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Research Article

Relations among Sums of Reciprocal Powers—Part II

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Some formulas relating the classical sums of reciprocal powers are derived in a compact way by using generating functions. These relations can be conveniently written by means of certain numbers which satisfy simple summation formulas. The properties of the generating functions can be further used to easily calculate several series involving the classical sums of reciprocal powers.

1. Introduction

In [1], we studied some arithmetic relations among the classical numbers:

$$\lambda(n) = \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^n} \quad (n \geq 2), \qquad L(n) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)^n} \quad (n \geq 1). \tag{1.1}$$

In this paper, we extend this analysis to the remaining

$$\zeta(n) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^n} \quad (n \geq 2), \qquad \eta(n) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu^n} \quad (n \geq 1). \tag{1.2}$$

Although the numbers $\lambda(n)$, $\zeta(n)$, and $\eta(n)$ are related to each other through the identities

$$\lambda(n) = (1-2^{-n})\zeta(n), \quad \eta(n) = (1-2^{1-n})\zeta(n), \quad (n \geq 2) \tag{1.3}$$

($\eta(1) = \ln 2$) and thus

$$\eta(n) + \zeta(n) = 2\lambda(n) \quad (n \geq 2), \tag{1.4}$$

we will use all of them in order to keep the algebraic expressions as simple as possible.

For $k \geq 2$, define

$$D(k) = 2\pi^{k-1}\sum_{j=1}^{\infty}\frac{\lambda(2j)}{2j\cdots(2j+k-1)}. \tag{1.5}$$

It will be shown below that these numbers can be alternatively expressed as

$$D(k) = \frac{1}{2(k-1)!}\int_0^{\pi} t^{k-1}\cot\frac{t}{2}\,dt = \frac{2^{k-1}}{(k-1)!}\int_0^{\pi/2} t^{k-1}\cot t\,dt. \tag{1.6}$$

Indefinite integrals of this type were considered by Ramanujan [2, page 260]. The constants $D(k)$ have the property of relating the values of ζ-numbers (eventually, η- or λ-numbers) with odd argument to the elementary values $\zeta(2n) = (2\pi)^{2n}|B_{2n}|/2(2n)!$ (where B_{2n} are the Bernoullian numbers) as, for example, in

$$\sum_{j=0}^{n}(-1)^{n-j}\frac{\pi^{2n-2j}}{(2n-2j)!}\eta(2j+1) = \frac{2}{\pi}\sum_{j=1}^{n}(-1)^{j}D(2j)\zeta(2n-2j+2) + \frac{(-1)^{n}}{\pi}D(2n+2) \tag{1.7}$$

(if $n = 0$, the first term on the right hand side of (1.7) has to be dropped). This and other formulas expressing the numbers $D(k)$ by means of sums of reciprocal powers with odd arguments and, conversely, $\eta(2n+1)$, $\zeta(2n+1)$, and $\lambda(2n+1)$ via $D(k)$ will be proved in Section 2 (see Propositions 2.2 and 2.4, and Corollary 2.6 below) using some generating functions defined by the numbers $\lambda(n)$, $\eta(n)$, $\zeta(n)$, and $D(n)$. The properties of these generating functions, which are given as expansions both in powers and in partial fractions, will be instrumental for most of the subsequent results.

Sections 3, 4 deal with the numbers $D(k)$ and with the classical sums of reciprocal powers, respectively. In particular, Section 3 is mainly devoted to the calculation of several series containing $D(k)$. In Section 4, the focus changes onto some particular series whose terms contain λ-, η-, ζ-, and L-numbers like, for example,

$$\sum_{n=1}^{\infty}(\zeta(2n)-\zeta(2n+1)) = \frac{1}{2}, \qquad \sum_{n=1}^{\infty}(\zeta(2n+1)-\lambda(2n)) = 0. \tag{1.8}$$

Finally, in the brief Section 5, the generating function of the numbers $D(k)$ is expressed using the Psi (Digamma) function $\psi(x) = \Gamma'(x)/\Gamma(x)$.

2. Main statements

Define the generating functions $\Lambda(x)$, $\mathcal{E}(x)$, and $\mathcal{Z}(x)$ by

$$\Lambda(x) = \sum_{n=2}^{\infty}\lambda(n)x^{n-1}, \qquad \mathcal{E}(x) = \sum_{n=1}^{\infty}\eta(n)x^{n-1}, \qquad \mathcal{Z}(x) = \sum_{n=2}^{\infty}\zeta(n)x^{n-1}, \tag{2.1}$$

where, in principle, $x \in \mathbb{C}$ (since $\lim_{n\to\infty}\lambda(n) = \lim_{n\to\infty}\eta(n) = \lim_{n\to\infty}\zeta(n) = 1$, these formal power series converge only for $|x| < 1$). Furthermore, denote by $\Lambda_+(x)$ and $\Lambda_-(x)$ the even and odd parts, respectively, of $\Lambda(x)$ and similarly for $\mathcal{E}_{\pm}(x)$ and $\mathcal{Z}_{\pm}(x)$. Then, the following identities hold [3, 4.3.67/68/70]:

$$\tan\frac{\pi x}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty}\lambda(2n)x^{2n-1} = \frac{4}{\pi}\Lambda_{-}(x) \quad (|x| < 1), \tag{2.2}$$

$$\csc \pi x = \frac{1}{\pi x} + \frac{2}{\pi}\sum_{n=1}^{\infty}\eta(2n)x^{2n-1} = \frac{1}{\pi x} + \frac{2}{\pi}\mathcal{E}_{-}(x) \quad (0 < |x| < 1), \tag{2.3}$$

$$\cot \pi x = \frac{1}{\pi x} - \frac{2}{\pi}\sum_{n=1}^{\infty}\zeta(2n)x^{2n-1} = \frac{1}{\pi x} - \frac{2}{\pi}\mathcal{Z}_{-}(x) \quad (0 < |x| < 1). \tag{2.4}$$

Owing to (1.3) and (1.4), the above generating functions fulfill the trivial relations

$$\begin{gathered}\mathcal{E}(x) = \mathcal{Z}(x) - \mathcal{Z}\left(\frac{x}{2}\right) + \ln 2, \qquad \Lambda(x) = \mathcal{Z}(x) - \frac{1}{2}\mathcal{Z}\left(\frac{x}{2}\right), \\ \mathcal{E}(x) + \mathcal{Z}(x) = 2\Lambda(x) + \ln 2,\end{gathered} \tag{2.5}$$

respectively.

On substituting the definitions of $\lambda(n)$, $\eta(n)$, and $\zeta(n)$ into the corresponding generating functions, we find the following expansions in partial fractions:

$$\begin{gathered}\Lambda(x) = \sum_{\nu=0}^{\infty}\left(\frac{1}{2\nu+1-x} - \frac{1}{2\nu+1}\right), \qquad \mathcal{E}(x) = \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu - x}, \\ \mathcal{Z}(x) = \sum_{\nu=1}^{\infty}\left(\frac{1}{\nu - x} - \frac{1}{\nu}\right).\end{gathered} \tag{2.6}$$

In particular, the expansions

$$\begin{aligned}\mathcal{E}_{+}(x) &= \frac{1}{2}(\mathcal{E}(x) + \mathcal{E}(-x)) = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{\nu}{\nu^2 - x^2}, \\ \mathcal{Z}_{+}(x) &= \frac{1}{2}(\mathcal{Z}(x) + \mathcal{Z}(-x)) = \sum_{\nu=1}^{\infty}\left(\frac{\nu}{\nu^2 - x^2} - \frac{1}{\nu}\right)\end{aligned} \tag{2.7}$$

will be used below.

