} := \mathcal{K}^r(\mathcal{A})$ with $\mathcal{M}$.

Theorem 5.1. *The dual space of $\mathcal{K}$ is isometrically isomorphic to $\mathcal{M}$.*

*Proof.* Let $\varphi \in (\mathcal{K})^\#$. For each $j,k \in \mathbb{N}$, define $\varphi_{jk}$ on $\mathcal{A}$ by

$$\varphi_{jk}(a) = \varphi(E_{jk}(a)) \quad \forall a \in \mathcal{A}, \tag{5.1}$$

where $E_{jk}(a)$ is the matrix whose $(j,k)$ entry is $a$ and all others are 0. Then it is readily seen that $\varphi_{jk} \in \mathcal{A}^\#$ for all $(j,k)$. We show that $[\varphi_{jk}] \in \mathcal{M}$ and that $\|\varphi\| (= \|\varphi\|_{(\mathcal{K})^\#}) = \|[\varphi_{jk}]\|_{\mathcal{M}}$. Let $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A})$. Let $\widetilde{A} = [(\operatorname{sgn}(\varphi_{jk}(a_{jk}))a_{jk})]$. Then $\widetilde{A} \in \mathcal{S}^r(\mathcal{A})$. For each $\nu \in \mathbb{N}$, since $A_{\nu\lrcorner} \in \mathcal{K}^r$, $\varphi((\widetilde{A})_{\nu\lrcorner})$ is defined and

$$\begin{aligned} \sum_{1\leq j,k\leq\nu} |\varphi_{jk}(a_{jk})| &= \varphi((\widetilde{A})_{\nu\lrcorner}) \leq \|\varphi\| \|(\widetilde{A})_{\nu\lrcorner}\| \\ &= \|\varphi\| \|A_{\nu\lrcorner}\| \leq \|\varphi\| \|A\|. \end{aligned} \tag{5.2}$$

Since this is true for all $\nu \in \mathbb{N}$, we have $\sum_{j,k=1}^{\infty} |\varphi_{jk}(a_{jk})| < \infty$, and hence $[\varphi_{jk}] \in \mathcal{M}$ by the arbitrariness of $A \in \mathcal{S}^r(\mathcal{A})$. By the inequalities in (5.2), we also see that $\|[\varphi_{jk}]\| \leq \|\varphi\|$. To see the opposite inequality, let $\epsilon > 0$, and choose $A = [a_{jk}] \in \mathcal{K}$ such that $\|A\| = 1$ and $|\varphi(A)| \geq \|\varphi\| - \epsilon$. It suffices to show that $|\varphi(A_{\nu\lrcorner}) - \varphi(A)| \to 0$, as $\nu \to \infty$. We first note that $\varphi(A_{\nu\lrcorner}) = [\varphi_{jk}] \bullet A_{\nu\lrcorner} = \sum_{j,k=1}^{\nu} \varphi_{jk}(a_{jk})$ for all $\nu \in \mathbb{N}$. Since $A \in \mathcal{K}$, $\|A_{\nu\lrcorner} - A\| \to 0$ in $\mathcal{K}$, as $\nu \to \infty$, and hence $\varphi(A_{\nu\lrcorner}) \to \varphi(A)$ as required. Though not required for the proof of this theorem, we also note that by the absolute convergence of the series $\sum_{j,k=1}^{\infty} \varphi_{jk}(a_{jk})$, which follows from (5.2), $\varphi(A) = \sum_{j,k=1}^{\infty} \varphi_{jk}(a_{jk})$. □

A bounded linear functional $\psi \in (\mathcal{S}^r(\mathcal{A}))^\#$ on $\mathcal{S}^r(\mathcal{A})$ is said to be *singular* if $\psi \in (\mathcal{K}^r(\mathcal{A}))^\perp$. Denote by $(\mathcal{S}^r(\mathcal{A}))_s^\#$ the space of all bounded singular linear functionals on

$\mathcal{S}^r(\mathcal{A})$. Elements $\Phi = [\varphi_{jk}] \in \mathcal{M}$ will also be regarded as elements in $(\mathcal{S}^r(\mathcal{A}))^{\#}$ by the convention $\Phi(A) = \sum_{j,k} \varphi_{jk}(a_{jk})$ for all $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A})$, where the convergence of the series is guaranteed by the absolute convergence of the series from the membership $\Phi \in \mathcal{M}$. The following is an analog of a theorem by Schatten.

