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Research Article

Classification of All Associative Mono-n-ary Algebras with 2 Elements

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We consider algebras with a single n-ary operation and a certain type of associativity. We prove that (up to isomorphism) there are exactly 5 of these associative mono-n-ary algebras with 2 elements for even $n \geq 2$ and 6 for odd $n \geq 3$. These algebras are described explicitly. It is shown that a similar result is impossible for algebras with at least 4 elements. An application concerning the assignment of a control bit to a string is given.

1. Introduction

One of the most demanding tasks of combinatorics consists in counting finite algebraic structures with certain properties or even in classifying them up to isomorphism. Various types of structures can be enumerated by the number of their elements using the methods of Pólya [1, 2]. However, there are some very difficult combinatorial problems that have not been solved until now. One of these problems is to determine the number of semigroups with k elements, where k is a positive integer. An asymptotic formula for the number of labelled semigroups with k elements was found in [3]. But a different problem consists in counting up to isomorphism. Let us reformulate this problem into the language of universal algebra. An *n-ary operation* μ on a set A is a function $A^n \rightarrow A$. In this article the ground set A will always be finite. (A, μ) is called *mono- n-ary algebra (with $\#A$ elements)*. Two mono-n-ary algebras (A, μ) and (B, ν) are *isomorphic* if there is a bijection f from A to B, so that for all $x_1, x_2, \ldots, x_n \in A$

$$f\left(\mu(x_1, x_2, \ldots, x_n)\right) = \nu\left(f(x_1), f(x_2), \ldots, f(x_n)\right). \tag{1.1}$$

This is the standard notion of isomorphism for algebras, (cf. [4])

Table 1: Number of nonisomorphic associative mono-n-ary algebras with k elements.

n	k = 1	2	3	4	5	6	7	8	$\cdots$
1	1	3	7	19	47	130	343	951	$\cdots$
2	1	5	24	188	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
3	1	6	$\cdots$	$\cdots$					
$\cdots$	$\cdots$	$\cdots$							

Hence we have asked how many associative mono-2-ary algebras with k elements exist up to isomorphism. We may consider the more general problem of determining the number of associative mono-n-ary algebras with k elements, where n and k are positive integers. But what does it mean for an algebra with a single operation to be associative?

In fact, there are many different ways to generalize associativity from binary to n-ary operations. A well-known example is superassociativity, introduced by Menger [5]. Another example is the associativity of diagonal algebras [6]. This paper will treat only a natural kind of associativity which could be called *left-right-pushing*. Let (A, μ) be a mono-n-ary algebra. It is called *associative* (in the left-right sense) if for all $1 \leq i < n$ and $a_1, a_2, \ldots, a_{2n-1} \in A$

$$\begin{aligned} &\mu(a_1, \ldots, a_{i-1}, \mu(a_i, a_{i+1}, \ldots, a_{n+i-1}), a_{n+i}, a_{n+i+1}, \ldots, a_{2n-1}) \\ &\quad = \mu(a_1, \ldots, a_{i-1}, a_i, \mu(a_{i+1}, \ldots, a_{n+i-1}, a_{n+i}), a_{n+i+1}, \ldots, a_{2n-1}). \end{aligned} \tag{1.2}$$

Note that a list of the type $a_{m+1}, \ldots, a_m$ denotes an empty list.

This type of associativity implies a general associativity law that will be explained in Section 2.

Left-right-pushing was introduced by Dörnte [7] who generalized the notion of groups to m-groups where the binary operation is replaced by an m-ary. These m-groups (or polyadic groups) were investigated further by Post [8]. Later the concept was widened to *polyadic semigroups* by Zupnik [9] or *associatives* as Gluskin and Shvarts [10] called them. These are other names for the associative mono-n-ary algebras we study here. Whilst Zupnik, Gluskin, and Shvarts investigated the representation of certain polyadic semigroups by (associative) binary and unary operations, we are interested in counting polyadic semigroups with no restrictions on their structure, only depending on their order and arity.

We would like to complete Table 1 which lists the numbers of associative mono-n-ary algebras with k elements (up to isomorphism) for some small values of n and k. The entries of the table were computed by the author using brute force algorithms.

Since it seems to be very difficult to find general formulas for the rows even for small values of n, one might try to find formulas for the columns for small values of k. Surprisingly we will find a simple formula for the case $k = 2$. Moreover, we explicitly classify all associative mono-n-ary algebras with 2 elements using purely elementary methods. Note that we even need not make use of the methods of Zupnik [9] to prove our result. The occurring algebras will be introduced in Section 3.1. In Section 3.2 we prove that every associative mono-n-ary algebra with 2 elements is isomorphic to one of these algebras which will be the main subject of this article. Similar results for the columns with $k \geq 3$ could not be found, but in Section 4 we will see that the entries of the columns with $k \geq 4$ are at least exponentially increasing.