Proposition 2.1. *The integral representation* (1.6) *holds.*

Proof. Let $f(t) = \tan(\pi t/2)$. By repeated partial integration, one gets

$$\begin{aligned}\int_0^1 t^{k-1}f(1-t)dt &= \frac{f(0)}{k} + \frac{f'(0)}{k(k+1)} + \frac{f''(0)}{k(k+1)(k+2)} + \cdots \\ &= \sum_{\nu=0}^{\infty}\frac{f^{(\nu)}(0)}{k(k+1)\cdots(k+\nu)},\end{aligned} \tag{2.8}$$

where according to the Taylor expansion (2.2),

$$f^{(\nu)}(0) = \begin{cases} 0, & \text{if } \nu = 2j,\ j \in \mathbb{N}, \\ 4(2j-1)!\lambda(2j)/\pi, & \text{if } \nu = 2j-1,\ j \in \mathbb{N}. \end{cases} \tag{2.9}$$

Hence,

$$\begin{aligned} \frac{2^{k-1}}{(k-1)!}\int_0^{\pi/2} t^{k-1}\tan\left(\frac{\pi}{2}-t\right)dt &= \frac{\pi^k}{2(k-1)!}\int_0^1 t^{k-1}\tan\frac{\pi}{2}(1-t)dt \\ &= \frac{\pi^k}{2(k-1)!}\sum_{j=1}^{\infty}\frac{4(2j-1)!\lambda(2j)/\pi}{k(k+1)\cdots(k+2j-1)} \\ &= 2\pi^{k-1}\sum_{j=1}^{\infty}\frac{\lambda(2j)}{2j\cdots(2j+k-1)} \\ &= D(k). \end{aligned} \tag{2.10}$$

□

Since, for $0 \le x \le \pi/2$ and $n \in \mathbb{N}$,

$$x^n \cot x \lesssim \frac{1}{2n}\left(\frac{\pi}{2}\right)^{n+1}, \tag{2.11}$$

it follows that $\lim_{n\to\infty} D(n+1) = 0$ faster (in fact, much faster) than $\pi^{n+2}/8nn!$.

Proposition 2.2. *The numbers $D(k)$ $(k \ge 2)$ can be evaluated in terms of $\eta(1), \eta(3), \ldots, \eta(2\lfloor k/2\rfloor - 1)$, supplemented with $\lambda(k)$ if k is odd, by the formula*

$$D(k) = \sum_{j=1}^{\lfloor k/2\rfloor}(-1)^{j-1}\frac{\pi^{k-2j+1}}{(k-2j+1)!}\eta(2j-1) + \begin{cases} 0, & (k \text{ even}), \\ (-1)^{(k-1)/2}2\lambda(k), & (k \text{ odd}). \end{cases} \tag{2.12}$$

Proof. For each $\nu \in \mathbb{N}$, we have upon $k-1$ integrations by parts

$$\begin{aligned} &\frac{2^k}{(k-1)!}\int_0^{\pi/2} t^{k-1}\sin 2\nu t\,dt \\ &\quad = \sum_{j=1}^{\lfloor k/2\rfloor}(-1)^j\frac{\pi^{k-2j+1}}{(k-2j+1)!}\frac{(-1)^\nu}{\nu^{2j-1}} + \begin{cases} 0, & (k \text{ even}), \\ (-1)^{(k-1)/2}\dfrac{1-(-1)^\nu}{\nu^k}, & (k \text{ odd}). \end{cases} \end{aligned} \tag{2.13}$$

Sum now both sides on ν, $1 \le \nu \le N$, and use the identity [4, 1.342(1)]

$$\sum_{\nu=1}^{N}\sin 2\nu t = \sin(N+1)t\ \sin\ Nt\csc t = \frac{1}{2}(\cos t - \cos(2N+1)t)\csc t, \tag{2.14}$$

to get on the left side

$$\begin{aligned} &\frac{2^k}{(k-1)!}\sum_{\nu=1}^{N}\int_0^{\pi/2} t^{k-1}\sin 2\nu t\,dt \\ &\quad = \frac{2^{k-1}}{(k-1)!}\int_0^{\pi/2}\left(t^{k-1}\cot t - \frac{t^{k-1}}{\sin t}\cos(2N+1)t\right)dt. \end{aligned} \tag{2.15}$$

To finish the proof, let $N \to \infty$ and use the Riemann-Lebesgue lemma. □

In particular,

$$D(2) = \frac{\pi}{1!}\eta(1) = \pi \ln 2 \tag{2.16}$$

as in [4, 3.747(7)], and

$$D(3) = \frac{\pi^2}{2}\ln 2 - 2\lambda(3) = \frac{\pi^2}{2}\ln 2 - \frac{7}{4}\zeta(3) \tag{2.17}$$

as in [5, 4.2(3)]. Other expressions for $D(k)$ different from (2.12) can be found in [4, 3.748(2)].

Due to the definition (1.5) and formula (2.12), each number $D(k)$ embodies a relation among all elementary values $\lambda(2j)$ and a finite number of their nonelementary counterparts $\lambda(2j+1)$, namely,

$$\begin{aligned} \sum_{j=1}^{\infty}\frac{\lambda(2j)}{2j\cdots(2j+2k-1)} &= \frac{1}{2}\sum_{j=1}^{k}(-1)^{j-1}\frac{(1/\pi)^{2j-2}}{(2k-2j+1)!}\eta(2j-1), \\ \sum_{j=1}^{\infty}\frac{\lambda(2j)}{2j\cdots(2j+2k)} &= \frac{1}{2}\sum_{j=1}^{k}(-1)^{j-1}\frac{(1/\pi)^{2j-2}}{(2k-2j+2)!}\eta(2j-1) + (-1)^k\frac{\lambda(2k+1)}{\pi^{2k}}, \end{aligned} \tag{2.18}$$

for $k \geq 1$.

Define next the generating function

$$\mathfrak{D}(x) = \sum_{n=2}^{\infty} D(n)(ix)^{n-1}. \tag{2.19}$$

Owing to the vanishing rate of the coefficients $D(n)$, this power series is convergent for all $x \in \mathbb{C}$. Then,

$$\mathfrak{D}(x) = \int_0^{\pi/2} dt \cot t \sum_{n=2}^{\infty}\frac{(i2xt)^{n-1}}{(n-1)!} = \int_0^{\pi/2}\left(e^{i2xt}-1\right)\cot t\, dt. \tag{2.20}$$

Furthermore, let $\mathfrak{D}_+(x)$ and $\mathfrak{D}_-(x)$ be the even and odd parts of $\mathfrak{D}(x)$, that is,

$$\begin{aligned} \mathfrak{D}_+(x) &= \sum_{n=1}^{\infty} D(2n+1)(ix)^{2n} = \sum_{n=1}^{\infty}(-1)^n D(2n+1)x^{2n} \\ &= \int_0^{\pi/2}(\cos 2xt - 1)\cot t\, dt, \\ \mathfrak{D}_-(x) &= \sum_{n=1}^{\infty} D(2n)(ix)^{2n-1} = i\sum_{n=1}^{\infty}(-1)^{n-1}D(2n)x^{2n-1} \\ &= i\int_0^{\pi/2}\sin 2xt \cot t\, dt. \end{aligned} \tag{2.21}$$

Thus, if x is meant to be real, $\mathfrak{D}_+(x)$ and $(1/i)\mathfrak{D}_-(x)$ are the real and imaginary part, respectively, of $\mathfrak{D}(x)$.