Theorem 5.2. *The dual space $(\mathcal{S}^r(\mathcal{A}))^{\#}$ of $\mathcal{S}^r(\mathcal{A})$ has the Banach space direct sum decomposition*

$$(\mathcal{S}^r(\mathcal{A}))^{\#} = \mathcal{M} \oplus (\mathcal{S}^r(\mathcal{A}))^{\#}_s. \tag{5.3}$$

*Proof.* Let $\Psi \in (\mathcal{S}^r(\mathcal{A}))^{\#}$. Set $\Psi_1 := \Psi|_{\mathcal{K}^r(\mathcal{A})} \in (\mathcal{K}^r(\mathcal{A}))^{\#} = \mathcal{M}$, and $\Psi_2 = \Psi - \Psi_1 \in (\mathcal{K}^r(\mathcal{A}))^{\perp} = (SrA)^{\#}_s$. Then we clearly have the decomposition $\Psi = \Psi_1 + \Psi_2$. □

Schatten's theorem also states that $\|\Psi_0\| + \|\Psi_s\| = \|\Psi\|$ for each $\Psi$ in the dual of the algebra of bounded linear operators on a Hilbert space, where $\Psi_0$ is given by a trace-class operator and $\Psi_s$ is singular such that $\Psi = \Psi_0 + \Psi_s$. We do not know, however, whether this is true in this setting. In [3, 5] it has been proved to hold in their respective settings.

## 6. The Schur algebra as a dual space

It is not hard to see that if $\mathcal{S}^r(\mathcal{A})$ is to be the dual space of some normed space, then so must be $\mathcal{A}$ itself. Therefore, for the discussions in this section to make sense, we make the standing assumption that the $C^*$-algebra $\mathcal{A}$ is the dual space of some normed space (and hence the dual of the Banach space completion of the normed space). That is, we make the standing assumption in this section that $\mathcal{A}$ is a von Neumann algebra.

Let $\mathcal{A}_{\#}$ be the predual of $\mathcal{A}$. From the results in [6, 7, pages 454–485], instead of using the whole dual space $\mathcal{A}^{\#}$ and the set $s(\mathcal{A})$ of all states on $\mathcal{A}$ to define $\mathcal{S}^r(\mathcal{A})$, here we will use the predual of $\mathcal{A}$ (the space of all normal, or ultraweakly continuous, linear functionals on $\mathcal{A}$) and $s(\mathcal{A})$ will be the set of all normal states on $\mathcal{A}$. Then by polarization each element in $\mathcal{A}_{\#}$ is a linear combination of at most four normal states. Since $\mathcal{A}_{\#}$ is also complete (as a space of bounded linear functionals on $\mathcal{A}$), and each selfadjoint $a \in \mathcal{A}$ also has $\|a\| = \sup_{\varphi \in s(\mathcal{A})} \varphi(a)$, what holds true for $\mathcal{S}^r(\mathcal{A})$ defined previously holds for this new setting. We will show that the Schur algebra $\mathcal{S}^r(\mathcal{A})$ is also the dual space of some Banach space, another analog of a theorem of Schatten.

Since each $\xi \in \mathcal{A}_{\#}$ is an ultraweakly continuous linear functional on $\mathcal{A}$; that is, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in the underlying Hilbert space on which $\mathcal{A}$ acts, such that $\sum_{n=1}^{\infty}(\|x_n\|^2 + \|y_n\|^2) < \infty$ and $\xi(a) = \sum_{n=1}^{\infty}\langle ax_n, y_n\rangle$ for each $a \in \mathcal{A}$, thus rather than writing each element in $\mathcal{A}$ as a function on $\mathcal{A}_{\#}$, we will use this fact to express each element in $\mathcal{A}_{\#}$ as a function on $\mathcal{A}$. For each infinite matrix $\Xi = [\xi_{jk}]$ with entries from $\mathcal{A}_{\#}$, and for each $A = [a_{jk}]$ with entries from $\mathcal{A}$, denote by $A \bullet \Xi$ the matrix whose $(j,k)$ entry is $\xi_{jk}(a_{jk})$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$A \bullet \Xi = [\xi_{jk}(a_{jk})]. \tag{6.1}$$

Note that $A \bullet \Xi$ is just an infinite matrix, and it may not be "bounded" in any sense.

