Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 814861, 14 pages
doi:10.1155/2009/814861

Research Article

$\mathcal{N}$-Fuzzy Ideals in Ordered Semigroups

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Received 9 July 2009; Accepted 26 October 2009

Recommended by David Dobbs

We introduce the concept of $\mathcal{N}$-fuzzy left (right) ideals in ordered semigroups and characterize ordered semigroups in terms of $\mathcal{N}$-fuzzy left (right) ideals. We characterize left regular (right regular) and left simple (right simple) ordered semigroups in terms of $\mathcal{N}$-fuzzy left ($\mathcal{N}$-fuzzy right) ideals. The semilattice of left (right) simple semigroups in terms of $\mathcal{N}$-fuzzy left (right) ideals is discussed.

1. Introduction

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \to [0,1]$, where $[0,1]$ is the usual closed interval of real numbers. This fundamental concept was first introduced by Zadeh in his pioneering paper [1] of 1965, which provides a natural framework for the generalizations of some basic notions of algebra, for example, set theory, group theory, ring theory, groupoids, real analysis, measure theory, topology, and differential equations, and so forth. Rosenfeld (see [2]) was the first who introduced the concept of a fuzzy set in a group. The concept of a fuzzy ideal in semigroups was first developed by Kuroki (see [3–8]). He studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals, and fuzzy semiprime ideals of semigroups. Fuzzy ideals and Green's relations in semigroups were studied by McLean and Kummer in [9]. Dib and Galham in [10] introduced the definitions of a fuzzy groupoid and a fuzzy semigroup and studied fuzzy ideals and fuzzy bi-ideals of a fuzzy semigroup. Ahsan et al. in [11] characterized semisimple semigroups in terms of fuzzy ideals. A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [12], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines, and fuzzy languages. The monograph by Mordeson and Malik (see [13]) deals with the applications of fuzzy approach to the concept of automata and formal languages. Fuzzy sets in ordered semigroups/ordered groupoids were first introduced by Kehayopulu and Tsingelis in [14]. They also introduced the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups in [14, 15].

In [16], Shabir and Khan introduced the concept of a fuzzy generalized bi-ideal of ordered semigroups and characterized different classes of ordered semigroups by using fuzzy generalized bi-ideals. They also gave the concept of fuzzy left (resp., bi-) filters in ordered semigroups and gave the relations of fuzzy bi-filters and fuzzy bi-ideal subsets of ordered semigroups in [17].

In this paper, we introduce the concept of $\mathcal{N}$-fuzzy left (resp., right) ideals and characterize regular, left, and right simple ordered semigroups and completely regular ordered semigroups in terms of $\mathcal{N}$-fuzzy left (resp., right) ideals. In this respect, we prove that a regular ordered semigroup S is left simple if and only if every $\mathcal{N}$-fuzzy left ideal f of S is a constant function. We also prove that S is left regular if and only if for every $\mathcal{N}$-fuzzy left ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$. Next we characterize semilattice of left simple ordered semigroups in terms of $\mathcal{N}$-fuzzy left ideals. We prove that an ordered semigroup S is a semilattice of left simple semigroups if and only if for every $\mathcal{N}$-fuzzy left ideal f of S, $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$.

2. Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S, \cdot, \leq)$ in which

(OS1) $(S, \cdot)$ is a semigroup,

(OS2) $(S, \leq)$ is a poset,

(OS3) $(\forall a, b, x \in S)$ $(a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb)$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of S (see [14]) if

(i) $SA \subseteq A$ (resp., $AS \subseteq A$),

(ii) $(\forall a \in A)$ $(\forall b \in S)$ $(b \leq a \Rightarrow b \in A)$.

A is called a *two-sided ideal* or simply an *ideal* if A is both left and right ideal of S.

For $A \subseteq S$, denote $(A] := \{t \in S \mid t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $\{(a]\}$. Let $A, B \subseteq S$, then $A \subseteq (A]$, $(A](B] \subseteq (AB]$, and $((A]] = (A]$.

Let S be an ordered semigroup and f a fuzzy subset of S. Then f is called a *fuzzy left* (resp., *right*) *ideal* of S if

(i) $(\forall x, y \in S)$ $(x \leq y \Rightarrow f(x) \geq f(y))$,

(ii) $(x, y \in S)$ $(f(xy) \geq f(y)$ (resp., $f(xy) \geq f(x)))$.

