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*Research Article*

# On Some Fractional Stochastic Integrodifferential Equations in Hilbert Space

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We study a class of fractional stochastic integrodifferential equations considered in a real Hilbert space. The existence and uniqueness of the Mild solutions of the considered problem is also studied. We also give an application for stochastic integropartial differential equations of fractional order.



## 1. Introduction

Let $H$ and $K$ denote real Hilbert spaces equipped with norms $\|\cdot\|_H$ and $\|\cdot\|_K$, respectively, and let the space of bounded linear operators from $K$ to $H$ be denoted by $\mathrm{BL}(K;H)$. For Banach space $X$ and $Y$, the space of continuous functions from $X$ into $Y$ (equipped with the usual sup-norm) will be denoted by $C(X;Y)$, while $L^p(0,T;X)$ will represent the space of $X$-valued functions that are $p$-integrable on $[0,T]$. Let $(\Omega, Z, P)$ be a complete probability space equipped with a normal filtration $\{Z_t : 0 \le t \le T\}$. An $H$-valued random variable is an $Z$-measurable function $X : \Omega \to H$, and a collection of random variables $\psi = \{X(t;\omega) : \Omega \to H : 0 \le t \le T\}$ is called a stochastic process. The collection of all strongly measurable square integrable $H$-valued random variables, denoted by $L^2(\Omega;H)$, is a Banach space equipped with norm $\|X(\cdot)\|_{L^2(\Omega;H)} = (E\|X(\cdot;\omega)\|_H^2)^{1/2}$.

An important subspace is given by $L_0^2(\Omega;H) = \{f \in L^2(\Omega;H) : f \text{ is } Z_0 \text{ measurable}\}$. Next we define the space $\gamma((0,T);H)$ to be the set $\{v \in C([0,T];L^2(\Omega;H) : v \text{ is } Z_t\text{-adapted}\}$ with norm

$$\|v\|_\gamma = \sup_{0\le t\le T} \left(E\|v(t)\|_H^2\right)^{1/2} \tag{1.1}$$

(see in [1–5]). In this paper we study the existence and uniqueness of the mild solution of the fractional stochastic integrodifferential equation of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + F(x)(t) + \int_0^t G(x)(s)dW(s), \quad 0 \le t \le T,$$
$$x(0) = h(x) + x_0, \tag{1.2}$$

in a real separable Hilbert space $H$. Here, $1/2 \le \alpha \le 1$, $A : D(A) \subset H \to H$ is a linear closed operator generating semigroup, $F : \gamma([0,T];H) \to L^p([0,T];L^2(\Omega;H))$ $(1 \le p < \infty)$, $G : \gamma([0,T];H) \to C([0,T];L^2(\Omega;\mathrm{BL}(K;H)))$ (where $K$ is a real separable Hilbert space), $W$ is a $K$-valued Wiener process with incremental covariance described by the nuclear operator $Q$, $x_0$ is an $Z_0$-measurable $H$-valued random variable independent of $W$ and $h : \gamma([0,T];H) \to L_0^2(\Omega;H)$.

*Definition 1.1.* An $Z_t$-adapted stochastic process $x : [0,T] \to H$ is called a mild solution of (1.2) if $x(t)$ is measurable, for all $t \in [0,T]$,

$$\int_0^T \|x(s)\|_H^2 ds < \infty,$$

$$x(t) = \int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)(h(x) + x_0)d\theta + \alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)F(x)(\eta)d\theta\, d\eta$$
$$+ \alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)\left[\int_0^\eta G(x)(\tau)dW(\tau)\right]d\theta\, d\eta, \quad 0 \le t \le T, \tag{1.3}$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$,

$$\int_0^\infty \xi_\alpha(\theta)d\theta = 1 \tag{1.4}$$

(see [6–12]). In the next section, we will prove the existence and uniqueness of the mild solutions to (1.2).

## 2. Existence and Uniqueness

Consider the initial value problem (1.2) in a real separable Hilbert space $H$ under the following assumptions:

(I) the linear operator $A : D(A) \subset H \to H$ generates a $C_0$-semigroup$\{S(t) : t \ge 0\}$ on $H$;

(II) $F : \gamma([0,T];H) \to L^p(0,T;L^2(\Omega;H))$ is such that there exists $M_F > 0$ for which

$$\|F(x) - F(y)\|_{L^p} \le M_F\|x-y\|_\gamma, \quad \forall x,y \in \gamma([0,T];H); \tag{2.1}$$

(III) $G : \gamma([0,T];H) \to C([0,T];L^2(\Omega;\mathrm{BL}(K;H)))$ $(= \gamma_{\mathrm{BL}})$ is such that there exists $M_G > 0$ for which

$$\|G(x) - G(y)\|_{\gamma_{\mathrm{BL}}} \le M_G\|x-y\|_\gamma \quad \forall x,y \in \gamma([0,T];H); \tag{2.2}$$

(IV) $h : \gamma([0,T];H) \to L_0^2(\Omega;H)$ is such that there exists $M_h > 0$ for which

$$\|h(x) - h(y)\|_{L_0^2} \le M_h \|x - y\|_\gamma \quad \forall x, y \in \gamma([0,T];H); \tag{2.3}$$

(V) $x_0 \in L_0^2(\Omega;H)$.