These results might be interesting for some aspects of nonlinear coding theory. An example of an application is given in Section 5.

2. The General Associativity Law

When denoting a (multi)product of several elements of a semigroup it is not necessary to write brackets in order to indicate the order of calculation since the value of the product only depends on the elements and their total order. This is called "general associativity law," and it may be generalized for multiproducts in associative mono-n-ary algebras.

Let us fix some terms. If (A,μ) is a mono-n-ary algebra, then a *product* is a formal expression $\mu(x_1,\dots,x_n)$ with some *entries* $x_1,\dots,x_n$. A *term* is defined recursively by the following:

(i) every element of the algebra is a term,

(ii) every product with terms as entries (the *subterms*) is a term.

Furthermore, a *right-tower-term* is a term where for each occurring product all entries are single elements of the algebra except for the last entry.

We can formulate now a basic result that was remarked by Dörnte [7]. Because of its importance for this article the idea of a formal proof is added.

Theorem 2.1. *In an associative mono-n-ary algebra the value of any term does not depend on the structure of the subterms, only on the total order of the entries.*

Proof. By induction on the number of subterms of a given term we prove that every term can be transformed into a right-tower-term. By induction we may assume that the left-most subterm that is not a single element is a right-tower-term; thus we have

$$\mu(x_1,\dots,x_{j-1},\mu(y_1,\dots,y_{n-1},\mu(\cdots)),t_{j+1},\dots,t_n), \tag{2.1}$$

where $x_i,\ y_i$ are single elements and t_i are terms. Applying associativity $n-j$ times we may shift the left-most inner product to the right and obtain

$$\mu(x_1,\dots,x_{j-1},y_1,\dots,y_{n-j},\mu(y_{n-j+1},\dots,y_{n-1},\mu(\cdots),t_{j+1},\dots,t_n)). \tag{2.2}$$

By using the induction hypothesis again the right subterm can be transformed into a right-tower-term, and we are done. □

3. The 2-Element-Column

Here we consider associative mono-n-ary algebras with 2 elements. This main section is organized as follows: In Section 3.1 the main result is presented whereas Section 3.2 is devoted to its proof.

3.1. The Occurring Algebras

We define the following types of algebras on the set $\{0,1\}$ with single n-ary operation μ:

type **0**:

$$\mu(x_1, x_2, \dots, x_n) := 0, \tag{3.1}$$

type **A**:

$$\begin{gathered} \mu(x_1, x_2, \dots, x_n) := 0 \qquad \text{if } \exists i: \ x_i = 0, \\ \mu(1, 1, \dots, 1) := 1, \end{gathered} \tag{3.2}$$

type **L**:

$$\mu(x_1, x_2, \dots, x_n) := x_1, \tag{3.3}$$

type **R**:

$$\mu(x_1, x_2, \dots, x_n) := x_n, \tag{3.4}$$

type **G0**:

$$\begin{aligned} \mu(x_1, x_2, \dots, x_n) &:= 0 \qquad \text{if } \#\{i \mid x_i = 0\} \equiv 1 \bmod 2, \\ \mu(x_1, x_2, \dots, x_n) &:= 1 \qquad \text{if } \#\{i \mid x_i = 0\} \equiv 0 \bmod 2, \end{aligned} \tag{3.5}$$

type **G1**:

$$\begin{aligned} \mu(x_1, x_2, \dots, x_n) &:= 0 \qquad \text{if } \#\{i \mid x_i = 0\} \equiv 0 \bmod 2, \\ \mu(x_1, x_2, \dots, x_n) &:= 1 \qquad \text{if } \#\{i \mid x_i = 0\} \equiv 1 \bmod 2. \end{aligned} \tag{3.6}$$

It is easy to see that these algebras are associative. (For type **G0** and type **G1** distinguish cases according to the parity of the number of zeros in the inner and outer brackets when verifying the associativity laws.)

In case of odd $n \geq 3$ these types are pairwise nonisomorphic. The same holds in case of even $n \geq 2$ except for type **G0** and type **G1** which are then isomorphic. In the latter case we also call type **G0** and type **G1** simply type **G**. In the trivial case $n = 1$ type **L**, type **R** and type **G0** are isomorphic to type **A**.

The types **G0** and **G1** are the only polyadic groups with 2 elements which can be seen easily [7]. However, the analogue classification of polyadic semigroups is more complicated because of the lack of invertibility.

The main result of this article will be as follows.