A fuzzy left and right ideal of S is called a *fuzzy two-sided ideal* or simply a *fuzzy ideal* of S.

3. $\mathcal{N}$-Fuzzy Left (Resp., Right) Ideals

Let S be an ordered semigroup. By a *negative fuzzy subset* (briefly *$\mathcal{N}$-fuzzy subset*) f of S we mean a *mapping* $f : S \to [-1, 0]$.

Definition 3.1. Let $(S, \cdot, \leq)$ be an ordered semigroups and f an $\mathcal{N}$-fuzzy subset of S. Then f is called an *$\mathcal{N}$-fuzzy left* (resp., right) ideal of S if

(1) $(\forall x, y \in S)$ $(x \leq y \Rightarrow f(x) \leq f(y))$,

(2) $(\forall x, y \in S)$ $(f(xy) \leq f(y)$ (resp., $f(xy) \leq f(x)))$.

An $\mathcal{N}$-fuzzy left and right ideal f of S is called an *$\mathcal{N}$-fuzzy two-sided ideal* of S.

For any $\mathcal{N}$-fuzzy subset f of S and $t \in [-1, 0)$ the set

$$L(f;t) := \{x \in S \mid f(x) \leq t\} \tag{3.1}$$

is called the $\mathcal{N}$-level subset of f.

Theorem 3.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. An $\mathcal{N}$-fuzzy subset f of S is an $\mathcal{N}$-fuzzy left (resp., right) ideal of S if and only if it satisfies*

$$\left(\forall t \in [-1,0)\ L(f;t)(\neq \emptyset) \text{ is a left ideal iff } f \text{ is an } \mathcal{N}\text{-fuzzy left ideal}\right). \tag{3.2}$$

Proof. Suppose that f is an $\mathcal{N}$-fuzzy left ideal of S. Let $x, y \in S$ be such that $x \leq y$. If $y \in L(f;t)$, then $f(y) \leq t$. Since $x \leq y$, we have $f(x) \leq f(y)$ and $f(x) \leq t$ and we have $x \in L(f;t)$. Let $x, y \in S$ be such that $y \in L(f;t)$. Then $f(y) \leq t$, since $f(xy) \leq f(y)$. Then one has $f(xy) \leq t$ implies that $xy \in L(f;t)$. Thus $SL(f;t) \subseteq L(f;t)$.

Conversely, assume that for all $t \in [-1,0)$ such that $L(f;t) \neq \emptyset$, the set $L(f;t)$ is a left ideal of S. Let $x, y \in S$ be such that $x \leq y$. If $f(y) = 0$, then since $f(x) \leq 0$ for all $x \in S$, we have $f(y) \leq f(x)$. If $f(y) = t$, then $y \in L(f;t)$ and since $x \leq y \in L(f;t)$, and $L(f;t)$ is a left ideal of S, we have $x \in L(f;t)$ and so $f(x) \leq t = f(y)$. Let $x, y \in S$. If $f(y) = 0$, then since $f(xy) \leq 0$ for all $x, y \in S$ we have $f(xy) \leq f(y)$. If $f(y) = t$, then $y \in L(f;t)$ and since $L(f;t)$ is a left ideal of S, we have $xy \in L(f;t)$. Then $f(xy) \leq t = f(y)$. □

By left-right dual of the above theorem, we have the following theorem.

Theorem 3.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup. An $\mathcal{N}$-fuzzy subset f of S is an $\mathcal{N}$-fuzzy left (resp., right) ideal of S if and only if it satisfies*

$$\left(\forall t \in [-1,0)\ L(f;t)(\neq \emptyset) \text{ is a right ideal iff } f \text{ is an } \mathcal{N}\text{-fuzzy right ideal}\right). \tag{3.3}$$

Example 3.4. Let $S = \{a, b, c, d, e, f\}$ be the ordered semigroup defined by the multiplication and the order as follows:

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

(3.4)