Let $\mathcal{M}(\mathcal{A}_{\#})$ be the space of all matrices $\Xi = [\xi_{jk}]$ over $\mathcal{A}_{\#}$ such that $A \bullet \Xi \in (\mathcal{A}\mathcal{S})$ for all $A \in \mathcal{S}^r(\mathcal{A})$, that is, $\sum_{jk} |\xi_{jk}(a_{jk})| < \infty$ for all $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A})$. Define, for each

$\Xi = [\xi_{jk}] \in \mathcal{M}(\mathcal{A}_{\#})$,

$$\begin{aligned} \|\Xi\|_{\#} = \|\Xi\| &:= \sup\{\|A \bullet \Xi\|_{(\mathcal{AS})} : A \in \mathcal{S}^r(\mathcal{A}),\ \|A\| \le 1\} \\ &= \sup\left\{ \sum_{j,k=1}^{\infty} |\xi_{jk}(a_{jk})| : A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A}),\ \|A\| \le 1 \right\}. \end{aligned} \tag{6.2}$$

We will prove that $\|\cdot\|_{\#}$ is indeed a norm on $\mathcal{M}(\mathcal{A}_{\#})$, and that $\mathcal{M}(\mathcal{A}_{\#})$ is a Banach space.

Proposition 6.1. (1) *The function as defined in (6.2) is a norm on the space* $\mathcal{M}(\mathcal{A}_{\#})$.

(2) *For each* $\Xi = [\xi_{jk}] \in \mathcal{M}(\mathcal{A}_{\#})$ *and for each* $(\mu, \nu) \in \mathbb{N} \times \mathbb{N}$,

$$\|\xi_{\mu\nu}\| \le \|\Xi\|. \tag{6.3}$$

(3) *For each* $\Xi = [\xi_{jk}] \in \mathcal{M}(\mathcal{A}_{\#})$, *and each* $\nu \in \mathbb{N}$,

$$\|\Xi_{\nu\lrcorner}\| \le \|\Xi\|. \tag{6.4}$$

(4) *The space* $\mathcal{M}(\mathcal{A}_{\#})$ *is a Banach space with this norm (and the usual entry-wise addition and scalar multiplication).*

*Proof.* (1) Let $\Xi = [\xi_{jk}] \in \mathcal{M}(\mathcal{A}_{\#})$. By the definition of $\mathcal{M}(\mathcal{A}_{\#})$,

$$\mathcal{T}_{\Xi}(A) = A \bullet \Xi = [\xi_{jk}(a_{jk})] \quad \forall A = [a_{jk}] \in \mathcal{S}^r(\mathcal{A}) \tag{6.5}$$

defines a linear map $\mathcal{T}_{\Xi} : \mathcal{S}^r(\mathcal{A}) \to (\mathcal{AS})$. We show that $\mathcal{T}_{\Xi}$ is a bounded linear transformation by the closed-graph theorem. Let $\{A_n, \mathcal{T}_{\Xi}(A_n)\}_{n=1}^{\infty}$ be a sequence in $\mathcal{S}^r(\mathcal{A}) \oplus (\mathcal{AS})$ such that $A_n \to A$ for some $A = [a_{jk}]$ in $\mathcal{S}^r(\mathcal{A})$ and $\mathcal{T}_{\Xi}(A_n) \to B$ for some $B = [b_{jk}] \in (\mathcal{AS})$. Let $A_n = [a_{jk}^{(n)}]$. Since $A \in \mathcal{S}^r(\mathcal{A})$, $A \bullet \Xi \in (\mathcal{AS})$. From $A_n \to A$, we see that $\|a_{jk}^{(n)} - a_{jk}\| \to 0$, and hence $a_{jk}^{(n)} \to a_{jk}$ ultraweakly, as $n \to \infty$, for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. Thus $\xi_{jk}(a_{jk}^{(n)}) \to \xi_{jk}(a_{jk})$, as $n \to \infty$, for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. Since $\mathcal{T}_{\Xi}(A_n) \to B$ in $(\mathcal{AS})$, for each $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$|\xi_{jk}(a_{jk}^{(n)}) - b_{jk}| \le \sum_{s,t=1}^{\infty} |\xi_{st}(a_{st}^{(n)}) - b_{st}| = \|\mathcal{T}_{\Xi}(A_n) - B\|_{(\mathcal{AS})} \longrightarrow 0 \tag{6.6}$$