Proposition 2.3. *The following relations hold*:

$$\mathfrak{D}_+(x) = \cos(\pi x)\mathcal{E}_+(x) + \mathcal{Z}_+(x) - \ln 2,$$
$$\mathfrak{D}_-(x) = i\sin(\pi x)\mathcal{E}_+(x). \tag{2.22}$$

Proof. We claim that

$$\mathfrak{D}(x) = e^{i\pi x}\mathcal{E}_+(x) + \mathcal{Z}_+(x) - \ln 2. \tag{2.23}$$

In fact, from

$$\mathfrak{D}(x) = \int_0^{\pi/2}\left(e^{i2xt}-1\right)\cot t\,dt = 2\sum_{\nu=1}^{\infty}\int_0^{\pi/2}\left(e^{i2xt}-1\right)\sin 2\nu t\,dt,$$
$$\int_0^{\pi/2} e^{i2xt}\sin 2\nu t\,dt = \frac{(-1)^{\nu-1}\nu e^{i\pi x}+\nu}{2(\nu^2-x^2)}, \qquad \int_0^{\pi/2}\sin 2\nu t\,dt = \frac{1+(-1)^{\nu-1}}{2\nu}, \tag{2.24}$$

we obtain

$$\mathfrak{D}(x) = e^{i\pi x}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}\nu}{\nu^2-x^2} + \sum_{\nu=1}^{\infty}\left(\frac{\nu}{\nu^2-x^2}-\frac{1}{\nu}\right) - \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}. \tag{2.25}$$

Comparison with (2.7) proves the claim. Take now even and odd parts. □

Solving for $\mathcal{E}_+(x)$ and $\mathcal{Z}_+(x)$ in (2.22), we get the identities

$$\mathcal{E}_+(x) = \frac{1}{i}\csc(\pi x)\mathfrak{D}_-(x) \quad (0<|x|<1), \tag{2.26}$$

$$\mathcal{Z}_+(x) = \mathfrak{D}_+(x) + i\cot(\pi x)\mathfrak{D}_-(x) + \ln 2 \quad (0<|x|<1), \tag{2.27}$$

which lead to a kind of converse of Proposition 2.2.

Proposition 2.4. *The numbers $\eta(2n+1)$ and $\zeta(2n+1)$, $n \geq 1$, can be expressed in terms of the $D(k)$ and the elementary values $\eta(2k)$ and $\zeta(2k)$ by*

$$\eta(2n+1) = \frac{2}{\pi}\sum_{j=1}^{n}(-1)^{j-1}\eta(2n-2j+2)D(2j) + \frac{(-1)^n}{\pi}D(2n+2),$$
$$\zeta(2n+1) = \frac{2}{\pi}\sum_{j=1}^{n}(-1)^{j-1}\zeta(2n-2j+2)D(2j) + (-1)^n D(2n+1) + \frac{(-1)^{n+1}}{\pi}D(2n+2). \tag{2.28}$$

Proof. (1) Equation (2.26) reads explicitly (use $\eta(0)=1/2$ in (2.3)),

$$\begin{aligned}
\sum_{n=0}^{\infty}\eta(2n+1)x^{2n}
&= \frac{1}{i}\csc(\pi x)\mathfrak{D}_-(x)\\
&= \frac{2}{i\pi}\sum_{n=0}^{\infty}\eta(2n)x^{2n-1}\sum_{j=1}^{\infty}i(-1)^{j-1}D(2j)x^{2j-1}\\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\sum_{j=1}^{n+1}(-1)^{j-1}\eta(2n-2j+2)D(2j)x^{2n}\\
&= \frac{1}{\pi}D(2) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(-1)^{j-1}\eta(2n-2j+2)D(2j) + \frac{(-1)^n}{2}D(2n+2)\right)x^{2n}.
\end{aligned} \tag{2.29}$$

(2) The second identity follows from (2.27) since (use $\zeta(0) = -1/2$ in (2.4)),

$$\begin{aligned}
&\cot(\pi x)\mathfrak{D}_-(x) \\
&\quad = -\frac{2}{\pi}\sum_{n=0}^{\infty}\zeta(2n)x^{2n-1}\sum_{j=1}^{\infty}i(-1)^{j-1}D(2j)x^{2j-1} \\
&\quad = -\frac{2i}{\pi}\sum_{n=0}^{\infty}\left(\sum_{j=1}^{n+1}(-1)^{j-1}\zeta(2n-2j+2)D(2j)\right)x^{2n} \\
&\quad = \frac{i}{\pi}D(2) - \frac{2i}{\pi}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(-1)^{j-1}\zeta(2n-2j+2)D(2j) + \frac{(-1)^{n+1}}{2}D(2n+2)\right)x^{2n},
\end{aligned} \tag{2.30}$$

□

Remark 2.5. Equation (1.7) is nothing else but the formula

$$\cos(\pi x)\mathcal{E}_+(x) = \frac{1}{i}\cot(\pi x)\mathfrak{D}_-(x) \tag{2.31}$$

written explicitly. Indeed,

$$\begin{aligned}
\cos(\pi x)\mathcal{E}_+(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}x^{2n}\sum_{j=1}^{\infty}\eta(2j-1)x^{2j-2} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(-1)^{n-j}\frac{\pi^{2n-2j}}{(2n-2j)!}\eta(2j+1)\right)x^{2n} \\
&= \eta(1) + \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n}(-1)^{n-j}\frac{\pi^{2n-2j}}{(2n-2j)!}\eta(2j+1)\right)x^{2n},
\end{aligned} \tag{2.32}$$

and $\cot(\pi x)\mathfrak{D}_-(x)$ was calculated in the last proof. For $n = 1$ one gets the representation

$$\zeta(3) = \frac{2}{9}\pi^2\ln 2 - \frac{4}{3\pi}D(4) = \frac{2}{9}\pi^2\ln 2 - \frac{16}{9\pi}\int_0^{\pi/2} t^3\cot t\,dt. \tag{2.33}$$

Equation (1.7) and also the first formula of Proposition 2.4 show that $\eta(2n+1)$ (and, for that case, $\zeta(2n+1)$ and $\lambda(2n+1)$) can be expressed in terms of only $D(2), D(4), \ldots, D(2n+2)$. Furthermore, adding the two formulas of Proposition 2.4 and using (1.4), we obtain the following result.

Corollary 2.6. *For $n \geq 1$,*

$$\lambda(2n+1) = \frac{2}{\pi}\sum_{j=1}^{n}(-1)^{j-1}\lambda(2n-2j+2)D(2j) + \frac{(-1)^n}{2}D(2n+1). \tag{2.34}$$

For $n = 1$, we recover (2.17).

3. Summation formulas for the $D(k)$s

Before deriving more relations involving the sums of reciprocal powers, we obtain next some "summation" formulas for the numbers $D(k)$. Two (integral) summation formulas follow trivially from the very definition of the generating function $\mathfrak{D}(x)$ and (2.20), namely,

$$\sum_{n=2}^{\infty} D(n) = \mathfrak{D}\left(\frac{1}{i}\right) = \int_0^{\pi/2} (e^{2t} - 1)\cot t\, dt,$$
$$\sum_{n=2}^{\infty} (-1)^n D(n) = -\mathfrak{D}(i) = \int_0^{\pi/2} (1 - e^{-2t})\cot t\, dt. \tag{3.1}$$

Also,

$$\sum_{n=1}^{\infty} (-1)^n D(2n+1) = \mathfrak{D}_+(1) = \int_0^{\pi/2} (\cos 2t - 1)\cot t\, dt = -1, \tag{3.2}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{D(2n+1)}{2^{2n}} = \mathfrak{D}_+\left(\frac{1}{2}\right) = \frac{1}{2}\int_0^{\pi/2} (\cos t - 1)\cot t\, dt = \ln 2 - 1, \tag{3.3}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} D(2n) = \frac{1}{i}\mathfrak{D}_-(1) = \int_0^{\pi/2} \sin 2t \cot t\, dt = \frac{\pi}{2}, \tag{3.4}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{D(2n)}{2^{2n-1}} = \frac{1}{i}\mathfrak{D}_-\left(\frac{1}{2}\right) = \int_0^{\pi/2} \sin t \cot t\, dt = 1. \tag{3.5}$$