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,e), (f,f)\}.$$

Then left ideals of S are $\{a\}, \{d\}, \{a,b\}, \{a,d\}, \{a,b,d\}, \{a,b,c,d\}, \{a,b,d,e,f\}$, and S (see [18]). Define $f : S \to -1,0]$ by

$$f(a) = -0.8, \quad f(b) = -0.6, \quad f(d) = -0.5, \quad f(c) = -0.4, \quad f(e) = f(f) = -0.2. \tag{3.5}$$

Then

$$L(f;t) := \begin{cases} S & \text{if } t \in [-0.2, 0), \\ \{a,b,c,d\} & \text{if } t \in [-0.4, -0.2), \\ \{a,b,d\} & \text{if } t \in [-0.5, -0.4), \\ \{a,b\} & \text{if } t \in [-0.6, -0.5), \\ \{a\} & \text{if } t \in [-0.8, -0.6), \\ \emptyset & \text{if } t \in [-1, -0.8). \end{cases} \tag{3.6}$$

Then $L(f;t)$ is a left ideal of S, and by Theorem 3.2, f is an $\mathcal{N}$-fuzzy left ideal of S.

Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. The *characteristic $\mathcal{N}$-function* $\kappa_A : S \to \{-1,0\}$ of A is defined by

$$\kappa_A : S \longrightarrow [-1,0], \quad x \longmapsto \kappa_A(x) := \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{3.7}$$

Theorem 3.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the followings are equivalent.*

(i) *A is a left (resp., right) ideal of S.*

(ii) *The characteristic $\mathcal{N}$-function*

$$\kappa_A(x) = \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases} \tag{3.8}$$

of A is an $\mathcal{N}$-fuzzy left (resp., right) ideal of S.

Proof. (i)⇒(ii) Suppose that A is a left ideal of S. Let $x, y \in S$, $x \leq y$. If $y \in A$, then $\kappa_A(y) = -1$. Since $x \leq y \in A$ and A is a left ideal of S, we have $x \in A$. Then $\kappa_A(x) = -1$ and hence $\kappa_A(x) \leq \kappa_A(y)$. If $y \notin A$, then $\kappa_A(y) = 0$. Since κ_A is an $\mathcal{N}$-fuzzy subset of S, we have $\kappa_A(x) \leq 0$ for all $x \in S$. Hence $\kappa_A(x) \leq \kappa_A(y)$.

Let $x \in S$ and $y \in A$. Then $\kappa_A(y) = -1$. Since A is a left ideal of S and $y \in A$, we have $xy \in A$. Then $\kappa_A(xy) = -1$, and we have $\kappa_A(xy) \leq \kappa_A(y)$. If $y \notin A$. Then $\kappa_A(y) = 0$. Since $\kappa_A(xy) \leq 0$ for all $x, y \in S$. Hence $\kappa_A(xy) \leq \kappa_A(y)$. Thus κ_A is an $\mathcal{N}$-fuzzy left ideal of S.

(ii)⇒(i) Assume that

$$\kappa_A(x) = \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases} \tag{3.9}$$

is an $\mathcal{N}$-fuzzy left ideal of S. Let $x, y \in S$, $x \leq y$. If $y \in A$, then $\kappa_A(y) = -1$. Since $x \leq y$, we have $\kappa_A(x) \leq \kappa_A(y)$. Then $\kappa_A(x) = -1$ and we have $x \in A$.

Let $x \in S$ and $y \in A$. Then $\kappa_A(y) = -1$. Since κ_A is an $\mathcal{N}$-fuzzy left ideal of S, we have $\kappa_A(xy) \leq \kappa_A(y)$. Hence $\kappa_A(xy) = -1$ and $xy \in A$. Thus A is a left ideal of S. □

A subset T of an ordered semigroup S is called *semiprime* (see [15]) if for every $a \in S$ such that $a^2 \in T$, we have $a \in T$. Equivalent definition: for each subset A of S such that $A^2 \subseteq T$, we have $A \subseteq T$.

4. Characterization of Left Simple and Left Regular Ordered Semigroups

Lemma 4.1 (cf. [15]). *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the followings are equivalent.*

(i) *$(x)_{\mathcal{N}}$ is a left simple subsemigroup of S, for every $x \in S$.*

(ii) *Every left ideal of S is a right ideal of S and semiprime.*

An ordered semigroup S is regular (see [19]) if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.