We can therefore state the following theorem.

**Theorem 2.1.** *Assume that (I)–(V) hold. Then* (1.2) *has a unique solution on* $[0,T]$*, provided that*

$$M_S\left[M_h + C_F T^\alpha + M_G C_G T^{\alpha+1/2}\right] < 1, \tag{2.4}$$

*where* $M_h > 0$*,* $M_S > 0$*, and* $C_G > 0$*.*

*Proof.* Define the solution map $J : \gamma([0,T];H) \to \gamma([0,T];H)$ by

$$\begin{aligned}(Jx)(t) = &\int_0^\infty \xi_\alpha(\theta) S(t^\alpha\theta)(h(x) + x_0)d\theta \\ &+ \alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)F(x)(\eta)d\theta\, d\eta \\ &+ \alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right) \\ &\qquad\times\left[\int_0^\eta G(x)(\tau)dW(\tau)\right]d\theta\, d\tau, \quad 0 \le t \le T.\end{aligned} \tag{2.5}$$

From Holder's inequality, we get

$$\begin{aligned}&\left[E\left\|\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)F(x)(\eta)d\theta\, d\eta\right\|_H^2\right]^{1/2} \\ &\quad\le M_S\left[\int_0^T \|(T-\eta)^{\alpha-1}F(x)(\eta)\|^2_{L^2(\Omega;H)}d\eta\right]^{1/2} \\ &\quad\le M_S\left[\int_0^T (T-\eta)^{2(\alpha-1)}d\eta\right]^{1/2}\left[\int_0^T \|F(x)(\eta)\|^2_{L^2(\Omega;H)}d\eta\right]^{1/2} \\ &\quad\le M_S\frac{T^{\alpha-1/2}}{(2\alpha-1)^{1/2}}\left[\int_0^T \|F(x)(\eta)\|^2_{L^2(\Omega;H)}d\eta\right]^{1/2} \\ &\quad\le C_F M_S T^{\alpha-1/2}\|F(x)\|_{L^p},\end{aligned} \tag{2.6}$$

where $C_F$ is a constant depending on $\alpha$.

Subsequently, an application of (II), together with Minkowski's inequality enables us to continue the string of inequalities in (2.6) to conclude that

$$\left[E\left\|\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)F(x)(\eta)\,d\theta\,d\eta\right\|_H^2\right]^{1/2} \le M_S C_F T^{\alpha-1/2}\left[M_F\|x\|_\gamma + \|F(0)\|_{L^p}\right]. \tag{2.7}$$

Taking the supermum over $[0,T]$ in (2.7) then implies that

$$\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)F(x)(\eta)\,d\theta\,d\eta \in \gamma([0,T];H), \tag{2.8}$$

for any $x \in \gamma([0,T];H)$. Furthermore for such $x$, $G(x)(\eta) \in \mathrm{BL}(K;H)$, and $h(x)+x_0 \in L_0^2(\Omega;H)$ (by (IV) and (V)). Consequently, one can argue as in [13–15] to conclude that $J$ is well defined.

Next we show that $J$ is a strict contraction.

Observe that for $x,y \in \gamma([0,T];H)$, we infer from (2.5) that

$$\begin{aligned}
&(Jx)(t)-(Jy)(t)\\
&\quad=\int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)\left(h(x)-h(y)\right)d\theta\\
&\qquad+\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)\left(F(x)(\eta)-F(y)(\eta)\right)d\theta\,d\eta\\
&\qquad+\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)\left[\int_0^\eta \left(G(x)(\tau)-G\left(y(\tau)\right)\right)dW(\tau)\right]d\theta\,d\eta, \quad 0\le t\le T.
\end{aligned} \tag{2.9}$$