Other similar series can be also straightforwardly deduced after differentiating (2.21),

$$\sum_{n=1}^{\infty} (-1)^{n+1} n D(2n+1) x^{2n-1} = \frac{1}{2}\mathfrak{D}'_+(x) = \int_0^{\pi/2} t \sin 2xt \cot t\, dt,$$
$$\sum_{n=1}^{\infty} (-1)^{n+1} (2n-1) D(2n) x^{2n-2} = \frac{1}{i}\mathfrak{D}'_-(x) = 2\int_0^{\pi/2} t \cos 2xt \cot t\, dt, \tag{3.6}$$

and substituting fixed values for x. In particular, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} (2n-1) D(2n) = \frac{1}{i}\mathfrak{D}'_-(1) = 2\int_0^{\pi/2} t \cos 2t \cot t\, dt = D(2) - \frac{\pi}{2} = \pi\left(\ln 2 - \frac{1}{2}\right) \tag{3.7}$$

(where $\cos 2t = 1 - 2\sin^2 t$, (1.6) and (2.16) were used) will be needed below. Note that from (3.7) and (3.4), it follows

$$D(2) = 2\sum_{n=2}^{\infty} (-1)^n n D(2n). \tag{3.8}$$

Furthermore, from (2.20), we have

$$\mathfrak{D}\left(x + \frac{1}{2}\right) - \mathfrak{D}\left(x - \frac{1}{2}\right) = 2i\int_0^{\pi/2} e^{i2xt}\cos t\, dt = 2i\frac{e^{i\pi x} - i2x}{1 - 4x^2}, \tag{3.9}$$

so, after separating real and imaginary parts, the equations

$$\mathfrak{D}_+\left(x+\frac{1}{2}\right)-\mathfrak{D}_+\left(x-\frac{1}{2}\right)=\frac{2}{1-4x^2}(2x-\sin\pi x), \tag{3.10}$$

$$\mathfrak{D}_-\left(x+\frac{1}{2}\right)-\mathfrak{D}_-\left(x-\frac{1}{2}\right)=i\frac{2}{1-4x^2}\cos\pi x \tag{3.11}$$

hold for $x\in\mathbb{R}$. Letting $x\to 1/2$ one recovers (3.2) and (3.4).

Proposition 3.1. *The following identities hold:*

(1) *For* $n=0,1,2,\ldots,$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\binom{2n+2k}{2n+1}\frac{D(2n+1+2k)}{2^{2n+1+2k}}=\frac{(-1)^n}{4}\left(\sum_{k=0}^{n}(-1)^k\frac{\pi^{2k+1}}{(2k+1)!}-2\right). \tag{3.12}$$

In particular, for $n=0$,

$$\sum_{k=1}^{\infty}(-1)^{k+1}k\frac{D(2k+1)}{2^{2k+1}}=\frac{1}{8}(\pi-2). \tag{3.13}$$

(2) *For* $n=0,1,2,\ldots,$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\binom{2n+2k-1}{2n}\frac{D(2n+2k)}{2^{2n+2k}}=\frac{(-1)^n}{2}\sum_{k=0}^{n}(-1)^k\frac{\pi^{2k}}{(2k)!}. \tag{3.14}$$

Proof. (1) In fact,

$$\begin{aligned}\mathfrak{D}_+\left(x+\frac{1}{2}\right)-\mathfrak{D}_+\left(x-\frac{1}{2}\right)&=\sum_{k=1}^{\infty}(-1)^kD(2k+1)\left(\left(x+\frac{1}{2}\right)^{2k}-\left(x-\frac{1}{2}\right)^{2k}\right)\\&=2\sum_{k=1}^{\infty}(-1)^kD(2k+1)\sum_{n=0}^{k-1}\binom{2k}{2n+1}x^{2n+1}2^{-(2k-2n-1)}\\&=2^{2n+3}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n+k}\binom{2n+2k}{2n+1}\frac{D(2n+1+2k)}{2^{2n+1+2k}}\right)x^{2n+1}.\end{aligned} \tag{3.15}$$

Comparison with (3.10), that is,

$$\begin{aligned}\frac{1}{1-4x^2}(2x-\sin\pi x)&=\left((2-\pi)x+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{2n+1}}{(2n+1)!}x^{2n+1}\right)\sum_{j=0}^{\infty}(2x)^{2j}\\&=\sum_{n=0}^{\infty}(2-\pi)(2x)^{2n+1}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{k+1}\frac{\pi^{2k+1}}{(2k+1)!}\right)(2x)^{2n+1}\\&=2^{2n+1}\sum_{n=0}^{\infty}\left(2+\sum_{k=0}^{n}(-1)^{k+1}\frac{\pi^{2k+1}}{(2k+1)!}\right)x^{2n+1},\end{aligned} \tag{3.16}$$

where $|x|<1$ yields the result.

(2) Analogously to (1), the summation formula follows comparing

$$\mathfrak{D}_-\left(x+\frac{1}{2}\right)-\mathfrak{D}_-\left(x-\frac{1}{2}\right) = 2^{2n+2}i\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n+k+1}\binom{2n+2k-1}{2n}\frac{D(2n+2k)}{2^{2n+2k}}\right)x^{2n} \tag{3.17}$$

with (3.11), that is,

$$\frac{2i}{1-4x^2}\cos\pi x = 2i\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}x^{2n}\sum_{j=0}^{\infty}(2x)^{2j} = 2^{2n+1}i\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\frac{\pi^{2k}}{(2k)!}\right)x^{2n}, \tag{3.18}$$

where $|x|<1$. □

Of course, all these summation formulas can be also checked using the integral representation (1.6). Finally, a finite summation formula can be derived in the following way. For $n\geq 1$,

$$\begin{aligned}\int_0^{\pi/2}\left[\left(\frac{\pi}{2}-t\right)^n-\left(\frac{\pi}{2}\right)^n\right]\tan\left(\frac{\pi}{2}-t\right)dt &= \int_0^{\pi/2}\left[\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\pi}{2}\right)^{n-k}(-t)^k\right]\tan\left(\frac{\pi}{2}-t\right)dt\\ &= \sum_{k=1}^{n}(-1)^k\binom{n}{k}\left(\frac{\pi}{2}\right)^{n-k}\int_0^{\pi/2}t^k\cot t\,dt\\ &= \sum_{k=1}^{n}(-1)^k\binom{n}{k}\left(\frac{\pi}{2}\right)^{n-k}\frac{k!}{2^k}D(k+1).\end{aligned} \tag{3.19}$$

Therefore, changing the variable in the integral, we get

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{\pi^{n-k}}{(n-k)!}D(k+1) = \frac{2^n}{n!}\int_0^{\pi/2}\left(\left(\frac{\pi}{2}\right)^n-t^n\right)\tan t\,dt. \tag{3.20}$$

4. Further relations

From the close-form expressions obtained in the previous sections, we can derive a number of interesting results for series containing, in turn, the series $\zeta(n)$, $\lambda(n)$, or $\eta(n)$. For completeness, we will also include the series $L(n)$, what requires the consideration of a new generating function $\mathcal{C}(x)$ defined in [1]. As a first example, we will prove the following proposition.