Equivalent definitions are

(1) *$(\forall a \in S)$ $(a \in (aSa])$,*

(2) *$(\forall A \subseteq S)$ $(A \subseteq (ASA])$.*

An ordered semigroup S is left (resp., right) simple (see [15]) if for every left (resp., right) ideal A of S, one has $A = S$, and S is called simple if it is left simple and right simple.

Lemma 4.2 (cf. [15]). *An ordered semigroup $(S, \cdot, \leq)$ is left (resp., right) simple if and only if for every $a \in S$, $(Sa] = S$ (resp., $(aS] = S$).*

Theorem 4.3. *For a regular ordered semigroup S, the following conditions are equivalent.*

(i) *S is left simple.*

(ii) *Every $\mathcal{N}$-fuzzy left ideal of S is a constant $\mathcal{N}$-function.*

Proof. (i)⇒(ii) Let S be a left simple ordered semigroup, f an $\mathcal{N}$-fuzzy left ideal of S, and $a \in S$. We consider the set

$$E_S := \left\{ e \in S \mid e^2 \geq e \right\}. \tag{4.1}$$

Then $E_S \neq \emptyset$. In fact, since S is regular and $a \in S$, there exists $x \in S$ such that $a \leq axa$. It follows from (OS3) that

$$(ax)^2 = (axa)x \geq ax, \tag{4.2}$$

and so $ax \in E_S$ and hence $E_S \neq \emptyset$.

(1) f is a constant $\mathcal{N}$-function on E_S. Let $t \in E_S$. Since S is left simple and $t \in S$ we have $(St] = S$. Since $e \in S$, $e \in (St]$, so there exists $z \in S$ such that $e \leq zt$. Hence $e^2 \leq (zt)(zt) = (ztz)t$. Since f is an $\mathcal{N}$-fuzzy left ideal of S, we have

$$f\left(e^2\right) \leq f((ztz)t) \leq f(t). \tag{4.3}$$

Since $e \in E_S$, we have $e^2 \geq e$. Since f is an $\mathcal{N}$-fuzzy left ideal of S, we have $f(e) \leq f(e^2)$. Thus $f(e) \leq f(t)$. Besides, since S is left simple and $e \in S$, we have $(Se] = S$. Since $t \in S = (Se]$, so there exists $y \in S$ such that $t \leq ye$. Hence $t^2 \leq (ye)(ye) = (yey)e$. Since f is an $\mathcal{N}$-fuzzy left ideal of S, we have

$$f\left(t^2\right) \leq f((yey)e) \leq f(e). \tag{4.4}$$

On the other hand, $t \in E_S$, we have $t^2 \geq t$ and so $f(t) \leq f(t^2) \leq f((yey)e) \leq f(e)$. Hence $f(t) = f(e)$.

(2) f is a constant $\mathcal{N}$-function on S. Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa$. We consider the element $xa \in S$. Then it follows by (OS3) that

$$(xa)^2 = x(axa) \geq xa. \tag{4.5}$$

Hence $xa \in E_S$ and by (1), we have $f(xa) = f(t)$. Besides, since f is an $\mathcal{N}$-fuzzy left ideal of S, we have $f(xa) \leq f(a)$. Thus $f(t) \leq f(a)$. On the other hand, since S is left simple and $t \in S$, $S = (St]$. Since $a \in S$, we have $a \leq st$ for some $s \in S$. Since f is an $\mathcal{N}$-fuzzy left ideal of S, we have $f(a) \leq f(st) \leq f(t)$. Thus $f(t) = f(a)$.

(ii)⇒(i) Let $a \in S$. Then the set $(Sa]$ is a left ideal of S. In fact, $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$. If $x \in (Sa]$ and $S \ni y \leq x$, then $y \in ((Sa]] = (Sa]$. Since $(Sa]$ is a left ideal of S, by Theorem 3.5, the characteristic $\mathcal{N}$-function $\kappa_{(Sa]}$ of $(Sa]$,

$$\kappa_{(Sa]} : S \longrightarrow [-1,0], \quad x \longmapsto \kappa_{(Sa]}(x) := \begin{cases} -1 & \text{if } x \in (Sa], \\ 0 & \text{if } x \notin (Sa], \end{cases} \tag{4.6}$$