Squaring both sides and taking the expectation in (2.9) yields, with the help of Young's inequality,

$$\begin{aligned}
&E\|(Jx)(t)-(Jy)(t)\|_H^2\\
&\quad\le 4E\left\|\int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)(h(x)+x_0)\,d\theta\right\|_H^2\\
&\qquad+4\alpha^2\left[E\left\|\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)\left(F(x)(\eta)-F(y)(\eta)\right)d\theta\,d\eta\right\|_H^2\right.\\
&\qquad\qquad+E\left\|\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S\left((t-\eta)^\alpha\theta\right)\right.\\
&\qquad\qquad\quad\left.\left.\times\left[\int_0^\eta (G(x)(\tau)-G(y)(\tau))\,dW(\tau)\right]d\theta\,d\eta\right\|_H^2\right],
\end{aligned} \tag{2.10}$$

and subsequently,

$$
\begin{aligned}
&\|(Jx)(t)-(Jy)(t)\|_{\gamma}\\
&\quad\le \left\|\int_0^{\infty}\xi_{\alpha}(\theta)S(t^{\alpha}\theta)(h(x)-h(y))d\theta\right\|_{\gamma}\\
&\quad\quad+4\alpha^2\left[\left\|\int_0^t\int_0^{\infty}\theta(t-\eta)^{\alpha-1}\xi_{\alpha}(\theta)S\big((t-\eta)^{\alpha}\theta\big)\big(F(x)(\eta)-F(y)(\eta)\big)d\theta\, d\eta\right\|_{\gamma}\right.\\
&\quad\quad\quad+\left\|\int_0^t\int_0^{\infty}\theta(t-\eta)^{\alpha-1}\xi_{\alpha}(\theta)S\big((t-\eta)^{\alpha}\theta\big)\right.\\
&\quad\quad\quad\quad\left.\left.\times\left[\int_0^{\eta}(G(x)(\tau)-G(y)(\tau))dW(\tau)\right]d\theta\, d\eta\right\|_{\gamma}\right].
\end{aligned}
\tag{2.11}
$$

Using reasoning similar to that which led to (2.6), one can show that

$$
\begin{aligned}
&\left\|\int_0^{\infty}\xi_{\alpha}(\theta)S(t^{\alpha}\theta)(h(x)-h(y))d\theta\right\|_{\gamma}\\
&\quad\quad= E\left\|\int_0^{\infty}\xi_{\alpha}(\theta)S(t^{\alpha}\theta)(h(x)+x_0)d\theta\right\|_H^2\le M_sM_h\|x-y\|_{\gamma},\\
&\left\|\int_0^t\int_0^{\infty}\theta(t-\eta)^{\alpha-1}\xi_{\alpha}(\theta)S\big((t-\eta)^{\alpha}\theta\big)\big(F(x)(\eta)-F(y)(\eta)\big)d\theta\, d\eta\right\|_{\gamma}\\
&\quad\quad=\left\|\int_0^t\int_0^{\infty}\theta(t-\eta)^{\alpha-1}\xi_{\alpha}(\theta)S((t-\eta)^{\alpha}\theta)(F(x)(\eta)-F(y)(\eta))d\theta\, d\eta\right\|_{\gamma}\\
&\quad\quad\le C_FM_ST^{\alpha}\|x-y\|_{\gamma},
\end{aligned}
\tag{2.12}
$$

where $C_F$ depending on $\alpha$ and $M_F$. We also infer that

$$
\begin{aligned}
&\left\|\int_0^t\int_0^{\infty}\theta(t-\eta)^{\alpha-1}\xi_{\alpha}(\theta)S\big((t-\eta)^{\alpha}\theta\big)\left[\int_0^{\eta}(G(x)(\tau)-G(y)(\tau))dW(\tau)\right]d\theta\, d\eta\right\|_{\gamma}\\
&\quad=\left[E\left\|\int_0^t\int_0^{\infty}\theta(t-\eta)^{\alpha-1}\xi_{\alpha}(\theta)S\big((t-\eta)^{\alpha}\theta\big)\left[\int_0^{\eta}(G(x)(\tau)-G(y)(\tau))dW(\tau)\right]d\theta\, d\eta\right\|_H^2\right]^{1/2}\\
&\quad\le \mathrm{Tr}(Q)\frac{T^{\alpha-1/2}}{(2\alpha-1)^{1/2}}M_S\left[\int_0^T\int_0^T\|G(x)(\tau)-G(y)(\tau)\|_{L^2(\Omega;H)}^2d\tau\, d\eta\right]^{1/2}\\
&\quad\le C_GT^{\alpha+1/2}M_sM_G\|x-y\|_{\gamma},
\end{aligned}
\tag{2.13}
$$

where $C_G$ is a constant depending on ($\alpha$ and $\mathrm{Tr}(Q)$). Using (2.12) and (2.13) in (2.11) enables us to conclude that $J$ is a strict contraction, provided that (2.4) is satisfied, and has a unique fixed point which coincides with a mild solution of (1.2). This completes the proof. □