as $n \to \infty$. Hence $b_{jk} = \xi_{jk}(a_{jk})$ for all $j,k \in \mathbb{N}$. Therefore $B = \mathcal{T}_{\Xi}(A)$. Thus $\mathcal{T}_{\Xi}$ has a closed graph, and hence $\mathcal{T}_{\Xi}$ is bounded.

We will identify each $\Xi \in \mathcal{M}(\mathcal{A}_{\#})$ with $\mathcal{T}_{\Xi} \in \mathcal{B}(\mathcal{S}^r(\mathcal{A}), (\mathcal{AS}))$. We then have $\mathcal{M}(\mathcal{A}_{\#}) \subseteq \mathcal{B}(\mathcal{S}^r(\mathcal{A}), (\mathcal{AS}))$. We also note that the norm $\|\Xi\|$ of $\Xi \in \mathcal{M}(\mathcal{A}_{\#})$ as defined in (6.2) is exactly the norm of $\mathcal{T}_{\Xi} \in \mathcal{B}(\mathcal{S}^r(\mathcal{A}), (\mathcal{AS}))$. Therefore, the function $\Xi \mapsto \|\Xi\|$ is a norm on $\mathcal{M}(\mathcal{A}_{\#})$.

(2) Let $a \in \mathcal{A}$ be such that $\|a\| \le 1$. For a fixed $(j,k) \in \mathbb{N} \times \mathbb{N}$, denote by $E_{jk}(a)$ the matrix whose $(j,k)$-entry is $a$ and all others are 0. Then $E_{jk}(a) \in \mathcal{S}^r(\mathcal{A})$ and $\|E_{jk}(a)\| = \|a\| \le 1$. We also have, for each $\Xi = [\xi_{jk}] \in \mathcal{M}_{\#}$ and $\mu, \nu \in \mathbb{N}$, $E_{\mu\nu}(a) \bullet \Xi = E_{\mu\nu}(\xi_{\mu\nu}(a))$, the

matrix whose $(\mu,\nu)$ entry is the number $\xi_{\mu\nu}(a)$ and all others are 0. Then

$$\begin{aligned} |\xi_{\mu\nu}(a)| &= \|E_{\mu\nu}(\xi_{\mu\nu}(a))\|_{(\mathcal{AS})} = \|E_{\mu\nu}(a)\bullet\Xi\|_{(\mathcal{AS})} \\ &\le \|E_{\mu\nu}(a)\|_{\mathcal{S}^r(\mathcal{A})}\|\Xi\|_{\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))} = \|\Xi\|. \end{aligned} \tag{6.7}$$

Since this is true for all $a\in\mathcal{A}$ with $\|a\|\le 1$, we have $\|\xi_{\mu\nu}\|\le\|\Xi\|$ for all $(\mu,\nu)\in\mathbb{N}\times\mathbb{N}$.

(3) We omit the straightforward modification of the preceding argument that can give a proof of this statement.

(4) To see that $\mathcal{M}(\mathcal{A}_{\#})$ is complete in the norm, let $\{\Xi_n=[\xi^{(n)}_{jk}]\}_{n=1}^{\infty}$ be a sequence in $\mathcal{M}(\mathcal{A}_{\#})$ such that $\mathcal{T}_{\Xi_n}\to\mathcal{T}$ for some $\mathcal{T}\in\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))$. Then, by the inequality just established in part (2), each $\{\xi^{(n)}_{jk}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{A}_{\#}$, and hence converges to some $\xi_{jk}$. Let $\Xi=[\xi_{jk}]$. For each $A=[a_{jk}]\in\mathcal{S}^r(\mathcal{A})$, since $\mathcal{T}(A)\in(\mathcal{AS})$, $\mathcal{T}(A)=[t_{jk}]$, an infinite matrix. For each $(\mu,\nu)\in\mathbb{N}\times\mathbb{N}$,

