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Research Article

On the Rational Recursive Sequence $x_{n+1} = (\alpha - \beta x_n)/(\gamma - \delta x_n - x_{n-k})$

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We study the global stability, the periodic character, and the boundedness character of the positive solutions of the difference equation $x_{n+1} = (\alpha - \beta x_n)/(\gamma - \delta x_n - x_{n-k})$, $n = 0, 1, 2, \ldots,$ $k \in \{1, 2, \ldots\}$, in the two cases: (i) $\delta \geq 0$, $\alpha > 0$, $\gamma > \beta > 0$; (ii) $\delta \geq 0$, $\alpha = 0$, $\gamma, \beta > 0$, where the coefficients α, β, γ, and δ, and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are real numbers. We show that the positive equilibrium of this equation is a global attractor with a basin that depends on certain conditions posed on the coefficients of this equation.

1. Introduction

The asymptotic stability of the rational recursive sequence,

$$x_{n+1} = \frac{\alpha + \beta x_n}{\gamma + \sum_{i=0}^{k} \gamma_i x_{n-i}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

was investigated when the coefficients α, β, γ, and γ_i are nonnegative real numbers (see [1–3]). Studying the asymptotic behavior of the rational sequence (1.1) when some of the coefficients are negative was suggested in [3]. Recently, Aboutaleb et al. [4] studied the rational recursive sequence,

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where α, β, and γ are nonnegative real numbers and obtained sufficient conditions for the global attractivity of the positive equilibria. Yan et al. [5] studied recently the rational recursive sequence,

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

where $\alpha \geq 0$, $\gamma, \beta > 0$ are real numbers while $k \geq 1$ is an integer number, and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are arbitrary real numbers. They proved that the positive equilibrium $\overline{x}$ of (1.3) is a global attractor with a basin that depends on certain conditions of the coefficients. He et al. [6] studied recently the rational recursive sequence,

$$x_{n+1} = \frac{a - b x_{n-k}}{A + x_n}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $a \geq 0$, $A, b > 0$ are real numbers while $k \geq 1$ is an integer number and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are arbitrary real numbers. They proved the global attractivity and periodic character of the positive solution of (1.4). Stević [7] studied recently the rational recursive sequence,

$$x_{n+1} = \frac{\alpha + \beta x_n}{\gamma - x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where the parameters α, β, and γ are nonnegative real numbers and $k \geq 1$ is an integer number while the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are arbitrary real numbers. Other related results can be found in [8–19].

Our aim in this paper is to study the global attractivity, the periodicity, and the boundedness of the positive solution of the following rational recursive sequence:

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - \delta x_n - x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

in the two cases (i) $\delta \geq 0$, $\alpha > 0$, $\gamma > \beta > 0$, (ii) $\delta \geq 0$, $\alpha = 0$, $\gamma, \beta > 0$, where the coefficients α, β, γ, and δ are real numbers and $k \geq 1$ is an integer number, while the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are arbitrary real numbers. We will prove that the positive equilibrium $\overline{x}$ of (1.6) is a global attractor with a basin that depends on certain conditions of these coefficients.

2. Local stability and permanence

We first recall some results which will be useful in the sequel. Let $\mathbf{I}$ be some real interval and let F be a continuous function defined on $\mathbf{I}^{k+1}$. Then, for initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in \mathbf{I}$, it is easy to see that the difference equation,

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \ k \geq 1, \tag{2.1}$$

has a unique solution $\{x_n\}$.

Definition 2.1. A point $\overline{x}$ is called an equilibrium of (2.1), if $\overline{x} = F(\overline{x}, \ldots, \overline{x})$. That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (2.1), or equivalently, is a fixed point of F.

Definition 2.2. An interval $\mathbf{J} \subset \mathbf{I}$ is called an invariant interval of (2.1) if the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in \mathbf{J}$ imply that the solution $x_n \in \mathbf{J}$ for $n > 0$. That is, every solution of (2.1) with initial conditions in $\mathbf{J}$ remains in $\mathbf{J}$.

Definition 2.3. The difference equation (2.1) is said to be permanent if there exist numbers P and Q with $0 < P \leq Q < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ there exists a positive integer N which depends on the initial conditions such that $P \leq x_n \leq Q$, for all $n \geq N$.