Proposition 4.1. *The following identities hold true:*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}}=\frac{1}{2}, \qquad \sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2^{2n+1}}=\ln 2-\frac{1}{2},\\
&\sum_{n=1}^{\infty}\frac{\lambda(2n)}{2^{2n}}=\frac{\pi}{8}, \qquad \sum_{n=1}^{\infty}\frac{\lambda(2n+1)}{2^{2n+1}}=\frac{1}{4}\ln 2,\\
&\sum_{n=1}^{\infty}\frac{\eta(2n)}{2^{2n}}=\frac{1}{2}\left(\frac{\pi}{2}-1\right), \qquad \sum_{n=0}^{\infty}\frac{\eta(2n+1)}{2^{2n+1}}=\frac{1}{2},\\
&\sum_{n=1}^{\infty}\frac{L(2n)}{2^{2n}}=\frac{\sqrt{2}}{4}\ln\left(1+\sqrt{2}\right), \qquad \sum_{n=0}^{\infty}\frac{L(2n+1)}{2^{2n+1}}=\frac{\pi\sqrt{2}}{8}.
\end{aligned} \tag{4.1}$$

Note, in particular, the series

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{2^{n}}=\ln 2, \qquad \sum_{n=1}^{\infty}\frac{\eta(n)}{2^{n}}=\frac{\pi}{4}. \tag{4.2}$$

Proof. (1) Setting $x = 1/2$ in (2.27), we get

$$\mathfrak{Z}_{+}\left(\frac{1}{2}\right)=\mathfrak{D}_{+}\left(\frac{1}{2}\right)+\ln 2. \tag{4.3}$$

Thus, by (3.3),

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2^{2n+1}}\equiv\frac{1}{2}\mathfrak{Z}_{+}\left(\frac{1}{2}\right)=\frac{1}{2}\mathfrak{D}_{+}\left(\frac{1}{2}\right)+\frac{1}{2}\ln 2=\ln 2-\frac{1}{2}. \tag{4.4}$$

On the other hand (see (2.4)),

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}}\equiv\frac{1}{2}\mathfrak{Z}_{-}\left(\frac{1}{2}\right)=\frac{1}{2}. \tag{4.5}$$

Adding up the even and odd parts, we get

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{2^{n}}=\ln 2, \qquad \sum_{n=2}^{\infty}(-1)^{n}\frac{\zeta(n)}{2^{n}}=1-\ln 2. \tag{4.6}$$

(2) Setting $x = 1/2$ in (2.26), we get

$$\mathcal{E}_{+}\left(\frac{1}{2}\right)=\frac{1}{i}\mathfrak{D}_{-}\left(\frac{1}{2}\right). \tag{4.7}$$

Hence, by (3.5),

$$\sum_{n=0}^{\infty}\frac{\eta(2n+1)}{2^{2n+1}}\equiv\frac{1}{2}\mathcal{E}_{+}\left(\frac{1}{2}\right)=\frac{1}{2i}\mathfrak{D}_{-}\left(\frac{1}{2}\right)=\frac{1}{2}. \tag{4.8}$$

On the other hand, by (2.3),

$$\sum_{n=1}^{\infty}\frac{\eta(2n)}{2^{2n}} \equiv \frac{1}{2}\mathcal{E}_{-}\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2}-1\right). \tag{4.9}$$

Adding up the even and odd parts, we get

$$\sum_{n=1}^{\infty}\frac{\eta(n)}{2^{n}} = \frac{\pi}{4}, \qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\eta(n)}{2^{n}} = 1-\frac{\pi}{4}. \tag{4.10}$$

(3) In [1], it is proved that, for $|x|<1$,

$$\sum_{n=1}^{\infty}\lambda(2n+1)x^{2n} \equiv \Lambda_{+}(x) = \mathcal{C}_{+}(x) + \frac{1}{i}\tan\frac{\pi x}{2}\mathcal{C}_{-}(x) \tag{4.11}$$

holds, where

$$\mathcal{C}_{+}(x) = \frac{1}{2}\int_{0}^{\pi/2}\frac{\cos xt-1}{\sin t}dt, \qquad \mathcal{C}_{-}(x) = \frac{i}{2}\int_{0}^{\pi/2}\frac{\sin xt}{\sin t}dt. \tag{4.12}$$

Hence,

$$\sum_{n=1}^{\infty}\frac{\lambda(2n)}{2^{2n}} \equiv \frac{1}{2}\Lambda_{-}\left(\frac{1}{2}\right) = \frac{\pi}{8} \tag{4.13}$$

by (2.2), and

$$\sum_{n=1}^{\infty}\frac{\lambda(2n+1)}{2^{2n+1}} \equiv \frac{1}{2}\Lambda_{+}\left(\frac{1}{2}\right) = \frac{1}{2}\mathcal{C}_{+}\left(\frac{1}{2}\right) + \frac{1}{2i}\mathcal{C}_{-}\left(\frac{1}{2}\right) = \frac{1}{4}\ln 2, \tag{4.14}$$

since

$$\begin{aligned}
\mathcal{C}_{+}\left(\frac{1}{2}\right) &= \frac{1}{2}\int_{0}^{\pi/2}\frac{\cos(t/2)-1}{\sin t}dt \\
&= \frac{1}{2}\ln\frac{2}{1+\sqrt{2}}, \\
\mathcal{C}_{-}\left(\frac{1}{2}\right) &= \frac{i}{2}\int_{0}^{\pi/4}\sec t\,dt \\
&= \frac{i}{2}\ln\tan\frac{3\pi}{8} \\
&= \frac{i}{2}\ln\left(1+\sqrt{2}\right).
\end{aligned} \tag{4.15}$$

Adding up the even and odd parts,

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{\lambda(n)}{2^{n}} &= \frac{1}{4}\left(\frac{\pi}{2}+\ln 2\right), \\
\sum_{n=2}^{\infty}(-1)^{n}\frac{\lambda(n)}{2^{n}} &= \frac{1}{4}\left(\frac{\pi}{2}-\ln 2\right).
\end{aligned} \tag{4.16}$$

(4) In [1], it is proved that

$$\sum_{n=0}^{\infty} L(2n+1)x^{2n} \equiv \mathcal{L}_{+}(x) = \frac{\pi}{4} \sec \frac{\pi x}{2}, \tag{4.17}$$

$$\sum_{n=1}^{\infty} L(2n)x^{2n-1} \equiv \mathcal{L}_{-}(x) = \frac{1}{i} \sec \frac{\pi x}{2} \mathcal{C}_{-}(x), \tag{4.18}$$

for $|x| < 1$. Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L(2n)}{2^{2n}} &\equiv \frac{1}{2}\mathcal{L}_{-}\left(\frac{1}{2}\right) = \frac{1}{2i} \sec \frac{\pi}{4} \mathcal{C}_{-}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} \ln\left(1+\sqrt{2}\right), \\ \sum_{n=0}^{\infty} \frac{L(2n+1)}{2^{2n+1}} &\equiv \frac{1}{2}\mathcal{L}_{+}\left(\frac{1}{2}\right) = \frac{\pi}{8} \sec \frac{\pi}{4} = \frac{\pi\sqrt{2}}{8}. \end{aligned} \tag{4.19}$$

Adding up the even and odd parts,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L(n)}{2^{n}} &= \frac{\pi\sqrt{2}}{8} + \frac{1}{2\sqrt{2}} \ln\left(1+\sqrt{2}\right), \\ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{L(n)}{2^{n}} &= \frac{\pi\sqrt{2}}{8} - \frac{1}{2\sqrt{2}} \ln\left(1+\sqrt{2}\right). \end{aligned} \tag{4.20}$$

□

The next theorem shows that more challenging results can be achieved by being slightly more sophisticated.