$$\begin{aligned} |\xi^{(n)}_{\mu\nu}(a_{\mu\nu})-t_{\mu\nu}| &= \|[\mathcal{T}_{\Xi_n}(A)]\bullet E_{\mu\nu}(1)-[\mathcal{T}(A)]\bullet E_{\mu\nu}(1)\| \\ &= \|[(\mathcal{T}_{\Xi_n}-\mathcal{T})(A)]\bullet E_{\mu\nu}(1)\| \\ &\le \|[(\mathcal{T}_{\Xi_n}-\mathcal{T})(A)]\| \\ &\le \|\mathcal{T}_{\Xi_n}-\mathcal{T}\|\|A\|\longrightarrow 0 \qquad \text{as } n\longrightarrow\infty. \end{aligned} \tag{6.8}$$

But we also have $\xi^{(n)}_{\mu\nu}(a_{\mu\nu})\to\xi_{\mu\nu}(a_{\mu\nu})$. Thus $t_{\mu\nu}=\xi_{\mu\nu}(a_{\mu\nu})$. Since this is true for all $\mu,\nu\in\mathbb{N}$, $\mathcal{T}(A)=\mathcal{T}_{\Xi}(A)$. Since this is true for all $A\in\mathcal{S}^r(\mathcal{A})$, $\mathcal{T}=\mathcal{T}_{\Xi}$ and $\mathcal{T}_{\Xi}\in\mathcal{B}(\mathcal{S}^r(\mathcal{A}),(\mathcal{AS}))$. By the definition of $\mathcal{M}(\mathcal{A}_{\#})$, $\Xi\in\mathcal{M}(\mathcal{A}_{\#})$. This completes the proof. □

Since the algebra of bounded linear operators on a Hilbert space is the dual of the trace class operators under the trace norm, which is the closure in the trace norm of the set of matrices with finitely many nonzero entries, by analogy we define $\mathcal{M}_{\#}$ to be the closure in the norm, defined above, of all matrices in $\mathcal{M}(\mathcal{A}_{\#})$ with finitely many nonzero entries. Then it is readily seen that, for each element $\Xi=[\xi_{jk}]\in\mathcal{M}_{\#}$, $\|\Xi_{\nu\lrcorner}-\Xi\|\to 0$ as $\nu\to\infty$, where $\|\cdot\|=\|\cdot\|_{\#}$ is the norm on $\mathcal{M}(\mathcal{A}_{\#})$ defined in (6.2). We do not know, however, whether the inclusion $\mathcal{M}_{\#}\subseteq\mathcal{M}(\mathcal{A}_{\#})$ is proper. However by the definition of $\mathcal{M}_{\#}$, we have the following.

Proposition 6.2. *The space $\mathcal{M}_{\#}$ is a Banach space under the norm $\|\cdot\|$ defined in (6.2) and the usual entry-wise addition and scalar multiplication.*

Theorem 6.3. *The Schur algebra $\mathcal{S}^r(\mathcal{A})$ is isometrically isomorphic to the dual of $\mathcal{M}_{\#}$.*

*Proof.* Note that each $A=[a_{jk}]\in\mathcal{S}^r(\mathcal{A})$ defines a linear functional $\Phi_A$ on $\mathcal{M}_{\#}$ as follows:

$$\Phi_A(\Xi)=\sum_{j,k=1}^{\infty}\xi_{jk}(a_{jk}) \quad \forall\Xi=[\xi_{jk}]\in\mathcal{M}_{\#}. \tag{6.9}$$

The series converges by its absolute convergence, which is guaranteed by the definition of membership in $\mathcal{M}_{\#}$. That $\Phi_A$ is also bounded on $\mathcal{M}_{\#}$ follows also from (6.2). Thus