Theorem 3.1. *Let (A, μ) be an associative mono-n-ary algebra with $\#A = 2$ and $n \geq 2$. Then (A, μ) is isomorphic to one of the types defined above.*

3.2. Proof of Theorem 3.1

Let $\mathcal{A} = (\{0, 1\}, \mu)$ be an associative mono-n-ary algebra, $n \geq 2$. In order to simplify the notation we denote the products only with brackets, that is,

$$(a_1 a_2 \cdots a_n) := \mu(a_1, a_2, \ldots, a_n). \tag{3.7}$$

The general idea of the proof is to distinguish cases according to the values of certain products. In each case $\mathcal{A}$ will be determined only by these values.

There are four possibilities for the values of the products $(0 \cdots 0)$ and $(1 \cdots 1)$. The case $1, 0$ is treated in Lemma 3.3, and the case $0, 0$ in Lemmas 3.4 and 3.5. Obviously, by exchanging 0 and 1, the case $1, 1$ leads to isomorphic algebras as in the case $0, 0$.

The case $(0 \cdots 0) = 0$ and $(1 \cdots 1) = 1$ is more complicated. Here we need to consider the products $(0 \cdots 01)$, $(10 \cdots 0)$, $(01 \ldots 1)$, and $(1 \cdots 10)$, too. The subcases 1, 0 and 0, 1, respectively, 0, 0 for $(0 \cdots 01), (10 \cdots 0)$ and the subcases 0, 1 and 1, 0, respectively, 1, 1 for $(01 \cdots 1), (1 \cdots 10)$ are examined by Lemma 3.6, respectively, Lemma 3.7. Note that these cases are not excluding each other and may be contradictory which has no effect on our proof. The remaining case is Lemma 3.8.

Lemma 3.2. *If $(1 \cdots 1) = 0$ in $\mathcal{A}$, then the value of a product depends only on the number of zeros in the product and not on the order of the elements.*

Proof. Indeed, whenever two products have the same number of zero entries, each zero can be replaced by the product $(1 \cdots 1)$ and the new expressions contain both the same number of ones and thus have the same value because of the general associativity law. □

Lemma 3.3. *If $(0 \cdots 0) = 1$ and $(1 \cdots 1) = 0$, then n is odd and $\mathcal{A}$ is isomorphic to type **G1**.*

Proof. Using the assumptions and the associativity of $\mathcal{A}$ we obtain

$$1 = (0 \cdots 0) = \Bigg(\underbrace{0 \ldots 0}_{n-1}(1 \ldots 1) \Bigg) = \Bigg(\Bigg(\underbrace{0 \ldots 01}_{n-1} \Bigg) \underbrace{1 \ldots 1}_{n-1} \Bigg). \tag{3.8}$$

Thus (in order to avoid the contradiction $1 = (1 \ldots 1)$) we conclude

$$\Bigg(\underbrace{0 \ldots 01}_{n-1} \Bigg) = 0. \tag{3.9}$$

Assume that n is even. Then, using (3.9) we have the contradiction

$$\begin{aligned}
0 = (1\ldots1) &= \Big(\underbrace{\underline{(0\ldots0)1}\ldots\underline{(0\ldots0)1}}_{n/2}\Big) \\
&= \left(\underbrace{\underline{0\Big(\underbrace{0\ldots0}_{n-1}1\Big)}\ldots\underline{0\Big(\underbrace{0\ldots0}_{n-1}1\Big)}}_{n/2}\right) \underset{(3.9)}{=} (0\ldots0) = 1.
\end{aligned} \tag{3.10}$$

So n is odd. We will now establish the following claim.

Claim 1. $(\underbrace{0\ldots0}_{2k}\underbrace{1\ldots1}_{n-2k}) = 0$ and $(\underbrace{0\ldots0}_{2k+1}\underbrace{1\ldots1}_{n-2k+1}) = 1$ for $k = 0,\ldots,(n-1)/2$.

We prove the claim by induction. The first assertion is clear for $k = 0$. Assume that the first assertion has been proved for all $k \leq K$ and that the second assertion has been proved for all $k < K$ and that $0 \leq 2K \leq n-1$. Then by assumption

$$0 = \Big(\underbrace{0\ldots0}_{2K}\underbrace{1\ldots1}_{n-2K}\Big) = \Big(\underbrace{0\ldots0}_{2K}(0\ldots0)\underbrace{1\ldots1}_{n-2K-1}\Big) = \Big(\underbrace{0\ldots0}_{n-1}\Big(\underbrace{0\ldots0}_{2K+1}\underbrace{1\ldots1}_{n-2K-1}\Big)\Big). \tag{3.11}$$

Let us fist consider the case

$$\Big(\underbrace{0\ldots0}_{2K+1}\underbrace{1\ldots1}_{n-2K-1}\Big) = 0. \tag{3.12}$$

Then in (3.11) we obtain the contradiction $0 = 1$. Thus we have

$$\Big(\underbrace{0\ldots0}_{2K+1}\underbrace{1\ldots1}_{n-2K-1}\Big) = 1. \tag{3.13}$$