is a fuzzy left ideal of S. By hypothesis, $\kappa_{(Sa]}$ is a constant $\mathcal{N}$-function; that is, there exists $c \in \{-1, 0\}$ such that

$$\kappa_{(Sa]}(x) = c, \quad \text{for every } x \in S. \tag{4.7}$$

Let $(Sa] \subset S$ and $t \in S$ be such that $t \notin (Sa]$. Then $\kappa_{(Sa]}(t) = 0$. On the other hand, since $a^2 \in (Sa]$, we have $\kappa_{(Sa]}(a^2) = 0$, a contradiction to the fact that $\kappa_{(Sa]}$ is a constant $\mathcal{N}$-mapping. Hence $S = (Sa]$. □

From left-right dual of Theorem 4.3, we have the following.

Theorem 4.4. *For a regular ordered semigroup. The following statements are equivalent.*

(1) *S is right simple.*

(2) *Every $\mathcal{N}$-fuzzy right ideal of S is a constant $\mathcal{N}$-function.*

An ordered semigroup $(S,\cdot,\leq)$ is left (resp., right) regular [20] if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp., $a \leq a^2x$).

Equivalent definitions are

(1) *$(\forall a \in S)$ $(a \in (Sa^2]$ (resp., $a \in (a^2S]$)),*

(2) *$(\forall A \subseteq S)$ $(A \subseteq (SA^2]$ (resp., $A \subseteq (A^2S]$)).*

An ordered semigroup $(S,\cdot,\leq)$ is intraregular (see [16]) if for every $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$.

Equivalent definitions are

(1) *$(\forall a \in S)$ $(a \in (Sa^2S])$,*

(2) *$(\forall A \subseteq S)$ $(A \subseteq (SA^2S])$.*

An ordered semigroup S is called completely regular (see [21]) if it is regular, left regular, and right regular.

Lemma 4.5 (cf. [21]). *An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Equivalently, $a \in (a^2Sa^2]$ for every $a \in S$.*

Theorem 4.6. *An ordered semigroup $(S,\cdot,\leq)$ is left regular if and only if for each $\mathcal{N}$-fuzzy left ideal f of S, one has*

$$f(a) = f\left(a^2\right) \quad \forall a \in S. \tag{4.8}$$

Proof. Suppose that f is an $\mathcal{N}$-fuzzy left ideal of S and let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leq xa^2$. Since f is an $\mathcal{N}$-fuzzy left ideal of S, we have

$$f(a) \leq f\left(xa^2\right) \leq f\left(a^2\right) \leq f(a). \tag{4.9}$$

Conversely, let $a \in S$. We consider the left ideal $L(a^2) = (a^2 \cup Sa^2]$ of S, generated by a^2. Then by Theorem 3.5, the characteristic $\mathcal{N}$-function

$$\kappa_{L(a^2)} : S \longrightarrow [-1,0] \mid x \longmapsto \kappa_{L(a^2)}(x) := \begin{cases} -1 & \text{if } x \in L(a^2), \\ 0 & \text{if } x \notin L(a^2) \end{cases} \tag{4.10}$$

is an $\mathcal{N}$-fuzzy left ideal of S. By hypothesis we have

$$\kappa_{L(a^2)}(a) = \kappa_{L(a^2)}\left(a^2\right). \tag{4.11}$$

Since $a^2 \in L(a^2)$, we have $\kappa_{L(a^2)}(a^2) = -1$ and $\kappa_{L(a^2)}(a) = -1$. Then $a \in L(a^2) = (a^2 \cup Sa^2]$ and $a \leq y$ for some $y \in a^2 \cup Sa^2$. If $y = a^2$, then $a \leq y = a^2 = aa = aa^2 \in Sa^2$ and $a \in (Sa^2]$. If $y = xa^2$ for some $x \in S$, then $a \leq y = xa^2 \in Sa^2$, and $a \in (Sa^2]$. □

From left-right dual of Theorem 4.6, we have the following theorem.