The linearized equation associated with (2.1) about the equilibrium $\overline{x}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \ldots, \overline{x})}{\partial u_i} y_{n-i}, \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Its characteristic equation is

$$\lambda^{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \ldots, \overline{x})}{\partial u_i} \lambda^{n-i}, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

Theorem 2.4 (see [3]). *Assume that F is a C^1-function and let $\overline{x}$ be an equilibrium of* (2.1). *Then, the following statements are true:*

(a) *if all the roots of* (2.3) *lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of* (2.1) *is locally asymptotically stable;*

(b) *if at least one root of* (2.3) *has absolute value greater than one, then the equilibrium $\overline{x}$ of* (2.1) *is unstable.*

Theorem 2.5 (see [3, 8]). *Assume that $p, q \in \mathbb{R}$, and $k \in \{1, 2, \ldots\}$. Then,*

$$|p| + |q| < 1 \tag{2.4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} - p x_n + q x_{n-k} = 0, \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

Suppose in addition that one of the following two cases holds: (i) *k is odd and $q < 0$, or* (ii) *k is even and $pq < 0$. Then,* (2.4) *is also a necessary condition for the asymptotic stability of* (2.5) *(see [6]).*

First, we study the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - \delta x_n - x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

together with the conditions

$$\delta \geq 0, \qquad \alpha > 0, \qquad \gamma > \beta > 0, \qquad k \in \{1, 2, \ldots\}. \tag{2.7}$$

The unique positive equilibrium point $\overline{x}$ of (2.6) is the solution of the equation

$$\overline{x} = \frac{\alpha - \beta\overline{x}}{\gamma - (\delta + 1)\overline{x}}, \tag{2.8}$$

which is given by

$$\overline{x} = \frac{(\gamma + \beta) \pm \sqrt{T}}{2(\delta + 1)}, \tag{2.9}$$

where

$$T = (\gamma + \beta)^2 - 4\alpha(\delta + 1). \tag{2.10}$$

If (2.7) holds and $\alpha = (\gamma + \beta)^2/4(\delta + 1)$, then (2.6) has a unique positive equilibrium $\overline{x}_0 = (\gamma + \beta)/2(\delta + 1)$. If (2.7) holds and $\alpha < (\gamma + \beta)^2/4(\delta + 1)$ then (2.6) has two positive equilibria $\overline{x}_{1,2}$ given by (2.9).

The linearized equation of (2.6) about the equilibrium $\overline{x}_i$ $(i = 0, 1, 2)$ is given by

$$y_{n+1} + \frac{\beta - \delta\,\overline{x}_i}{[\gamma - (\delta + 1)\overline{x}_i]} y_n - \frac{\overline{x}_i}{[\gamma - (\delta + 1)\overline{x}_i]} y_{n-k} = 0. \tag{2.11}$$

The characteristic equation associated with (2.6) about $\overline{x}_0$ is

$$\lambda^{k+1} + \left[\frac{2\beta}{\gamma - \beta} - \frac{\delta(\gamma + \beta)}{(\delta + 1)(\gamma - \beta)}\right]\lambda^k - \frac{\gamma + \beta}{(\delta + 1)(\gamma - \beta)} = 0. \tag{2.12}$$

Now, we have the following results:

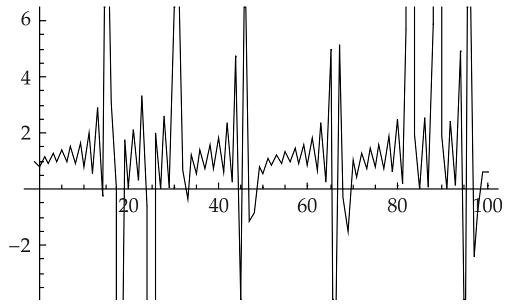
(a) if $0 \leq \delta < 2\beta/(\gamma - \beta)$, then $(\gamma + \beta)/(\delta + 1)(\gamma - \beta) > 1$ and hence the equilibrium $\overline{x}_0$ of (2.6) is unstable (see Figure 1);

(b) if $\delta > 2\beta/(\gamma - \beta)$, then

$$\left|\frac{2\beta}{\gamma - \beta} - \frac{\delta(\gamma + \beta)}{(\delta + 1)(\gamma - \beta)}\right| + \left|\frac{\gamma + \beta}{(\delta + 1)(\gamma - \beta)}\right| = 1. \tag{2.13}$$

Thus, the linearized stability analysis fails. On the other hand, the characteristic equation associated with (2.6) about $\overline{x}_1$ is

$$\lambda^{k+1} + \left[\frac{2\beta}{\gamma - \beta - \sqrt{T}} - \frac{\delta(\gamma + \beta + \sqrt{T})}{(\delta + 1)(\gamma - \beta - \sqrt{T})}\right]\lambda^k - \frac{\gamma + \beta + \sqrt{T}}{(\delta + 1)(\gamma - \beta - \sqrt{T})} = 0. \tag{2.14}$$



$\overline{x}_0 = 1.333$, $\alpha = 8/3$, $\beta = 1$, $\gamma = 3$, $\delta = 0.5$, $x_{-1} = e^{-1}$, $x_0 = 1$

Figure 1

Now, we have the following results:

(a) if $0 \le \delta < 2\beta/(\gamma - \beta)$, then it is obvious that

$$\left|\frac{\gamma+\beta+\sqrt{T}}{(\delta+1)(\gamma-\beta-\sqrt{T})}\right| \ge \left(\frac{\gamma-\beta}{\gamma+\beta}\right)\frac{[\gamma+\beta+\sqrt{T}]}{[\gamma-\beta-\sqrt{T}]} > 1, \tag{2.15}$$

hence the equilibrium $\overline{x}_1$ of (2.6) is unstable;

(b) if $\delta \ge 2\beta/(\gamma-\beta)$, then it is easy to see that $2\beta(\delta+1) < \delta[\gamma+\beta+\sqrt{T}]$, and consequently, we have

$$\left|\frac{2\beta}{\gamma-\beta-\sqrt{T}} - \frac{\delta\left[\gamma+\beta+\sqrt{T}\right]}{(\delta+1)\left[\gamma-\beta-\sqrt{T}\right]}\right| + \left|\frac{\gamma+\beta+\sqrt{T}}{(\delta+1)\left[\gamma-\beta-\sqrt{T}\right]}\right| = \frac{\gamma-\beta+\sqrt{T}}{\gamma-\beta-\sqrt{T}} > 1, \tag{2.16}$$

and hence the equilibrium $\overline{x}_1$ of (2.6) is unstable.

For the positive equilibrium $\overline{x}_2$, in view of conditions (2.7) and $\alpha < (\gamma+\beta)^2/4(\delta+1)$, we have

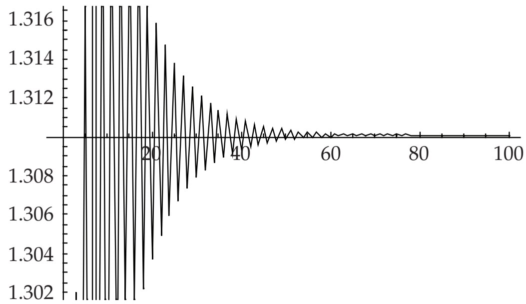
$$\overline{x}_2 = \frac{\gamma+\beta-\sqrt{T}}{2(\delta+1)} < \frac{\gamma+\beta}{2(\delta+1)} < \frac{\gamma}{\delta+1}. \tag{2.17}$$

Hence, if

$$0 < \alpha \le \frac{\beta(\gamma-\beta)}{\delta+1}, \tag{2.18}$$

then

$$\sqrt{T} \ge \sqrt{(\gamma+\beta)^2 - 4\beta(\gamma-\beta)} > \sqrt{(\gamma+\beta)^2 - (\gamma+3\beta)(\gamma-\beta)} = \sqrt{(\gamma+\beta)^2 - (\gamma+\beta)^2 + 4\beta^2} = 2\beta. \tag{2.19}$$



$\overline{x}_2 = 1.3101$, $\alpha = 5$, $\beta = 1.5$, $\gamma = 4.5$, $\delta = 2/3$, $x_{-1} = e^{-1}$, $x_0 = 1$

Figure 2

Consequently, we have

$$\left|\frac{\beta - \delta\overline{x}_2}{\gamma - (\delta+1)\overline{x}_2}\right| + \left|\frac{\overline{x}_2}{\gamma - (\delta+1)\overline{x}_2}\right| < \frac{\beta + (\delta+1)\overline{x}_2}{\gamma - (\delta+1)\overline{x}_2} = \frac{3\beta + \gamma - \sqrt{T}}{\gamma - \beta + \sqrt{T}} < \frac{3\beta + \gamma - 2\beta}{\gamma - \beta + 2\beta} = 1, \tag{2.20}$$

which by Theorem 2.5 implies that $\overline{x}_2$ is locally asymptotically stable (see Figure 2).

Lemma 2.6. *Let $f(u,v) = (\alpha - \beta u)/(\gamma - \delta u - v)$ and assume that conditions* (2.7) *and* (2.18) *hold. Then, the following statements are true:*

(a) $0 < \overline{x}_2 < \alpha/\beta$, $\alpha/\beta < \overline{x}_1 < \infty$;

(b) *$f(x,x)$ is a strictly decreasing function in $(-\infty, \alpha/\beta)$;*

(c) *let $u, v \in (-\infty, \alpha/\beta)$, then the function $f(u,v)$ is a strictly decreasing function in u and a strictly increasing function in v.*

Proof. We prove (a) only. The proofs of (b) and (c) are omitted here. In view of (2.7) and (2.18), we have

$$\overline{x}_2 = \frac{\gamma + \beta - \sqrt{T}}{2(\delta+1)} < \frac{\gamma + \beta}{2(\delta+1)} < \frac{\gamma}{\delta+1}. \tag{2.21}$$