Now assume that the first and the second assertions have been proved for all $0 \leq k < K$ and that $0 < 2K \leq n-1$. Then by (3.13) we conclude

$$1 \underset{(3.13)}{=} \Big(\underbrace{0\ldots0}_{2K-1}\underbrace{1\ldots1}_{n-2K+1}\Big) = \Big(\underbrace{0\ldots0}_{2K-1}(0\ldots0)\underbrace{1\ldots1}_{n-2K}\Big) = \Big(\underbrace{0\ldots0}_{n-1}\Big(\underbrace{0\ldots0}_{2K}\underbrace{1\ldots1}_{n-2K}\Big)\Big). \tag{3.14}$$

Let us first consider the case

$$\Big(\underbrace{0\dots0}_{2K}\,\underbrace{1\dots1}_{n-2K}\Big) = 1. \tag{3.15}$$

Then we obtain the contradiction

$$1 \underset{(3.14)}{=} \Big(\underbrace{0\dots01}_{n-1}\Big) \underset{(3.9)}{=} 0. \tag{3.16}$$

Therefore we have

$$\Big(\underbrace{0\dots0}_{2K}\,\underbrace{1\dots1}_{n-2K}\Big) = 0. \tag{3.17}$$

This proves the claim.

By Lemma 3.2 it follows immediately from Claim 1 that a product is 0 if and only if the number of its zero entries is even. Thus $\mathcal{A}$ is of type **G1**. □

Lemma 3.4. *If* $(0\dots0) = 0$ *and* $(1\dots1) = 0$ *and*

$$\Big(\underbrace{0\dots01}_{n-1}\Big) = 0, \tag{3.18}$$

*then $\mathcal{A}$ is isomorphic to type **0**.*

Proof. Prove by induction that

$$\Big(\underbrace{0\dots0}_{n-k}\,\underbrace{1\dots1}_{k}\Big) = 0 \quad \text{for } k = 1,\dots,n-1. \tag{3.19}$$

The induction step ($1 \le k \le n-2$) is

$$\begin{aligned} 0 &= \Big(\underbrace{0\dots0}_{n-k}\,\underbrace{1\dots1}_{k}\Big) = \Big(\underbrace{0\dots0}_{n-k-1}\Big(\underbrace{0\dots01}_{n-1}\Big)\underbrace{1\dots1}_{k}\Big) \\ &= \Big((0\dots0)\underbrace{0\dots0}_{n-k-2}\,\underbrace{1\dots1}_{k+1}\Big) = \Big(\underbrace{0\dots0}_{n-k-1}\,\underbrace{1\dots1}_{k+1}\Big). \end{aligned} \tag{3.20}$$

Use Lemma 3.2 again and conclude that every product is 0. □

Lemma 3.5. *If* $(0\dots0)=0$ *and* $(1\dots1)=0$ *and*

$$\Big(\underbrace{0\dots0}_{n-1}1\Big)=1, \tag{3.21}$$

then n *is even and* $\mathcal{A}$ *is isomorphic to type* $\boldsymbol{G}$.

Proof.

$$1=\Big(\underbrace{0\dots0}_{n-1}1\Big)=\Big(\underbrace{0\dots0}_{n-2}(1\dots1)1\Big)=\Big(\Big(\underbrace{0\dots0}_{n-2}11\Big)\underbrace{1\dots1}_{n-1}\Big). \tag{3.22}$$

In order to avoid the contradiction $(1\dots1)=1$ we have

$$\Big(\underbrace{0\dots0}_{n-2}11\Big)=0, \tag{3.23}$$

$$\Big(0\underbrace{1\dots1}_{n-1}\Big)=1. \tag{3.24}$$

Claim 1. $(\underbrace{0\dots0}_{n-2k}\underbrace{1\dots1}_{2k})=0$ for $k=1,\dots,\lfloor(n-1)/2\rfloor$.

For $k=1$ this is (3.23). Assume the claim has been proved for $1\le k\le\lfloor(n-3)/2\rfloor$. Then

$$\begin{aligned}0&=\Big(\underbrace{0\dots0}_{n-2k}\underbrace{1\dots1}_{2k}\Big)\underset{(3.23)}{=}\Big(\underbrace{0\dots0}_{n-2k-1}\Big(\underbrace{0\dots0}_{n-2}11\Big)\underbrace{1\dots1}_{2k}\Big)\\&=\Big((0\dots0)\underbrace{0\dots0}_{n-2k-3}\underbrace{1\dots1}_{2k+2}\Big)=\Big(\underbrace{0\dots0}_{n-2k-2}\underbrace{1\dots1}_{2k+2}\Big),\end{aligned} \tag{3.25}$$

that is, the claim is true for $k+1$.

Claim 2. $(\underbrace{0\dots0}_{2k-1}\underbrace{1\dots1}_{n-2k+1})=1$ for $k=1,\dots,\lfloor(n-1)/2\rfloor$.