Theorem 4.2. *The following equalities hold true:*

$$\begin{aligned}
&\sum(\zeta(2n) - \zeta(2n+1)) = \frac{1}{2}, && \sum\left(\eta(2n) - \eta(2n-1)\right) = \ln 2 - \frac{1}{2}, \\
&\sum(\lambda(2n) - \lambda(2n+1)) = \ln 2 - 1, && \sum\left(L(2n) - L(2n-1)\right) = \frac{1}{2}(1 - \ln 2), \\
&\sum(\zeta(2n+1) - \lambda(2n)) = 0, && \sum\left(\zeta(2n+1) - \eta(2n)\right) = \frac{1}{2}, \\
&\sum(\zeta(2n+1) - L(2n+1)) = \frac{1}{2}\left(\frac{\pi}{2} - 1\right), && \sum(\zeta(2n) - \lambda(2n+1)) = \ln 2, \\
&\sum(\lambda(2n+1) - \eta(2n)) = 1 - \ln 2, && \sum(\lambda(2n+1) - L(2n+1)) = \frac{\pi}{4} - \ln 2, \\
&\sum(\zeta(2n) - \eta(2n-1)) = \frac{1}{2} + \ln 2, && \sum(\lambda(2n) - \eta(2n-1)) = \ln 2, \\
&\sum(L(2n-1) - \eta(2n-1)) = \ln 2 - \frac{1}{2}, && \sum(\zeta(2n) - L(2n)) = \frac{1}{2}(1 + \ln 2), \\
&\sum(\zeta(2n+1) - L(2n)) = \frac{1}{2}\ln 2, && \sum(\lambda(2n) - L(2n)) = \frac{1}{2}\ln 2, \\
&\sum(\lambda(2n+1) - L(2n)) = \frac{1}{2}(1 - \ln 2), && \sum(L(2n) - \eta(2n)) = \frac{1}{2}(1 - \ln 2), \\
&\sum(L(2n) - \eta(2n-1)) = \frac{1}{2}\ln 2,
\end{aligned} \tag{4.21}$$

where all the series start with $n = 1$.

Proof. We will proceed left to right and top to bottom.

(1) From (2.4) and (2.27), it follows

$$\sum_{n=1}^{\infty}(\zeta(2n)-\zeta(2n+1))x^{2n} = x\mathcal{Z}_-(x)-\mathcal{Z}_+(x) = \frac{1}{2}-\frac{\pi}{2}x\cot(\pi x)-\mathfrak{D}_+(x)-i\cot(\pi x)\mathfrak{D}_-(x)-\ln 2 \tag{4.22}$$

for $|x|<1$. Thus, by the Tauber theorem [6], (3.2) and (3.7),

$$\begin{aligned}\sum_{n=1}^{\infty}(\zeta(2n)-\zeta(2n+1)) &= \lim_{x\to 1-}\sum_{n=1}^{\infty}(\zeta(2n)-\zeta(2n+1))x^{2n}\\ &= \frac{1}{2}-\mathfrak{D}_+(1)-\ln 2-\lim_{x\to 1-}\cot(\pi x)\left(\frac{\pi}{2}x+i\mathfrak{D}_-(x)\right)\\ &= \frac{1}{2}+1-\ln 2-\left(\frac{1}{2}+\frac{i}{\pi}\mathfrak{D}_-'(1)\right)=\frac{1}{2},\end{aligned} \tag{4.23}$$

where we have used the L'Hopital rule in the last line. We will use L'Hopital's rule also in the sequel to resolve indeterminacies.

(2) From (2.3) and (2.26), it follows

$$\sum_{n=1}^{\infty}(\eta(2n)-\eta(2n-1))x^{2n-1} = \mathcal{E}_-(x)-x\mathcal{E}_+(x) = -\frac{1}{2x}+\frac{\pi}{2}\csc(\pi x)+ix\csc(\pi x)\mathfrak{D}_-(x) \tag{4.24}$$

for $0<|x|<1$. Thus, by Tauber's theorem, (3.4) and (3.7),

$$\begin{aligned}\sum_{n=1}^{\infty}(\eta(2n)-\eta(2n-1)) &= \lim_{x\to 1-}\sum_{n=1}^{\infty}(\eta(2n)-\eta(2n-1))x^{2n-1}\\ &= -\frac{1}{2}+\lim_{x\to 1-}\csc(\pi x)\left(\frac{\pi}{2}+ix\mathfrak{D}_-(x)\right)\\ &= -\frac{1}{2}-\frac{i}{\pi}\left(\mathfrak{D}_-(1)+\mathfrak{D}_-'(1)\right)=\ln 2-\frac{1}{2}.\end{aligned} \tag{4.25}$$

(3) Analogously, from (2.2) and (4.11),

$$\begin{aligned}\sum_{n=1}^{\infty}(\lambda(2n)-\lambda(2n+1)) &= \lim_{x\to 1-}(x\Lambda_-(x)-\Lambda_+(x))\\ &= \lim_{x\to 1-}\left(\frac{\pi}{4}x\tan\frac{\pi x}{2}-\mathcal{C}_+(x)+i\tan\frac{\pi x}{2}\mathcal{C}_-(x)\right)\\ &= -\mathcal{C}_+(1)+\lim_{x\to 1-}\tan\frac{\pi x}{2}\left(\frac{\pi}{4}x+i\mathcal{C}_-(x)\right)\\ &= -\mathcal{C}_+(1)-\left(\frac{1}{2}+\frac{2i}{\pi}\mathcal{C}_-'(1)\right)\\ &= \ln 2-1,\end{aligned} \tag{4.26}$$

since (see (4.12))

$$\mathcal{C}_+(1) = \frac{1}{2}\int_0^{\pi/2}(\cot t - \csc t)dt = -\frac{1}{2}\ln 2, \tag{4.27}$$

$$\mathcal{C}_-'(1) = i\frac{1}{2}\int_0^{\pi/2} t\cot t\, dt = iD(2) = i\frac{\pi}{4}\ln 2. \tag{4.28}$$

(4) Using (4.18), (4.17), and (4.28), it follows

$$\begin{aligned}\sum_{n=1}^{\infty}(L(2n) - L(2n-1)) &= \lim_{x\to 1-}(\mathcal{L}_-(x) - x\mathcal{L}_+(x))\\ &= \lim_{x\to 1-}\left(\frac{1}{i}\sec\frac{\pi x}{2}\mathcal{C}_-(x) - \frac{\pi}{4}x\sec\frac{\pi x}{2}\right)\\ &= \lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{1}{i}\mathcal{C}_-(x) - \frac{\pi}{4}x\right)\\ &= -\frac{2}{i\pi}\mathcal{C}_-'(1) + \frac{1}{2} = \frac{1}{2}(1 - \ln 2).\end{aligned} \tag{4.29}$$

(5) From (2.27), (2.2), and (3.7), we get

$$\begin{aligned}&\sum_{n=1}^{\infty}(\dot{\zeta}(2n+1) - \lambda(2n))\\ &\quad= \lim_{x\to 1-}(\mathcal{Z}_+(x) - x\Lambda_-(x))\\ &\quad= \lim_{x\to 1-}\left(\mathfrak{D}_+(x) + i\cot(\pi x)\mathfrak{D}_-(x) + \ln 2 - \frac{\pi}{4}x\tan\frac{\pi x}{2}\right)\\ &\quad= \mathfrak{D}_+(1) + \ln 2 + \frac{1}{2}\lim_{x\to 1-}\tan\frac{\pi x}{2}\left(i\left(\cot^2\frac{\pi x}{2} - 1\right)\mathfrak{D}_-(x) - \frac{\pi}{2}x\right)\\ &\quad= -1 + \ln 2 + \frac{1}{2}\left(i\frac{2}{\pi}\mathfrak{D}_-'(1) + 1\right) = 0.\end{aligned} \tag{4.30}$$