From (2.8) and (2.21), we have $\alpha - \beta\overline{x}_2 > 0$ and so $\overline{x}_2 < \alpha/\beta$. Also, in view of (2.7) and (2.18), we have

$$\begin{aligned} 0 &< \frac{\alpha - \beta\overline{x}_1}{\gamma - (\delta+1)\overline{x}_1} = \overline{x}_1 = \frac{\gamma + \beta + \sqrt{T}}{2(\delta+1)} \geq \frac{\gamma + \beta + \sqrt{(\gamma+\beta)^2 - 4\beta(\gamma - \beta)}}{2(\delta+1)} \\ &= \frac{\gamma + \beta + \sqrt{(\gamma-\beta)^2 + 4\beta^2}}{2(\delta+1)} > \frac{\gamma + \beta + \sqrt{(\gamma-\beta)^2}}{2(\delta+1)} = \frac{\gamma}{\delta+1}, \end{aligned} \tag{2.22}$$

and so $\gamma - \overline{x}_1(\delta + 1) < 0$. Consequently, $\alpha - \beta\overline{x}_1 < 0$ which implies that $\overline{x}_1 > \alpha/\beta$. The proof is completed. □

Theorem 2.7. *Assume that the conditions* (2.7) *and* (2.18) *hold. Let* $\{x_n\}$ *be any solution of* (2.6)*. If* $x_i \in (-\infty, \alpha/\beta]$, *for* $i = -k, -k+1, \dots, -1$ *and if* $x_0 \in [0, \alpha/\beta]$, *then*

$$0 \le x_n \le \frac{\alpha}{\beta}, \quad n = 1, 2, \dots. \tag{2.23}$$

That is the solution $\{x_n\}$ *is bounded.*

Proof. By part (c) of Lemma 2.6, we have

$$0 = \frac{\alpha - \beta\cdot(\alpha/\beta)}{\gamma - \delta x_0 - x_{-k}} \le x_1 = \frac{\alpha - \beta x_0}{\gamma - \delta x_0 - x_{-k}} \le \frac{\alpha - \beta\cdot 0}{\gamma - \delta\cdot 0 - \alpha/\beta} = \frac{\alpha}{\gamma - \alpha/\beta} = \frac{\beta\alpha}{\gamma\beta - \alpha}. \tag{2.24}$$

From (2.18), we deduce that $\gamma\beta - \alpha > \beta^2$, and then we have

$$0 \le x_1 \le \frac{\alpha}{\beta}. \tag{2.25}$$

Also, we have

$$0 = \frac{\alpha - \beta\cdot(\alpha/\beta)}{\gamma - \delta x_1 - x_{-k+1}} \le x_2 = \frac{\alpha - \beta x_1}{\gamma - \delta x_1 - x_{-k+1}} \le \frac{\alpha - \beta\cdot 0}{\gamma - \delta\cdot 0 - \alpha/\beta} < \frac{\alpha}{\beta}. \tag{2.26}$$

Thus,

$$0 \le x_2 \le \frac{\alpha}{\beta}. \tag{2.27}$$

The result (2.23) now follows by induction. The proof is completed. □

3. Global attractivity

In this section, we will study the global attractivity of positive solutions of (2.6). We show that the positive equilibrium $\overline{x}$ of (2.6) is a global attractor with a basin that depends on certain conditions imposed on the coefficients.

Theorem 3.1. *Assume that conditions* (2.7) *and* (2.18) *hold. Then, the equilibrium point* $\overline{x}_2$ *of* (2.6) *is globally asymptotically stable.*

Proof. In Section 2, we have shown under the assumptions (2.7) and (2.18) that the equilibrium $\overline{x}_2$ is locally asymptotically stable. It remains to prove that the equilibrium $\overline{x}_2$ is a global attractor. To this end, set $I = \lim_{n\to\infty} \inf x_n$ and $S = \lim_{n\to\infty} \sup x_n$ which by Theorem 2.7 exist and are positive numbers. Then, from (2.6) we deduce that

$$S \le \frac{\alpha - \beta S}{\gamma - (\delta + 1)I}, \qquad I \ge \frac{\alpha - \beta I}{\gamma - (\delta + 1)S}. \tag{3.1}$$

Consequently, we have

$$-\alpha + (\gamma + \beta)S \le (\delta + 1)IS \le -\alpha + (\gamma + \beta)I, \tag{3.2}$$

from which it follows that $I = S$. Thus, the the proof of Theorem 3.1 is completed. □

Lemma 3.2 (see [8]). *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-k}), \quad k \geq 1, \; n = 0, 1, 2, \ldots. \tag{3.3}$$

Let $[a, b]$ be some interval of real numbers, and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is a nonincreasing function in u, and a nondecreasing function in v;*

(b) *if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(M, m), \qquad M = f(m, M), \tag{3.4}$$

then, $m = M$.