## 3. Application

Let $D$ be a bounded domain in $R^N$ with smooth boundary $\partial D$, and consider the initial boundary value problem:

$$\begin{aligned}\frac{\partial^\alpha(t,z)}{\partial t^\alpha} &= \Delta_z x(t,z) + \int_0^T a(t,s) f_1\left(s, x(s,z), \int_0^s k(s,\tau,x(\tau,z))d\tau\right)ds \\ &\quad + \int_0^T b(t,s) f_2(s,x(s,z))dW(s), \quad \text{on } (0,T)\times D,\end{aligned} \tag{3.1}$$

$$\begin{aligned} x(0,z) &= \sum_{i=1}^n g_i(z)x(t_i,z) + \int_0^T c(s) f_3(s,x(s,z))ds, \quad \text{on } D, \\ x(t,z) &= 0, \quad \text{on } (0,T)\times\partial D,\end{aligned} \tag{3.2}$$

where $0 \le t_1 < t_2 \cdots < t_n \le T$ are given and $W$ is an $L^2(D)$-valued Wiener process. We consider the equation (3.1) under the following conditions.

(H1) $f_1 : [0,T]\times R\times R \to R$ satisfies the Caratheodory conditions as well as

- (i) $f_1(\cdot,0,0) \in L^2(0,T)$,
- (ii) $|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \le M_{f_1}[|x_1 - x_2| + |y_1 - y_2|]$, for all $x_1, x_2, y_1, y_2 \in R$ and almost $t \in (0,T)$ for some $M_{f_1} > 0$,

(H2) $f_2 : [0,T]\times R \to \mathrm{BL}(L^2(D))$ where $\mathrm{BL}(L^2(D))$ is the space of bounded linear operator from $L^2(D)$ to $L^2(D)$ satisfies the Caratheodory conditions as well as

- (i) $f_2(\cdot,0) \in L^2(0,T)$,
- (ii) $|f_2(t,x) - f_2(t,y)|_{\mathrm{BL}(H)} \le M_{f_2}|x-y|$, for all $x, y \in R$ and almost all $t \in (0,T)$, for some $M_{f_2} > 0$.

(H3) $f_3 : [0,T]\times R \to R$ satisfies the Caratheodory conditions as well as

- (i) $f_3(\cdot,0) \in L^2(0,T)$,
- (ii) $|f_3(t,x) - f_3(t,y)| \le M_{f_3}|x-y|$, for all $x, y \in R$ and almost $t \in (0,T)$ for some $M_{f_3} > 0$,

(H4) $a \in L^2((0,T)^2)$,

(H5) $b \in L^\infty((0,T)^2)$,

(H6) $c \in L^2((0,T)^2)$,

(H7) $k : \Upsilon \times R \to R$, where $\Upsilon = \{(t,s) : 0 < s < t < T\}$, satisfies $|k(t,s,x_1) - k(t,s,x_2)| \le M_k|x_1 - x_2|$, for all $x_1, x_2 \in R$, and almost $(t,s) \in \Upsilon$,

(H8) $g_i \in L^2(D)$, $i = 1, \ldots, n$.

The stochastic integropartial differential equation (3.1) can be written in the abstract form (1.2), where $K = H = L^2(D)$, $A = \Delta_z$, with domain $D(A) = H^2(D) \cup H_0^1((D))$. It is well known that $A$ is a closed linear operator which generates a $C_0$-semigroup. We also introduce the mappings $F$, $G$, and $h$ defined by, respectively,

$$F(x)(t,\cdot) = \int_0^T a(t,s) f_1\left(s, x(s,z), \int_0^s k(s,\tau,x(\tau,z))d\tau\right)ds,$$
$$G(x)(t,\cdot) = b(t,s) f_2(s, x(s,\cdot)), \tag{3.3}$$
$$h(x)(\cdot) = x(0,z) = \sum_{i=1}^n g_i(\cdot)x(t_i,\cdot) + \int_0^T c(s) f_3(s, x(s,\cdot))ds.$$

One can use (H1)–(H8) to verify that $F$, $G$, and $h$ satisfy (II)–(IV) in the last section, respectively, with

$$M_F = 2M_{f_1} T |a|_{L^2((0,T)^2)} \left(1 + M_k T^3\right)^{1/2},$$
$$M_G = M_{f_2}, \tag{3.4}$$
$$M_h = 2\sum_{i=1}^n \|g_i\|_{L^2(D)} + M_{f_3}\sqrt{m(D)}|G|_{L^2(0,T)}.$$

Consequently theorem (2.4) can be applied for (3.1).

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