For $k = 1$ this is (3.24). Assume that the claim has been proved for $1 \le k \le \lfloor (n-3)/2 \rfloor$. Then

$$\begin{aligned}1 &= \Big(\underbrace{0\ldots0}_{2k-1}\underbrace{1\ldots1}_{n-2k+1}\Big) \underset{(3.24)}{=} \Big(\underbrace{0\ldots0}_{2k-1}\Big(0\underbrace{1\ldots1}_{n-1}\Big)\underbrace{1\ldots1}_{n-2k}\Big) \\ &= \Big(\underbrace{0\ldots0}_{2k}(1\ldots1)\underbrace{1\ldots1}_{n-2k-1}\Big) = \Big(\underbrace{0\ldots0}_{2k+1}\underbrace{1\ldots1}_{n-2k-1}\Big),\end{aligned} \tag{3.26}$$

that is, the claim is true for $k+1$.

If n is odd, then the two claims contradict each other. So n is even and according to Lemma 3.2, the premises, and Claims 1 and 2 a product is 0 if and only if it contains an even number of zeros; that is, $\mathcal{A}$ is of type **G**.

□

Lemma 3.6. *If* $(0\ldots0) = 0$ *and* $(1\ldots1) = 1$ *and*

$$\Big(\underbrace{0\ldots0}_{n-1}1\Big) = 1, \qquad \Big(1\underbrace{0\ldots0}_{n-1}\Big) = 0, \tag{3.27}$$

then $\mathcal{A}$ *is isomorphic to type* **R**.

Proof. For all $x_1, x_2, \ldots, x_{n-1} \in \{0,1\}$ we have

$$(x_1\cdots x_{n-1}0) = \Big(x_1\cdots x_{n-1}\Big(1\underbrace{0\ldots0}_{n-1}\Big)\Big) = \Big((x_1\cdots x_{n-1}1)\underbrace{0\ldots0}_{n-1}\Big) = 0 \tag{3.28}$$

$$\begin{aligned}(x_1\cdots x_{n-1}1) &= \Big(x_1\cdots x_{n-1}\Big(\underbrace{0\ldots0}_{n-1}1\Big)\Big) = \Big((x_1\cdots x_{n-1}0)\underbrace{0\ldots0}_{n-2}1\Big) \\ &\underset{(3.28)}{=} \Big(\underbrace{0\ldots0}_{n-1}1\Big) = 1.\end{aligned} \tag{3.29}$$

Thus $\mathcal{A}$ is isomorphic to type **R**. □

If we had instead of $(0\ldots01) = 1$ and $(10\ldots0) = 0$ the alternative conditions $(1\ldots10) = 0$ and $(01\ldots1) = 1$ in Lemma 3.6, we also would obtain type **R** since exchanging 0 and 1 does not affect this type. On the other hand, if we exchanged the order of the entries of all products, that is, if we had $(0\ldots01) = 0$ and $(10\ldots0) = 1$ (or $(1\ldots10) = 1$ and $(01\ldots1) = 0$), $\mathcal{A}$ would be isomorphic to type **L**.

By a similar argument (transposition of 0 and 1) one could change the conditions $(0\ldots01) = 0$ and $(10\ldots0) = 0$ in the next lemma into $(1\ldots10) = 1$ and $(01\ldots1) = 1$.

Lemma 3.7. *If* $(0\ldots0)=0$ *and* $(1\ldots1)=1$ *and*

$$\Big(\underbrace{0\ldots0}_{n-1}1\Big)=0,\qquad \Big(1\underbrace{0\ldots0}_{n-1}\Big)=0, \tag{3.30}$$

then $\mathcal{A}$ *is isomorphic to type* $\boldsymbol{A}$.

Proof. For all $x_1,x_2,\ldots,x_n\in\{0,1\}$ we have

$$(0x_2\cdots x_n)=\left(\left(\underbrace{0\ldots0}_{n-1}1\right)x_2\cdots x_n\right)=\left(\underbrace{0\ldots0}_{n-1}(1x_2\cdots x_n)\right)=0, \tag{3.31}$$

$$(x_1\cdots x_{n-1}0)=\left(x_1\cdots x_{n-1}\left(1\underbrace{0\ldots0}_{n-1}\right)\right)=\left((x_1\cdots x_{n-1}1)\underbrace{0\ldots0}_{n-1}\right)=0, \tag{3.32}$$

and for arbitrary $1\le k\le n-2$

$$\begin{aligned}\left(\underbrace{1\ldots1}_{k}0x_{k+2}\cdots x_{n-1}1\right)&=\left(\underbrace{1\ldots1}_{k}\left(1\underbrace{0\ldots0}_{n-1}\right)x_{k+2}\cdots x_{n-1}1\right)\\&=\left(\underbrace{1\ldots1}_{k+1}\underbrace{0\ldots0}_{n-k-2}\left(\underbrace{0\ldots0}_{k+1}x_{k+2}\cdots x_{n-1}1\right)\right)\\&\underset{(3.31)}{=}\left(\underbrace{1\ldots1}_{k+1}\underbrace{0\ldots0}_{n-k-1}\right)\underset{(3.32)}{=}0.\end{aligned} \tag{3.33}$$