Theorem 4.7. *An ordered semigroup $(S, \cdot, \leq)$ is right regular if and only if for each $\mathcal{N}$-fuzzy right ideal f of S, one has*

$$f(a) = f\left(a^2\right) \quad \forall a \in S. \tag{4.12}$$

From [22] and by Theorems 4.6, 4.7, and Lemma 4.5, we have the following characterization theorem for completely regular ordered semigroups.

Theorem 4.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following statements are equivalent.*

(i) *S is completely regular.*

(ii) *For each $\mathcal{N}$-fuzzy bi-ideal f of S one has*

$$f(a) = f\left(a^2\right) \quad \forall a \in S. \tag{4.13}$$

(iii) *For each $\mathcal{N}$-fuzzy left ideal g and each $\mathcal{N}$-fuzzy right ideal h of S we have*

$$g(a) = g\left(a^2\right), \quad h(a) = h\left(a^2\right) \quad \forall a \in S. \tag{4.14}$$

An ordered semigroup $(S, \cdot, \leq)$ is called left (resp., right) duo if every left (resp., right) ideal of S is a two-sided ideal of S. An ordered semigroup S is called duo if it is both left and right duo.

Definition 4.9. An ordered semigroup $(S, \cdot, \leq)$ is called *$\mathcal{N}$-fuzzy left* (resp., *right*) *duo* if every $\mathcal{N}$-fuzzy left (resp., right) ideal of S is an $\mathcal{N}$-fuzzy two-sided ideal of S. An ordered semigroup S is called *$\mathcal{N}$-fuzzy duo* if it is both $\mathcal{N}$-fuzzy left and $\mathcal{N}$-fuzzy right duo.

Theorem 4.10. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup. Then the followings are equivalent.*

(i) *S is left duo.*

(ii) *S is $\mathcal{N}$-fuzzy left duo.*

Proof. (i)$\Rightarrow$(ii) Let S be left duo and f an $\mathcal{N}$-fuzzy left ideal of S. Let $a, b \in S$. Then the set $(Sa]$ is a left ideal of S. In fact, $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$ and if $x \in (Sa]$ and $S \ni y \leq x$, then $y \in ((Sa]] = (Sa]$. Since S is left duo, then $(Sa]$ is a two-sided ideal of S. Since S is regular, there exists $x \in S$ such that $a \leq axa$,

$$ab \leq (axa)b \in (aSa)b \subseteq (Sa)S \subseteq (Sa]S \subseteq (Sa]. \tag{4.15}$$

Thus $ab \in ((Sa]] = (Sa]$ and $ab \leq xa$ for some $x \in S$. Since f is an $\mathcal{N}$-fuzzy left ideal of S, we have

$$f(ab) \leq f(xa) \leq f(a). \tag{4.16}$$

Let $x, y \in S$ be such that $x \leq y$. Then $f(x) \leq f(y)$, because f is an $\mathcal{N}$-fuzzy left ideal of S. Thus f is an $\mathcal{N}$-fuzzy right deal of S and S is $\mathcal{N}$-fuzzy left duo.

(ii)⇒(i) Let S be $\mathcal{N}$-fuzzy left duo and A a left ideal of S. Then the characteristic $\mathcal{N}$-function κ_A of A is an $\mathcal{N}$-fuzzy left ideal of S. By hypothesis, κ_A is an $\mathcal{N}$-fuzzy right ideal of S, and by Theorem 3.5, A is a right ideal of S. Thus S is left duo. □

By the left-right dual of Theorem 4.10, we have the following.

Theorem 4.11. *Let $(S,\cdot,\leq)$ be a regular ordered semigroup. Then the followings are equivalent.*

(i) *S is right duo.*

(ii) *S is $\mathcal{N}$-fuzzy right duo.*

Theorem 4.12. *Let $(S,\cdot,\leq)$ be a regular ordered semigroup. Then the followings are equivalent.*

(i) *Every bi-ideal of S is a right ideal of S.*

(ii) *Every $\mathcal{N}$-fuzzy bi-ideal of S is an $\mathcal{N}$-fuzzy right ideal of S.*