Then, (3.3) *has a unique equilibrium point $\overline{x}$ and every solution of* (3.3) *converges to $\overline{x}$.*

Theorem 3.3. *Assume that conditions* (2.7) *and* (2.18) *hold. Then, the positive equilibrium $\overline{x}$ of* (2.6) *is a global attractor with a basin $S^* = [0, \alpha/\beta]^{k+1}$.*

Proof. For $u, v \in [0, \alpha/\beta]$, set

$$f(u, v) = \frac{\alpha - \beta u}{\gamma - \delta u - v}. \tag{3.5}$$

We claim that $f : [0, \alpha/\beta] \times [0, \alpha/\beta] \to [0, \alpha/\beta]$. In fact, if we set $a = 0, \; b = \alpha/\beta$, then

$$f(b, a) = \frac{\alpha - \beta b}{\gamma - \delta b - a} = \frac{\alpha - \alpha}{\gamma - \delta(\alpha/\beta)} = 0 = a, \tag{3.6}$$

and in view of the condition (2.18), we have

$$f(a, b) = \frac{\alpha - \beta a}{\gamma - \delta a - b} = \frac{\alpha}{\gamma - \alpha/\beta} = \frac{\beta\alpha}{\gamma\beta - \alpha} < \frac{\alpha}{\beta} = b. \tag{3.7}$$

Since $f(u, v)$ is decreasing in u and increasing in v, it follows that $a \leq f(u, v) \leq b$, for all $u, v \in [a, b]$, which implies that our assertion is true. On the other hand, conditions (a) and (b) of Lemma 3.2 are clearly true. Let $\{x_n\}$ be a solution of (2.6) with the initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0) \in S$. By Lemma 3.2, we have $\lim_{n\to\infty} x_n = \overline{x}$. The proof is completed. □

Theorem 3.4. *Assume that the conditions* (2.7) *and* (2.18) *hold. Then, the positive equilibrium $\overline{x}$ of* (2.6) *is a global attractor with a basin $S^* = (-\infty, \alpha/\beta]^k \times [0, \alpha/\beta]$.*

Proof. Let $\{x_n\}$ be a solution of (2.6) with the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in S^*$. Then, by Theorem 2.7, we have

$$x_n \in \left[0, \frac{\alpha}{\beta}\right], \quad n = 1, 2, \ldots. \tag{3.8}$$

By Theorem 3.3, we have $\lim_{n\to\infty} x_{n+k} = \overline{x}$ and so $\lim_{n\to\infty} x_n = \overline{x}$. The proof is completed. □

Theorem 3.5. *Assume that conditions* (2.7) *hold with* $0 \le \delta < 1$*. Also, assume that k is an odd positive integer. Then, the necessary and sufficient condition for* (2.6) *to have positive solutions of prime period two is that*

$$\beta(\gamma-\beta) < \alpha < \frac{(\gamma-\beta)}{4}\left[\gamma+3\beta-\delta(\gamma-\beta)\right]. \tag{3.9}$$

Proof. First, suppose that there exist distinctive positive solutions of prime period two,

$$\ldots, P, Q, P, Q, \ldots, \tag{3.10}$$

of the difference equation (2.6).

If k is odd, then $x_{n+1} = x_{n-k}$. It follows from the difference equation (2.6) that

$$P = \frac{\alpha-\beta Q}{\gamma-\delta Q-P}, \qquad Q = \frac{\alpha-\beta P}{\gamma-\delta P-Q}. \tag{3.11}$$

Consequently, we obtain

$$P+Q = \gamma-\beta, \qquad PQ = \frac{\alpha-\beta(\gamma-\beta)}{1-\delta}. \tag{3.12}$$

Thus, we deduce that

$$\alpha > \beta(\gamma-\beta), \qquad 0 \le \delta < 1. \tag{3.13}$$

Now it is clear that P, Q are two positive distinct real roots of the quadratic equation

$$t^2 - (P+Q)t + PQ = 0. \tag{3.14}$$

Therefore, we have

$$(\gamma-\beta)^2 > \frac{4\left[\alpha-\beta(\gamma-\beta)\right]}{1-\delta}. \tag{3.15}$$

From (3.13) and (3.15) we obtain condition (3.9). Conversely, suppose that the condition (3.9) is valid. Then, we deduce that (3.13) and (3.15) hold. Consequently, there exists two positive distinct real numbers P and Q such that

$$P = \frac{\gamma-\beta}{2} - \frac{\sqrt{K}}{2}, \tag{3.16}$$

$$Q = \frac{\gamma-\beta}{2} + \frac{\sqrt{K}}{2}, \tag{3.17}$$

where $K > 0$ is given by

$$K = (\gamma-\beta)^2 - 4\left[\frac{\alpha-\beta(\gamma-\beta)}{1-\delta}\right], \quad 0 \le \delta < 1. \tag{3.18}$$

Thus, P and Q given by (3.16) and (3.17) represent two positive distinct real roots of the quadratic equation (3.14). Now, we are going to prove that P and Q given by (3.16) and (3.17) are positive solutions of prime period two of the difference equation (2.6). To this end, we assume that $x_{-k} = P,\ x_{-k+1} = Q, \dots,\ x_{-1} = P,\ x_0 = Q$. We wish to prove that $x_1 = P$ and $x_2 = Q$.