(6) From (2.27), (2.3), and (3.7), we get

$$\begin{aligned}&\sum_{n=1}^{\infty}(\dot{\zeta}(2n+1) - \eta(2n))\\ &\quad= \lim_{x\to 1-}(\mathcal{Z}_+(x) - x\mathcal{E}_-(x))\\ &\quad= \lim_{x\to 1-}\left(\mathfrak{D}_+(x) + i\cot(\pi x)\mathfrak{D}_-(x) + \ln 2 + \frac{1}{2} - \frac{\pi}{2}x\csc\pi x\right)\\ &\quad= \mathfrak{D}_+(1) + \ln 2 + \frac{1}{2} + \lim_{x\to 1-}\csc\pi x\left(i\cos\pi x\mathfrak{D}_-(x) - \frac{\pi}{2}x\right)\\ &\quad= \mathfrak{D}_+(1) + \ln 2 + \frac{1}{2} + \frac{i}{\pi}\mathfrak{D}_-'(1) + \frac{1}{2} = \frac{1}{2}.\end{aligned} \tag{4.31}$$

(7) From (2.27), (4.17), and (3.7), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\zeta(2n+1)-L(2n+1))\\
&\quad=\lim_{x\to 1-}(\mathcal{Z}_{+}(x)-\mathcal{L}_{+}(x)+L(1))\\
&\quad=\lim_{x\to 1-}\left(\mathfrak{D}_{+}(x)+i\cot(\pi x)\mathfrak{D}_{-}(x)+\ln 2-\frac{\pi}{4}\sec\frac{\pi x}{2}+\frac{\pi}{4}\right)\\
&\quad=\mathfrak{D}_{+}(1)+\ln 2+\frac{\pi}{4}+\frac{1}{2}\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(i\sin\frac{\pi x}{2}\left(\cot^{2}\frac{\pi x}{2}-1\right)\mathfrak{D}_{-}(x)-\frac{\pi}{2}\right)\\
&\quad=-1+\ln 2+\frac{\pi}{4}+\frac{i}{\pi}\mathfrak{D}_{-}'(1)=\frac{1}{2}\left(\frac{\pi}{2}-1\right).
\end{aligned}\tag{4.32}$$

(8) From (2.4), (4.11), and (4.28), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\zeta(2n)-\lambda(2n+1))\\
&\quad=\lim_{x\to 1-}(x\mathcal{Z}_{-}(x)-\Lambda_{+}(x))\\
&\quad=\lim_{x\to 1-}\left(\frac{1}{2}-\frac{\pi}{2}x\cot\pi x-\mathcal{C}_{+}(x)-\frac{1}{i}\tan\frac{\pi x}{2}\mathcal{C}_{-}(x)\right)\\
&\quad=\frac{1}{2}+\frac{1}{2}\ln 2-\lim_{x\to 1-}\tan\frac{\pi x}{2}\left(\frac{\pi}{4}x\left(\cot^{2}\frac{\pi x}{2}-1\right)+\frac{1}{i}\mathcal{C}_{-}(x)\right)\\
&\quad=\frac{1}{2}(1+\ln 2)-\left(\frac{1}{2}-\frac{2}{i\pi}\mathcal{C}_{-}'(1)\right)=\ln 2.
\end{aligned}\tag{4.33}$$

(9) From (4.11), (2.3), and (4.28), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\lambda(2n+1)-\eta(2n))\\
&\quad=\lim_{x\to 1-}(\Lambda_{+}(x)-x\mathcal{E}_{-}(x))\\
&\quad=\lim_{x\to 1-}\left(\mathcal{C}_{+}(x)+\frac{1}{i}\tan\frac{\pi x}{2}\mathcal{C}_{-}(x)+\frac{1}{2}-\frac{\pi}{2}x\csc\pi x\right)\\
&\quad=\mathcal{C}_{+}(1)+\frac{1}{2}+\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{1}{i}\sin\frac{\pi x}{2}\mathcal{C}_{-}(x)-\frac{\pi}{4}x\csc\frac{\pi x}{2}\right)\\
&\quad=-\frac{1}{2}\ln 2+\frac{1}{2}-\frac{2}{i\pi}\mathcal{C}_{-}'(1)+\frac{1}{2}=1-\ln 2.
\end{aligned}\tag{4.34}$$

(10) From (4.11), (4.17), and (4.28), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\lambda(2n+1)-L(2n+1)) \\
&\quad=\lim_{x\to 1-}(\Lambda_+(x)-\mathcal{L}_+(x)+L(1)) \\
&\quad=\lim_{x\to 1-}\left(\mathcal{C}_+(x)+\frac{1}{i}\tan\frac{\pi x}{2}\mathcal{C}_-(x)-\frac{\pi}{4}\sec\frac{\pi x}{2}+\frac{\pi}{4}\right) \\
&\quad=\mathcal{C}_+(1)+\frac{\pi}{4}+\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{1}{i}\sin\frac{\pi x}{2}\mathcal{C}_-(x)-\frac{\pi}{4}\right) \\
&\quad=-\frac{1}{2}\ln 2+\frac{\pi}{4}-\frac{2}{i\pi}\mathcal{C}_-^{'}(1)=\frac{\pi}{4}-\ln 2.
\end{aligned} \tag{4.35}$$

(11) From (2.4), (2.26), (3.4), and (3.7), we get

$$\begin{aligned}
\sum_{n=1}^{\infty}(\zeta(2n)-\eta(2n-1)) &= \lim_{x\to 1-}(\mathcal{Z}_-(x)-x\mathcal{E}_+(x)) \\
&=\lim_{x\to 1-}\left(\frac{1}{2x}-\frac{\pi}{2}\cot\pi x-\frac{1}{i}x\csc\pi x\mathfrak{D}_-(x)\right) \\
&=\frac{1}{2}-\lim_{x\to 1-}\csc\pi x\left(\frac{\pi}{2}\cos\pi x+\frac{1}{i}x\mathfrak{D}_-(x)\right) \\
&=\frac{1}{2}+\frac{1}{i\pi}(\mathfrak{D}_-(1)+\mathfrak{D}_-^{'}(1))=\frac{1}{2}+\ln 2.
\end{aligned} \tag{4.36}$$

(12) From (2.2), (2.26), (3.4), and (3.7), we get

$$\begin{aligned}
\sum_{n=1}^{\infty}(\lambda(2n)-\eta(2n-1)) &= \lim_{x\to 1-}(\Lambda_-(x)-x\mathcal{E}_+(x)) \\
&=\lim_{x\to 1-}\left(\frac{\pi}{4}\tan\frac{\pi x}{2}-\frac{1}{i}x\csc\pi x\mathfrak{D}_-(x)\right) \\
&=\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{\pi}{4}\sin\frac{\pi x}{2}-\frac{1}{2i}x\csc\frac{\pi x}{2}\mathfrak{D}_-(x)\right) \\
&=\frac{1}{i\pi}(\mathfrak{D}_-(1)+\mathfrak{D}_-^{'}(1))=\ln 2.
\end{aligned} \tag{4.37}$$

(13) From (4.17), (2.26), and (3.7), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}(L(2n+1)-\eta(2n+1)) &= \lim_{x\to 1-}(\mathcal{L}_+(x)-\mathcal{E}_+(x)) \\
&=\lim_{x\to 1-}\left(\frac{\pi}{4}\sec\frac{\pi x}{2}-\frac{1}{i}\csc(\pi x)\mathfrak{D}_-(x)\right) \\
&=\frac{1}{2}\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{\pi}{2}+i\csc\frac{\pi x}{2}\mathfrak{D}_-(x)\right) \\
&=\frac{1}{i\pi}\mathfrak{D}_-^{'}(1)=\ln 2-\frac{1}{2}.
\end{aligned} \tag{4.38}$$