Thus $\mathcal{A}$ is isomorphic to type **A**. □

Lemma 3.8. *If* $(0\ldots0)=0$ *and* $(1\ldots1)=1$ *and*

$$\Big(\underbrace{0\ldots0}_{n-1}1\Big)=1,\qquad \Big(1\underbrace{0\ldots0}_{n-1}\Big)=1,\qquad \Big(\underbrace{1\ldots1}_{n-1}0\Big)=0,\qquad \Big(0\underbrace{1\ldots1}_{n-1}\Big)=0, \tag{3.34}$$

then n *is odd and* $\mathcal{A}$ *is isomorphic to type* $\boldsymbol{G0}$.

Proof. We need some preparations before realizing that for all types of entries of a product the values are fixed by the premises. Note that we may not use the remark, since the algebra restricted on the main diagonal is the identical combination.

For $n = 2$ the premises

$$\begin{aligned} \Big(\underbrace{0\ldots0}_{n-1}1\Big) &= 1, \\ \Big(0\underbrace{1\ldots1}_{n-1}\Big) &= 0 \end{aligned} \tag{3.35}$$

are contradictory. So assume that $n \geq 3$. We will now establish three claims.

Claim 1. $(\underbrace{0\ldots0}_{n-2}11) = 0$.

Assume that the assertion is not true, that is, that

$$\Big(\underbrace{0\ldots0}_{n-2}11\Big) = 1. \tag{3.36}$$

Then we obtain the contradiction

$$\begin{aligned} 0 = \Big(0\underbrace{1\ldots1}_{n-1}\Big) &= \Big(0\Big(\underbrace{0\ldots0}_{n-2}11\Big)\underbrace{1\ldots1}_{n-2}\Big) \\ &= \Big(\Big(\underbrace{0\ldots0}_{n-1}1\Big)\underbrace{1\ldots1}_{n-1}\Big) = (1\ldots1) = 1. \end{aligned} \tag{3.37}$$

This proves Claim 1.

Claim 2. $(\underbrace{0\ldots0}_{n-2k+1}\,\underbrace{1\ldots1}_{2k-1}) = 1$ for $k = 1,\ldots,\lfloor n/2\rfloor$.

The proof is by induction. The case $k = 1$ is given by a premise. Now assume that $1 \leq k \leq \lfloor (n-2)/2\rfloor$ and that the assertion is true for k. Then by assumption and Claim 1

$$\begin{aligned} 1 = \Big(\underbrace{0\ldots0}_{n-2k+1}\,\underbrace{1\ldots1}_{2k-1}\Big) &= \Big(\underbrace{0\ldots0}_{n-2k}\Big(\underbrace{0\ldots0}_{n-2}11\Big)\underbrace{1\ldots1}_{2k-1}\Big) \\ &= \Big((0\ldots0)\underbrace{0\ldots0}_{n-2k-2}\,\underbrace{1\ldots1}_{2k+1}\Big) = \Big(\underbrace{0\ldots0}_{n-2k-1}\,\underbrace{1\ldots1}_{2k+1}\Big), \end{aligned} \tag{3.38}$$

and thus the assertion is true for $k + 1$.

This proves Claim 2.

Claim 3. n is odd.

Indeed, if n was even, we would have

$$\left(0\underbrace{1\ldots1}_{n-1}\right)=0 \tag{3.39}$$

by premise and

$$\left(0\underbrace{1\ldots1}_{n-1}\right)=1 \tag{3.40}$$

by Claim 2 which is a contradiction.

This proves Claim 3.

We are now ready to prepare the main argument in the proof of Lemma 3.8. If we write $(\cdots x_1x_2\cdots x_k\cdots)$, the dots at the beginning and at the end denote an arbitrary number of arbitrary entries, so that the total number of entries is n. Let $2\le b\le n-1$ be even. By Claim 2 we obtain

$$\begin{aligned}\left(\ldots\underbrace{0\ldots0}_{b}\cdots\right)&=\left(\ldots\underbrace{0\ldots0}_{b-1}\left(0\underbrace{1\ldots1}_{n-1}\right)\ldots\right)\\&=\left(\ldots\left(\underbrace{0\ldots0}_{b}\underbrace{1\ldots1}_{n-b}\right)\underbrace{1\ldots1}_{b-1}\ldots\right)=\left(\ldots\underbrace{1\ldots1}_{b}\ldots\right).\end{aligned} \tag{3.41}$$