Proof. (i)⇒(ii) Let $a,b \in S$ and f an $\mathcal{N}$-fuzzy bi-ideal of S. Then $(aSa]$ is a bi-ideal of S. In fact, $(aSa]^2 \subseteq (aSa](aSa] \subseteq (aSa]$, $(aSa]S(aSa] = (aSa](S](aSa] \subseteq (aSa]$ and if $x \in (aSa]$ and $S \ni y \leq x \in (aSa]$, then $y \in ((aSa]] = (aSa]$. Since $(aSa]$ is a bi-ideal of S, by hypothesis $(aSa]$ is right ideal of S. Since $a \in S$ and S is regular, there exists $x \in S$ such that $a \leq axa$, then

$$ab \leq (axa)b \in (aSa)S \subseteq (aSa]S \subseteq (aSa]. \tag{4.17}$$

Then $ab \leq aza$ for some $z \in S$. Since f is an $\mathcal{N}$-fuzzy bi-ideal of S, we have

$$f(ab) \leq f(aza) \leq \max\{f(a), f(a)\} = f(a). \tag{4.18}$$

Let $x,y \in S$ be such that $x \leq y$. Then $f(x) \leq f(y)$ because f is an $\mathcal{N}$-fuzzy bi-ideal of S. Thus f is an $\mathcal{N}$-fuzzy right ideal of S.

(ii)⇒(i) Let A be a bi-ideal of S. Then by Theorem 3.5, κ_A is an $\mathcal{N}$-fuzzy bi-ideal of S. By hypothesis κ_A is an $\mathcal{N}$-fuzzy right ideal of S. By Theorem 3.5, A is a right ideal of S. □

By left-right dual of Theorem 4.12, we have the following.

Theorem 4.13. *Let $(S,\cdot,\leq)$ be a regular ordered semigroup. Then the followings are equivalent.*

(i) *Every bi-ideal of S is a left ideal of S.*

(ii) *Every $\mathcal{N}$-fuzzy bi-ideal of S is an $\mathcal{N}$-fuzzy left ideal of S.*

5. Characterization of Intraregular Ordered Semigroups in Terms of $\mathcal{N}$-Fuzzy Ideals

Definition 5.1 (cf. [22]). Let $(S,\cdot,\leq)$ be an ordered semigroup and f an $\mathcal{N}$-fuzzy subset of S. Then f is called an semiprime $\mathcal{N}$-fuzzy subset of S if

$$f(a) \leq f\left(a^2\right) \quad \forall a \in S. \tag{5.1}$$

Theorem 5.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the followings are equivalent.*

(i) *A is semiprime.*

(ii) *The characteristic $\mathcal{N}$-function κ_A of A is an $\mathcal{N}$-fuzzy semiprime subset.*

Proof. (i)⇒(ii) Suppose that A is semiprime subset. Let $\kappa_A(a^2) = 0$. Since $\kappa_A(a) \leq 0$ for all $a \in S$, thus $\kappa_A(a) \leq \kappa_A(a^2)$. If $\kappa_A(a^2) = -1$, then $a^2 \in A$. Since A is semiprime, we have $a \in A$. Then $\kappa_A(a) = -1$ and; hence $\kappa_A(a) \leq \kappa_A(a^2)$.

(ii)⇒(i) Assume that κ_A is $\mathcal{N}$-fuzzy semiprime subset. Let $a \in S$ be such that $a^2 \in A$. Then $\kappa_A(a^2) = -1$. Since $\kappa_A(a) \leq \kappa_A(a^2)$, we have $\kappa_A(a) = -1$, hence $a \in A$. □

Theorem 5.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup and let f be an $\mathcal{N}$-fuzzy subsemigroup of S. Then f is an $\mathcal{N}$-fuzzy semiprime if and only if for every $a \in S$, one has*

$$f(a) = f\left(a^2\right). \tag{5.2}$$

Proof. Suppose that f is an $\mathcal{N}$-fuzzy subsemigroup of S such that f is semiprime. Let $a \in S$. Then

$$f(a) \leq f\left(a^2\right) = f(aa) \leq \max\{f(a), f(a)\} = f(a). \tag{5.3}$$

The converse is obvious. □

Theorem 5.4. *An ordered semigroup S is intraregular if and only if every $\mathcal{N}$-fuzzy ideal of S is semiprime.*