$\mathcal{S}^r(\mathcal{A}) \hookrightarrow (\mathcal{M}_\#)^\#$. Furthermore, since

$$
\begin{aligned}
\|\Phi_A\| &= \sup\left\{\left|\sum_{j,k=1}^{\infty}\xi_{jk}(a_{jk})\right| : \Xi = [\xi_{jk}] \in \mathcal{M}_\#,\ \|\Xi\| \le 1\right\}\\
&\le \sup\left\{\sum_{j,k=1}^{\infty}|\xi_{jk}(a_{jk})| : \Xi = [\xi_{jk}] \in \mathcal{M}_\#,\ \|\Xi\| \le 1\right\}\\
&= \sup\{\|A \bullet \Xi\|_{(\mathcal{A}\mathcal{S})} : \Xi \in M_\#,\ \|\Xi\| \le 1\}\\
&\le \sup\{\|A\|\|\Xi\| : \Xi \in \mathcal{M}_\#,\ \|\Xi\| \le 1\} \le \|A\|,
\end{aligned}
\tag{6.10}
$$

we may identify each element $A$ of $\mathcal{S}^r(\mathcal{A})$ with the linear functional $\Phi_A \in (\mathcal{M}_\#)^\#$, treating them as the same element, then we have the actual inclusion $\mathcal{S}^r(\mathcal{A}) \subseteq (\mathcal{M}_\#)^\#$.

Let $\mathcal{B}'$ be the unit ball of the dual $(\mathcal{M}_\#)^\#$ of $\mathcal{M}_\#$, and let $\mathcal{B}$ be the unit ball of $\mathcal{S}^r(\mathcal{A})$ (under the norm on $\mathcal{S}^r(\mathcal{A})$). From the above convention, we see that $\mathcal{B} \subseteq \mathcal{B}'$. We show that $\mathcal{B} = \mathcal{B}'$, and the result follows. The weak$^*$ topology $\sigma((\mathcal{M}_\#)^\#, \mathcal{M}_\#)$ on $(\mathcal{M}_\#)^\#$ is a locally convex topology, and since $\mathcal{B}$ separates points in $\mathcal{M}_\#$, $\sigma((\mathcal{M}_\#)^\#, \mathcal{M}_\#)$ induces a locally convex topology $\sigma$ on $\mathcal{B}$. We show that $(\mathcal{B}, \sigma)$ is complete so that it is a closed convex subset of the locally convex space $((\mathcal{M}_\#)^\#, \sigma((\mathcal{M}_\#)^\#, \mathcal{M}_\#))$. To that end, let $\{A_\alpha\} = \{[a_{jk}^{(\alpha)}]\}_{\alpha\in\Lambda}$ be a Cauchy net in $(\mathcal{B}, \sigma)$. Then, for each fixed $(j,k) \in \mathbb{N}\times\mathbb{N}$, $\epsilon > 0$, and unit vectors $\xi_m \in \mathcal{A}_\#$ $(m = 1,2,\ldots,l)$ $\Xi_m = E_{j,k}(\xi_m) \in \mathcal{M}_\#$ $(m = 1,2,\ldots,l)$ are unit vectors. Thus there is a $\gamma$ such that for all $\alpha, \beta \succeq \gamma$,

$$
|\xi_m(a_{jk}^{(\alpha)}) - \xi_m(a_{jk}^{(\beta)})| = |\Phi_{A_\alpha}(\Xi_m) - \Phi_{A_\beta}(\Xi_m)| < \epsilon \quad \forall m = 1,2,\ldots,l. \tag{6.11}
$$

Thus, for each fixed $j,k \in \mathbb{N}$, $\{a_{jk}^{(\alpha)}\}_\alpha$ is a Cauchy net in $((\mathcal{A})_1, \sigma(\mathcal{A}, \mathcal{A}_\#))$, where $(\mathcal{A})_1$ is the unit ball of $\mathcal{A}$. By Alaoglu's theorem, $(\mathcal{A})_1$ is weak$^*$ compact (i.e., $(\mathcal{A})_1$ is compact in the topology $\sigma(\mathcal{A}, \mathcal{A}_\#)$). Thus there is an $a_{jk} \in (\mathcal{A})_1$ such that $\lim_\alpha(a_{jk}^{(\alpha)}) = a_{jk}$ in the topology $\sigma(\mathcal{A}, \mathcal{A}_\#)$. Let $A = [a_{jk}]$. We show that $A \in \mathcal{B}$ and $A^{(\alpha)} \to A$ in $\sigma$. For each $\nu \in \mathbb{N}$ and each $\Xi = [\xi_{jk}] \in \mathcal{M}_\#$,