According to (3.41) we may shift even numbers of zeros (resp., ones) as ones (resp., zeros) to the left without changing the value of a product, so that every product can be written in an equivalent *normal form*:

$$\left(\underbrace{0\ldots0}_{n-2k}\underbrace{\underline{10}\ldots\underline{10}}_{k}\right),\quad k=0,1,2,\ldots,\frac{n-1}{2}, \tag{3.42}$$

$$\left(\underbrace{1\ldots1}_{n-2k}\underbrace{\underline{01}\ldots\underline{01}}_{k)}\right),\quad k=0,1,2,\ldots,\frac{n-1}{2}. \tag{3.43}$$

For example, the product (0111011) can be transformed in three steps into the normal form:

$$\left(01110\underline{11}\right) = \left(0111\underline{000}\right) = \left(0\underline{1111}10\right) = (0000010). \tag{3.44}$$

The next claims determine the values of these normal forms.

Claim 4. $(\underbrace{1\ldots1}_{n-2a}\underbrace{\underline{01}\ldots\underline{01}}_{a}) = 0$ for $a \in \{1,\ldots,n-1/2\}$ odd and $(\underbrace{1\ldots1}_{n-2b}\underbrace{\underline{01}\ldots\underline{01}}_{b}) = 1$ for $b \in \{1,\ldots,(n-1)/2\}$ even.

Claim 5. $(\underbrace{0\ldots0}_{n-2a}\underbrace{\underline{10}\ldots\underline{10}}_{a}) = 1$ for $a \in \{1,\ldots,n-1/2\}$ odd and $(\underbrace{0\ldots0}_{n-2b}\underbrace{\underline{10}\ldots\underline{10}}_{b}) = 0$ for $b \in \{1,\ldots,(n-1)/2\}$ even.

By symmetry arguments it is sufficient to prove Claim 4. We use the same kind of double-step-induction as in the proof of Claim 1.

Suppose that

$$\left(\underbrace{1\ldots1}_{n-2}01\right) = 1. \tag{3.45}$$

Then we obtain the contradiction

$$\begin{aligned} 0 &= \left(\underbrace{1\ldots1}_{n-1}0\right) = \left(\underbrace{1\ldots1}_{n-2}\left(\underbrace{1\ldots1}_{n-2}01\right)\left(0\underbrace{1\ldots1}_{n-1}\right)\right) \\ &= \left(\underbrace{1\ldots1}_{n-3}\left(\underbrace{1\ldots1}_{n-1}0\right)\left(10\underbrace{1\ldots1}_{n-2}\right)1\right) \underset{(3.41)}{=} \left(\underbrace{1\ldots1}_{n-3}\left(\underbrace{1\ldots1}_{n-1}0\right)\left(\underbrace{1\ldots1}_{n-2}01\right)1\right) \\ &= \left(\underbrace{1\ldots1}_{n-3}\left(\underbrace{1\ldots1}_{n-1}0\right)11\right) = \left(\underbrace{1\ldots1}_{n-2}\left(\underbrace{1\ldots1}_{n-2}01\right)1\right) = (1\ldots1) = 1. \end{aligned} \tag{3.46}$$

Thus

$$\left(\underbrace{1\ldots1}_{n-2}01\right) = 0. \tag{3.47}$$

Now let $B \geq 2$ be even and, the assertion be true for all $a, b < B$. Then

$$\begin{aligned}
\left(\underbrace{1\dots1}_{n-2B}\underbrace{\underline{01}\dots\underline{01}}_{B}\right) &= \left(\underbrace{1\dots1}_{n-2B}\underbrace{\underline{01}\dots\underline{01}}_{B-1}\left(0\underbrace{1\dots1}_{n-1}\right)1\right) \\
&= \left(\underbrace{1\dots1}_{n-2B}0\left(\underbrace{\underline{10}\dots\underline{10}}_{B-1}\underbrace{1\dots1}_{n-2B+2}\right)\underbrace{1\dots1}_{2B-2}\right) \\
&\underset{(3.41)}{=} \left(\underbrace{1\dots1}_{n-2B}0\left(\underbrace{1\dots1}_{n-2B+2}\underbrace{\underline{01}\dots\underline{01}}_{B-1}\right)\underbrace{1\dots1}_{2B-2}\right) \\
&= \left(\underbrace{1\dots1}_{n-2B}00\underbrace{1\dots1}_{2B-2}\right) \underset{(3.41)}{=} (1\dots1) = 1.
\end{aligned} \tag{3.48}$$