It follows from the difference equation (2.6) and the formulas (3.16) and (3.17) that

$$\begin{aligned} x_1 &= \frac{\alpha - \beta x_0}{\gamma - \delta x_0 - x_{-k}} \\ &= \frac{\alpha - \beta Q}{\gamma - \delta Q - P} \\ &= \frac{2\alpha - \beta\left[\gamma - \beta + \sqrt{K}\right]}{2\gamma - \delta\left[\gamma - \beta + \sqrt{K}\right] - \left[\gamma - \beta - \sqrt{K}\right]} \\ &= \frac{\beta\left[2\alpha/\beta - (\gamma - \beta) - \sqrt{K}\right]}{2\gamma - (1+\delta)(\gamma - \beta) + (1-\delta)\sqrt{K}} \\ &= \left(\frac{\beta}{1-\delta}\right) \frac{\left\{2\alpha/\beta - (\gamma - \beta) - \sqrt{K}\right\}\left\{2\gamma/(1-\delta) - ((1+\delta)/(1-\delta))(\gamma - \beta) - \sqrt{K}\right\}}{\left\{2\gamma/(1-\delta) - ((1+\delta)/(1-\delta))(\gamma - \beta) + \sqrt{K}\right\}\left\{2\gamma/(1-\delta) - ((1+\delta)/(1-\delta))(\gamma - \beta) - \sqrt{K}\right\}}. \end{aligned} \tag{3.19}$$

After some reduction, we deduce that

$$x_1 = \left(\frac{\beta}{1-\delta}\right) \frac{\left[\alpha(1-\delta) + \beta^2\right]\left[\gamma - \beta - \sqrt{K}\right]/\beta(1-\delta)}{2\left[\alpha(1-\delta) + \beta^2\right]/(1-\delta)^2} = \frac{\gamma - \beta - \sqrt{K}}{2} = P. \tag{3.20}$$

Similarly, we can show that,

$$x_2 = \frac{\alpha - \beta x_1}{\gamma - \delta x_1 - x_{-k+1}} = \frac{\alpha - \beta P}{\gamma - \delta P - Q} = Q. \tag{3.21}$$

By using the induction, we have

$$x_n = P, \quad x_{n+1} = Q, \quad \forall n \geq -k. \tag{3.22}$$

Thus, the difference equation (2.6) has positive solutions of prime period two. Hence, the proof of Theorem 3.5 is completed. □

Theorem 3.6. *Assume that the conditions* (2.7) *hold. If k is even, then* (2.6) *has no positive solutions of prime period two.*

Proof. Suppose that there exists distinctive positive solutions of prime period two,

$$\dots, P, Q, P, Q, \dots, \tag{3.23}$$

of the difference equation (2.6).

If k is even, then $x_n = x_{n-k}$. It follows from the difference equation (2.6) that

$$P = \frac{\alpha - \beta Q}{\gamma - (\delta + 1)Q}, \qquad Q = \frac{\alpha - \beta P}{\gamma - (\delta + 1)P}. \tag{3.24}$$

From which we have $(\gamma - \beta)(P - Q) = 0$ and by using (2.7), we deduce that $P = Q$. This is a contradiction. Thus, the proof of Theorem 3.6 is completed. □

4. The case $\alpha = 0$

Secondly, we study the rational recursive sequence

$$x_{n+1} = \frac{-\beta x_n}{\gamma - \delta x_n - x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

where $\delta \geq 0$, $\gamma, \beta > 0$ are real numbers and $k \in \{1, 2, \ldots\}$. By putting $x_n = \beta y_n$, (4.1) yields

$$y_{n+1} = \frac{-y_n}{A - \delta y_n - y_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{4.2}$$

where $A = \gamma/\beta$. Equation (4.2) has two equilibrium points

$$\overline{y}_1 = 0, \qquad \overline{y}_2 = \frac{1+A}{1+\delta}. \tag{4.3}$$

The linearized equation associated with (4.2) about the equilibria $\overline{y}_i$, $(i = 1, 2)$ is

$$z_{n+1} + \frac{1 - \delta \overline{y}_i}{A - (\delta + 1)\overline{y}_i} z_n - \frac{\overline{y}_i}{A - (\delta + 1)\overline{y}_i} z_{n-k} = 0. \tag{4.4}$$