$$
\lim_\alpha(\Phi_{(A_\alpha)_{\nu\lrcorner}}(\Xi)) = \lim_\alpha\left(\sum_{j,k=1}^{\nu}\xi_{jk}(a_{jk}^{(\alpha)})\right) = \sum_{j,k=1}^{\nu}\xi_{jk}(a_{jk}) = \Phi_{A_{\nu\lrcorner}}(\Xi). \tag{6.12}
$$

Thus $(A_\alpha)_{\nu\lrcorner} \to A_{\nu\lrcorner}$ in $\sigma$. Since $\|\Phi_{(A_\alpha)_{\nu\lrcorner}}\| \le \|(A_\alpha)_{\nu\lrcorner}\| \le \|A_\alpha\| \le 1$ for all $\alpha$, $\|\Phi_{A_{\nu\lrcorner}}\| \le 1$ also follows.

Here we digress to show that $\|B_{\nu\lrcorner}\| = \|\Phi_{B_{\nu\lrcorner}}\|$ for all $B \in \mathcal{S}^r(\mathcal{A})$ and all $\nu \in \mathbb{N}$. Let $\mathcal{B}'_{\nu\lrcorner}$ and $\mathcal{B}_{\nu\lrcorner}$ be the sets of all matrices in $\mathcal{B}'$ and $\mathcal{B}$ that have all $(j,k)$-entries 0 for $j > \nu$ or $k > \nu$. The duality between the upper $\nu\times\nu$ corners of $\mathcal{M}_\#$ and $\mathcal{S}^r(\mathcal{A})$ will be established once we prove that $\mathcal{B}'_{\nu\lrcorner} = \mathcal{B}_{\nu\lrcorner}$. Suppose the inclusion $\mathcal{B}_{\nu\lrcorner} \subset \mathcal{B}'_{\nu\lrcorner}$ is proper. Since $\mathcal{B}'_{\nu\lrcorner}$ is a closed convex subset (from what we just established), and there is a $\Phi \in \mathcal{B}'_{\nu\lrcorner} \setminus \mathcal{B}_{\nu\lrcorner}$, by a version of Hahn-Banach separation theorem [6, 7, Theorem 1.2.10], there are an element $\Xi \in (\mathcal{M}_\#)_{\nu\lrcorner}$ (the space of $\nu\times\nu$ truncations of elements in $\mathcal{M}_\#$), an $\epsilon > 0$, and a constant

$c \in \mathbb{R}$ such that

$$\mathcal{R}e(\Phi_B(\Xi)) \le c < c+\epsilon \le \mathcal{R}e(\Phi(\Xi)) \quad \forall B \in \mathcal{B}_{\nu_\lrcorner}. \tag{6.13}$$

For each $B \in \mathcal{B}_{\nu_\lrcorner}$, let $\widetilde{B} = \overline{(\operatorname{sgn}(\Phi_B(\Xi)))}B$. Then $\widetilde{B} \in \mathcal{B}_{\nu_\lrcorner}$, and

$$\|\Xi\| = \sup_{B\in\mathcal{B}} |\Phi_B(\Xi)| = \sup_{B\in\mathcal{B}} \Phi_{\widetilde{B}}(\Xi) \le c < c+\epsilon \le \mathcal{R}e(\Phi(\Xi)) \le |\Phi(\Xi)| \le \|\Xi\|, \tag{6.14}$$

a contradiction. Therefore $\mathcal{B}_{\nu_\lrcorner} = \mathcal{B}'_{\nu_\lrcorner}$, and hence $\|\Phi_{B_{\nu_\lrcorner}}\| = \|B_{\nu_\lrcorner}\|$ for all $B \in \mathcal{S}^r(\mathcal{A})$.