Finally let $A \geq 3$ be odd and, the assertion be true for all $a, b < A$. Then

$$\begin{aligned}
\left(\underbrace{1\dots1}_{n-2A}\underbrace{\underline{01}\dots\underline{01}}_{A}\right) &= \left(\underbrace{1\dots1}_{n-2A}\underbrace{\underline{01}\dots\underline{01}}_{A-1}\left(0\underbrace{1\dots1}_{n-1}\right)1\right) \\
&= \left(\underbrace{1\dots1}_{n-2A}0\left(\underbrace{\underline{10}\dots\underline{10}}_{A-1}\underbrace{1\dots1}_{n-2A+2}\right)\underbrace{1\dots1}_{2A-2}\right) \\
&\underset{(3.41)}{=} \left(\underbrace{1\dots1}_{n-2A}0\left(\underbrace{1\dots1}_{n-2A+2}\underbrace{\underline{01}\dots\underline{01}}_{A-1}\right)\underbrace{1\dots1}_{2A-2}\right) \\
&= \left(\underbrace{1\dots1}_{n-2A}0\underbrace{1\dots1}_{2A-1}\right) \underset{(3.41)}{=} \left(\underbrace{1\dots1}_{n-2}01\right) = 0.
\end{aligned} \tag{3.49}$$

This completes the proof of Claim 4. Now the value of every product is determined. Observe that these values are 0 if and only if the number of zero entries is odd; so $\mathcal{A}$ is isomorphic to type **G0**.

This completes the proof of the lemma. □

Now we continue with the proof of the main theorem. As described in the beginning of this subsection and by the notice between Lemma 3.6 and Lemma 3.7 the case distinctions are complete, which proves Theorem 3.1.

4. The Columns with Several Elements

We have seen that, for any fixed n, the number of nonisomorphic associative mono-n-ary algebras with 2 elements is bounded by the global constant 6, which does not depend on n. A classification of the associative mono-n-ary algebras with 3 elements has not been completed successfully until now. It would be an interesting question whether the number of these algebras, for fixed n, is bounded by a global constant independent from n.

If we consider the case of 4 or more elements, the numbers in the column of Table 1 are not bounded but at least exponentially growing. Indeed every mono-n-ary algebra with at least four elements 0, 1, 2, 3 of the following type is associative.

(i) If there is a 0 or an 1 entry, then the product is 0.

(ii) Otherwise the product is 0 or 1.

There are at least 2^n such algebras, and at least 2^{n-1} pairwise nonisomorphic among them.

The question arises whether or not a classification of the associative mono-n-ary algebras with more than 3 elements in finitely many series of algebras exists.

5. An Application: Control Bits

As an application of Theorem 3.1 we study *recursive allocation of control bits* for a given word of length $1 + \ell(n-1)$ over the alphabet $\{0,1\}$ *taking into consideration equal inner structures* of length n.

To be precise we want to assign a single control bit i to a very long string $a_1 a_2 \cdots a_{1+\ell(n-1)}$, $(n \geq 2,\ \ell \gg 1)$. Regarding our available memory space we do not want to use a map $\{0,1\}^{1+\ell(n-1)} \to \{0,1\}$, but only a map

$$\mu : \{0,1\}^n \longrightarrow \{0,1\} \tag{5.1}$$

which we apply recursively on substrings, replacing these substrings by the value of the mapping. For reasons of efficiency it might be advantageous not to choose the substrings in canonical order, rather in a randomlike order. This may be the case if it is cheaper to find and calculate two identical substrings than to calculate two different substrings. In order to have for each possible word a well-defined control bit i (i.e., independent of the choice of the substrings) we must claim that μ is associative (in the left-right sense). So only associative mono-n-ary algebras $(\{0,1\}, \mu)$ are a solution to this (very special) control bit allocation problem.

From Theorem 3.1 we know that there are only 8 of these algebras. (Note that type **A** and type **0** appear twice since their automorphism group is trivial.) The control bits resulting from type **0**, type **A**, type **L**, and type **R** are somewhat pathological, for example, for type **L** it is simply the repetition of the first bit of the string. In this context the commonly used control bits are those depending on the parity of the stringsum and stringlength produced by the polyadic groups **G0** and **G1**, and by Theorem 3.1 there is no sensible alternative.

The situation changes for an alphabet with $k \geq 3$ letters. In this case we should speak rather of control characters than of control bits. Even there the majority of associative mono-n-ary algebras which are not polyadic groups will lead to pathological control characters, see the algebras in Section 4. But there exist associative mono-n-ary algebras that are not polyadic groups with interesting associated control characters, for example, for $n = 2$ and $k = 3$ consider the addition of a neutral element to the group $\mathbb{Z}/2\mathbb{Z}$:

$$\begin{array}{c|ccc} \mu & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{array} \tag{5.2}$$

Although the control character produced by this semigroup contains hardly any information about the number of the letters 2 and 0 in the string it provides a lot of information about letter 1, namely the parity of its occurrence.

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