The characteristic equation of (4.4) about the equilibrium $\overline{y}_2 = (1 + A)/(1 + \delta)$ is

$$\lambda^{k+1} + \left[\frac{\delta A - 1}{\delta + 1}\right]\lambda^k + \frac{A+1}{\delta+1} = 0. \tag{4.5}$$

Now, we deduce from (4.5) the following results:

(a) if $\delta = 0$, and since $A + 1 > 1$, then the equilibrium $\overline{y}_2$ is unstable (see [7]);

(b) if $A > \delta > 0$, and since $(A + 1)/(\delta + 1) > 1$, then the equilibrium $\overline{y}_2$ is unstable;

(c) if $A = \delta$, then

$$\left|\frac{\delta A - 1}{\delta + 1}\right| + \left|\frac{\delta + 1}{\delta + 1}\right| = |\delta - 1| + 1. \tag{4.6}$$

Now, we have the following results from case (c): (i) if $A = \delta > 1$, then the equilibrium $\overline{y}_2$ is unstable; (ii) if $0 < A = \delta < 1$, then the equilibrium $\overline{y}_2$ is unstable; (iii) if $A = \delta = 1$, then the linearized stability analysis fails;

(d) if $1 < A < \delta$,

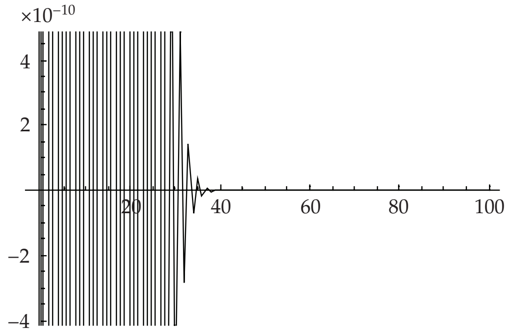
$$\left|\frac{\delta A - 1}{\delta + 1}\right| + \left|\frac{A + 1}{\delta + 1}\right| = \frac{\delta A - 1}{\delta + 1} + \frac{A+1}{\delta+1} = \frac{A(\delta + 1)}{\delta + 1} = A > 1, \tag{4.7}$$

and hence the equilibrium $\overline{y}_2$ is unstable;

(e) if $A < \delta \leq 1$,

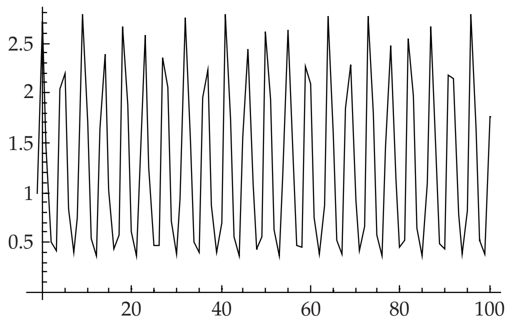
$$\left|\frac{\delta A - 1}{\delta + 1}\right| + \left|\frac{A + 1}{\delta + 1}\right| = \frac{1 - \delta A + 1 + A}{\delta + 1} \geq 1 + \frac{A}{2}(1 - \delta) \geq 1, \tag{4.8}$$

and hence the equilibrium $\overline{y}_2$ is unstable.



$\overline{y}_1 = 0$, $A = 2$, $\delta = 1/2$, $y_{-1} = e^{-1}$, $y_0 = 1$

Figure 3



$\overline{y}_2 = 1$, $A = \delta = 0.5$, $y_{-1} = e^{-1}$, $y_0 = 1$

Figure 4

The characteristic equation of (4.4) about the equilibrium $\overline{y}_1 = 0$ is

$$\lambda^{k+1} + \frac{1}{A}\lambda^k = 0. \tag{4.9}$$

This equation has two roots

$$\lambda_1 = 0, \qquad \lambda_2 = -\frac{1}{A}. \tag{4.10}$$

Now, we deduce from (4.10) the following results:

(i) if $A > 1$, then the equilibrium $\overline{y}_1 = 0$ is locally asymptotically stable (see Figure 3);

(ii) if $0 < A < 1$, then the equilibrium $\overline{y}_1 = 0$ is unstable (see Figure 4);

(iii) if $A = 1$, then the linearized stability analysis fails.

In the following results, we assume that $A \geq \delta + 2$, where $\delta \geq 0$.

Lemma 4.1. *Assume that the initial conditions* $y_{-i} \in [-1,1]$, *for* $i = 1,2,\ldots,k$ *and* $y_0 \in [-1,0]$. *Then,* $\{y_{2n-1}\}$ *is nonnegative and monotonically decreasing to zero, while* $\{y_{2n}\}$ *is nonpositive and monotonically increasing to zero.*

Proof. Suppose that $y_{-i} \in [-1,1]$, for $i = 1,2,\ldots,k$ and $y_0 \in [-1,0]$. Clearly, $0 \le y_1 \le 1$ and $-1 \le y_2 \le 0$. By induction, we can see that $0 \le y_{2n-1} \le 1$ and $-1 \le y_{2n} \le 0$ for $n \ge 1$.