Now we have, for the limit $A$ above, $\|A\| = \sup_{\nu\in\mathbb{N}} \|A_{\nu_\lrcorner}\| = \sup_{\nu\in\mathbb{N}} \|\Phi_{A_{\nu_\lrcorner}}\| \le 1$. Thus $A \in \mathcal{B}$. Let $\Xi = [\xi_{jk}] \in \mathcal{M}_\#$, and $\epsilon > 0$. Then there exists $\gamma$ such that

$$|\Phi_{A_\alpha}(\Xi) - \Phi_{A_\beta}(\Xi)| = \left| \sum_{j,k=1}^{\infty} \left(\xi_{jk}(a_{jk}^{(\alpha)}) - \xi_{jk}(a_{jk}^{(\beta)})\right) \right| < \frac{\epsilon}{3} \quad \forall \alpha, \beta \succeq \gamma. \tag{6.15}$$

By the definition of $\mathcal{M}_\#$, there is a $\nu_0 \in \mathbb{N}$ such that $\|\Xi - \Xi_{\nu_\lrcorner}\| < \epsilon/3$ for all $\nu \ge \nu_0$. Fix a $\nu \ge \nu_0$. Since $a_{jk}^{(\alpha)} \to a_{jk}$ in $\sigma(\mathcal{A}, \mathcal{A}_\#)$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$, and since $\nu$ is finite, we may choose $\beta_0 \succeq \gamma$ such that for all $\beta \succeq \beta_0$

$$\left| \sum_{j,k=1}^{\nu} \left(\xi_{jk}(a_{jk}^{(\beta)}) - \xi_{jk}(a_{jk})\right) \right| < \frac{\epsilon}{3}. \tag{6.16}$$

Thus for $\beta \succeq \beta_0$,

$$\begin{aligned} |\Phi_{A_\beta}(\Xi) - \Phi_A(\Xi)| &= \left| \sum_{j,k=1}^{\infty} \left(\xi_{jk}(a_{jk}^{(\beta)}) - \xi_{jk}(a_{jk})\right) \right| \\ &\le |\Phi_{A_\beta}(\Xi) - \Phi_{A_\beta}(\Xi_{\nu_\lrcorner})| + |\Phi_{A_\beta}(\Xi_{\nu_\lrcorner}) - \Phi_A(\Xi_{\nu_\lrcorner})| \\ &\quad + |\Phi_A(\Xi_{\nu_\lrcorner}) - \Phi_A(\Xi)| \\ &\le \|A_\beta\| \|\Xi - \Xi_{\nu_\lrcorner}\| + \left| \sum_{j,k=1}^{\nu} \left(\xi_{jk}(a_{jk}^{(\beta)}) - \xi_{jk}(a_{jk})\right) \right| \\ &\quad + \|A\| \|\Xi_{\nu_\lrcorner} - \Xi\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned} \tag{6.17}$$

Therefore, $A_\alpha \to A$ in $\sigma$, and hence $\mathcal{B}$ is a $\sigma((\mathcal{M}_\#)^\#, \mathcal{M}_\#)$ closed convex subset of $\mathcal{B}'$. The proof of $\mathcal{B} = \mathcal{B}'$ is exactly the same as that of $\mathcal{B}_{\nu_\lrcorner} = \mathcal{B}'_{\nu_\lrcorner}$ above. This completes the proof. □

We conclude with some natural questions, for which we do not know the answers.

(1) Is it possible to express $\mathcal{S}^r(\mathcal{A})$ as a topological tensor product of $\mathcal{S}^r$ and $\mathcal{A}$?

(2) When $\mathcal{A}$ is a von Neumann algebra, do we have

$$\mathcal{M}(\mathcal{A}_\#) = \ell^1(\mathbb{N} \times \mathbb{N}) \otimes_\gamma A_\#, \tag{6.18}$$

(where $\otimes_\gamma$ denotes the projective Banach space tensor product)?

(3) For a von Neumann algebra $\mathscr{A}$, is $\mathcal{M}_{\#}$ the *unique* predual of $\mathscr{S}^r(\mathscr{A})$?
(4) When is $\mathscr{S}^r(\mathscr{A})$ an operator algebra? With this regard, the paper "On quotients of function algebras and operator algebra structures on $\ell^p$," *J. Operator Theory* 34 (1995), 315-346, by D. P. Bleecher and C. Le Merdy, may be of interest.
(5) To what extent does the theory of Schatten ideals on a Hilbert space carry over to the present context?

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