(14) From (2.4), (4.18), and (4.28), we get

$$\begin{aligned}\sum_{n=1}^{\infty}(\zeta(2n)-L(2n)) &= \lim_{x\to 1-}(\mathcal{Z}_-(x)-\mathcal{L}_-(x))\\ &= \lim_{x\to 1-}\left(\frac{1}{2x}-\frac{\pi}{2}\cot\pi x+i\sec\frac{\pi x}{2}\mathcal{C}_-(x)\right)\\ &= \frac{1}{2}-\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{\pi}{4}\cos\pi x\csc\frac{\pi x}{2}-i\mathcal{C}_-(x)\right)\\ &= \frac{1}{2}-\frac{2i}{\pi}\mathcal{C}_-^{'}(1)=\frac{1}{2}(1+\ln 2).\end{aligned} \tag{4.39}$$

(15) From (2.27), (4.18), (3.7), (4.12), and (4.28), we get

$$\begin{aligned}&\sum_{n=1}^{\infty}(\zeta(2n+1)-L(2n))\\ &\quad= \lim_{x\to 1-}(\mathcal{Z}_+(x)-x\mathcal{L}_-(x))\\ &\quad= \lim_{x\to 1-}\left(\mathfrak{D}_+(x)+i\cot(\pi x)\mathfrak{D}_-(x)+\ln 2-\frac{1}{i}x\sec\frac{\pi x}{2}\mathcal{C}_-(x)\right)\\ &\quad= \mathfrak{D}_+(1)+\ln 2+i\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\sin\frac{\pi x}{2}\left(\cot^2\frac{\pi x}{2}-1\right)\mathfrak{D}_-(x)+x\mathcal{C}_-(x)\right)\\ &\quad= -1+\ln 2+i\frac{2}{\pi}(\mathfrak{D}_-^{'}(1)-\mathcal{C}_-(1)-\mathcal{C}_-^{'}(1))=\frac{1}{2}\ln 2.\end{aligned} \tag{4.40}$$

(16) From (2.2), (4.18), and (4.28), we get

$$\begin{aligned}\sum_{n=1}^{\infty}(\lambda(2n)-L(2n)) &= \lim_{x\to 1-}(\Lambda_-(x)-\mathcal{L}_-(x))\\ &= \lim_{x\to 1-}\left(\frac{\pi}{4}\tan\frac{\pi x}{2}+i\sec\frac{\pi x}{2}\mathcal{C}_-(x)\right)\\ &= \lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\frac{\pi}{4}\sin\frac{\pi x}{2}+i\mathcal{C}_-(x)\right)\\ &= \frac{2}{i\pi}\mathcal{C}_-^{'}(1)=\frac{1}{2}\ln 2.\end{aligned} \tag{4.41}$$

(17) From (4.11), (4.18), (4.27), and (4.12), we get

$$\begin{aligned}&\sum_{n=1}^{\infty}(\lambda(2n+1)-L(2n))\\ &\quad= \lim_{x\to 1-}(\Lambda_+(x)-x\mathcal{L}_-(x))\\ &\quad= \lim_{x\to 1-}\left(\mathcal{C}_+(x)+\frac{1}{i}\tan\frac{\pi x}{2}\mathcal{C}_-(x)-\frac{1}{i}x\sec\frac{\pi x}{2}\mathcal{C}_-(x)\right)\\ &\quad= \mathcal{C}_+(1)+\frac{1}{i}\mathcal{C}_-(1)\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\sin\frac{\pi x}{2}-x\right)=\frac{1}{2}(1-\ln 2).\end{aligned} \tag{4.42}$$

(18) From (4.18), (2.3), and (4.28), we get

$$\begin{aligned}\sum_{n=1}^{\infty}(L(2n)-\eta(2n)) &= \lim_{x\to 1-}(\mathcal{L}_-(x)-\mathcal{E}_-(x))\\ &= \lim_{x\to 1-}\left(\frac{1}{i}\sec\frac{\pi x}{2}\mathcal{C}_-(x)+\frac{1}{2x}-\frac{\pi}{2}\csc\pi x\right)\\ &= \frac{1}{2}-\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(i\mathcal{C}_-(x)+\frac{\pi}{4}\csc\frac{\pi x}{2}\right)\\ &= \frac{1}{2}-\frac{2}{i\pi}\mathcal{C}_-'(1)=\frac{1}{2}(1-\ln 2).\end{aligned}\tag{4.43}$$

(19) From (4.18), (2.26), (4.28), (3.4), and (3.7), we get

$$\begin{aligned}\sum_{n=1}^{\infty}(L(2n)-\eta(2n-1)) &= \lim_{x\to 1-}(\mathcal{L}_-(x)-x\mathcal{E}_+(x))\\ &= \lim_{x\to 1-}\left(\frac{1}{i}\sec\frac{\pi x}{2}\mathcal{C}_-(x)-\frac{1}{i}x\csc\pi x\mathfrak{D}_-(x)\right)\\ &= \frac{1}{i}\lim_{x\to 1-}\sec\frac{\pi x}{2}\left(\mathcal{C}_-(x)-\frac{x}{2}\csc\frac{\pi x}{2}\mathfrak{D}_-(x)\right)\\ &= -\frac{2}{i\pi}\mathcal{C}_-'(1)+\frac{1}{i\pi}\mathfrak{D}_-(1)+\frac{1}{i\pi}\mathfrak{D}_-'(1)=\frac{1}{2}\ln 2.\end{aligned}\tag{4.44}$$

□

5. Enter $\psi(x)$

Let $\psi(x),\ x\in\mathbb{C}$, be the logarithmic derivative of the Gamma function. Then, if $|x|<1$ [7, 1.17(5)]

$$\psi(1+x)=-\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)x^{n-1},\tag{5.1}$$

hence [7, 1.7(11)]

$$\begin{aligned}\mathfrak{Z}(x)&=-\psi(1-x)-\gamma=-\psi(x)-\pi\cot\pi x-\gamma,\\ \mathcal{E}(x)&=\mathfrak{Z}(x)-\mathfrak{Z}\left(\frac{x}{2}\right)+\ln 2=-\psi(x)+\psi\left(\frac{x}{2}\right)+\pi\csc\pi x+\ln 2.\end{aligned}\tag{5.2}$$

It follows

$$\begin{aligned}\mathfrak{Z}_+(x)&=-\psi_+(x)-\gamma,\\ \mathcal{E}_+(x)&=-\psi_+(x)+\psi_+\left(\frac{x}{2}\right)+\ln 2,\end{aligned}\tag{5.3}$$

where $\psi_+(x)\equiv(\psi(x)+\psi(-x))/2$.

Substitution of these expressions for $\mathcal{E}_+(x)$ and $\mathcal{Z}_+(x)$ in (2.22) and comparison with (2.21) yields

$$\begin{aligned}\mathfrak{D}_+(x) &= \int_0^{\pi/2} (\cos 2xt - 1)\cot t\,dt \\ &= -2\cos^2\frac{\pi x}{2}\psi_+(x) + \cos\pi x\left(\psi_+\left(\frac{x}{2}\right) + \ln 2\right) - \gamma - \ln 2, \\ \frac{1}{i}\mathfrak{D}_-(x) &= \int_0^{\pi/2} \sin 2xt\cot t\,dt \\ &= \sin\pi x\left(-\psi_+(x) + \psi_+\left(\frac{x}{2}\right) + \ln 2\right)\end{aligned} \tag{5.4}$$

respectively.

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