If $A \ge \delta + 2,\ \delta \ge 0$ we have

$$\frac{y_{2n-1}}{y_{2n+1}} = \left(A - \delta y_{2n} - y_{2n-k}\right)\left(A - \delta y_{2n-1} - y_{2n-k-1}\right) > 1, \tag{4.11}$$

and hence

$$y_{2n-1} > y_{2n+1}, \quad n = 1,2,\ldots. \tag{4.12}$$

Similarly, we can show that $y_{2n} < y_{2n+2},\ n = 1,2,\ldots$. The proof of Lemma 4.1 is completed. □

On using arguments similar to that used in Lemma 4.1, we can easily prove the following lemma.

Lemma 4.2. *Assume that the initial conditions* $y_{-i} \in [-1,1]$, *for* $i = 1,2,\ldots,k$ *and* $y_0 \in [0,1]$. *Then,* $\{y_{2n-1}\}$ *is nonpositive and monotonically increasing to zero, while* $\{y_{2n}\}$ *is nonnegative and monotonically decreasing to zero.*

Corollary 4.3. *The equilibrium point* $\overline{y}_1 = 0$ *of* (4.1) *is a global attractor with a basin* $S^* = [-1,1]^{k+1}$.

Theorem 4.4. *The equilibrium point* $\overline{y}_1 = 0$ *of* (4.1) *is a global attractor with a basin* $S^* = (-\infty,1]^k \times [(-A+1)/(\delta+1),(A-1)/(\delta+1)]$, *where* $\delta \ge 0$.

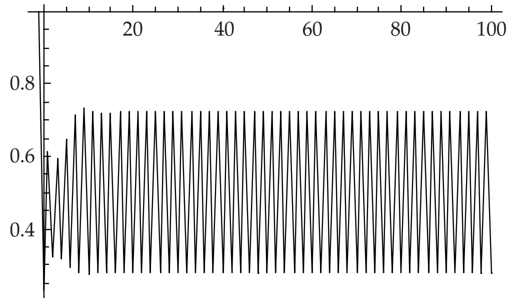
Proof. Assuming that the initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in S^*$. If $A \ge \delta + 2$, with $\delta \ge 0$, then we deduce that

$$\begin{aligned} -1 &\le \frac{(1-A)/(1+\delta)}{A - \delta y_0 - y_{-k}} \le y_1 = \frac{-y_0}{A - \delta y_0 - y_{-k}} \le \frac{(A-1)/(\delta+1)}{(A-1)/(\delta+1)} = 1, \\ -1 &\le \frac{-1}{A - \delta y_1 - y_{-k+1}} \le y_2 = \frac{-y_1}{A - \delta y_1 - y_{-k+1}} \le \frac{1}{A - \delta - 1} \le 1. \end{aligned} \tag{4.13}$$

By induction, it follows that $y_i \in [-1,1]$ for $i \ge 1$. Thus, the proof of Theorem 4.4 follows from Corollary 4.3. □

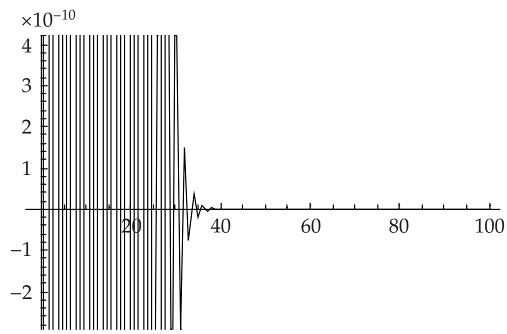
Theorem 4.5. *If* $A > 1$, *then the equilibrium point* $\overline{y}_1 = 0$ *of* (4.2) *is globally asymptotically stable.*

Finally, on using arguments similar to that used in Theorems 3.5 and 3.6, we can prove easily the following results.



$A = 2, k = 1, \delta = 6, y_{-1} = e^{-1}, y_0 = 1$

Figure 5



$A = 2, k = 2, \delta = 6, y_{-1} = e^{-1}, y_0 = y_{-2} = 1$

Figure 6

Theorem 4.6. *Assume that δ and $A > 1$. If k is an odd positive integer, then the necessary and sufficient condition for* (4.2) *to have positive solutions of prime period two is that (see Figure 5)*

$$(A - 1)\delta > A + 3. \tag{4.14}$$

Theorem 4.7. *If k is an even positive integer, then* (4.2) *has no positive solutions of prime period two (see Figure 6).*

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