Proof. Suppose that S is intraregular and f an $\mathcal{N}$-fuzzy ideal of S. Let $a \in S$. Then $f(a) \leq f(a^2)$. In fact, since S is intraregular, there exist $x, y \in S$ such that $a \leq xa^2y = x(a^2y)$. Then

$$f(a) \leq f\left(x\left(a^2y\right)\right) \leq f\left(a^2y\right) \leq f\left(a^2\right). \tag{5.4}$$

Assume that f is an $\mathcal{N}$-fuzzy ideal of S such that $f(a) \leq f(a^2)$ for all $a \in S$. Consider the ideal $I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$ of S generated by $a^2(a \in S)$. Then by Theorem 3.5, the characteristic $\mathcal{N}$-function $\kappa_{I(a^2)}$ is an $\mathcal{N}$-fuzzy ideal of S, and by hypothesis, we have $\kappa_{I(a^2)}(a) \leq \kappa_{I(a^2)}(a^2)$. Since $a^2 \in I(a^2)$, then $\kappa_{I(a^2)}(a^2) = -1$ and $\kappa_{I(a^2)}(a) = -1 \Rightarrow a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$. Then $a \leq x$ for some $x \in a^2 \cup Sa^2 \cup a^2S \cup Sa^2S$. If $x = a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aa^2a \in Sa^2S$ and $a \in (Sa^2S]$. If $x = ya^2$ for some $y \in S$, then $a \leq ya^2 = yaa \leq y(ya^2)a = yya^2a \in Sa^2S$ and $a \in (Sa^2S]$. If $x = a^2z$, then $a \leq a^2z = aaz \leq a(a^2z)z = aa^2zz \in Sa^2S$ and $a \in (Sa^2S]$. □

6. Some Semilattices of Left Simple Ordered Semigroups in Terms of $\mathcal{N}$-Fuzzy Left Ideals

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup F of S is called *filter* (see [15]) of S if

(1) $(\forall a, b \in S)$ $(ab \in F \Rightarrow a \in F$ and $b \in F)$,

(2) $(\forall c \in S)$ $(c \geq a \in F \Rightarrow c \in F)$.

For $x \in S$, we denote by $N(x)$ the filter of S generated by x ($x \in S$) (i.e., the least filter with respect to inclusion relation containing x). $\mathcal{N}$ denotes the equivalence relation on S defined by $\mathcal{N} := \{(x,y) \in S \times S \mid N(x) = N(y)\}$ (see [15]).

Definition 6.1 (cf. [15]). Let S be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a,b) \in \sigma$ implies $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2,a) \in \sigma$ and $(ab,ba) \in \sigma$ for each $a,b \in S$. If σ is a semilattice congruence on S, then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$.

An ordered semigroup S is called a semilattice of *left simple semigroups* if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left simple subsemigroup of S for every $x \in S$.

Equivalent definition: there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left simple subsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha,\beta \in Y$, $\alpha \neq \beta$,

(2) $S = \bigcup_{\alpha\in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha,\beta \in Y$.

In ordered semigroups the semilattice congruences are defined exactly same as in the case of semigroups—without order—so the two definitions are equivalent (see [15]).

Lemma 6.2 (cf. [22]). *An ordered semigroup $(S,\cdot,\leq)$ is a semilattice of left simple semigroups if and only if for all left ideals A, B of S one has*

$$\left(A^2\right] = A, \qquad (AB] = (BA]. \tag{6.1}$$

Theorem 6.3. *An ordered semigroup $(S,\cdot,\leq)$ is a semilattice of left simple semigroups if and only if for every $\mathcal{N}$-fuzzy left ideal f of S, one has*

$$f\left(a^2\right) = f(a), \quad f(ab) = f(ba) \quad \forall a,b \in S. \tag{6.2}$$

Proof. $\Rightarrow$ (A) Let S be a semilattice of left simple semigroups. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left simple subsemigroups of S such that

(i) $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha,\beta \in Y$, $\alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha\in Y} S_\alpha$,

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha,\beta \in Y$.

Let f be an $\mathcal{N}$-fuzzy left ideal of S and $a \in S$. Then $f(a) = f(a^2)$. In fact, by Theorem 4.6, it is enough to prove that $a \in (Sa^2]$ for every $a \in S$. Let $a \in S$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have $S_\alpha = (S_\alpha a]$ and

$$a \leq xa \quad \text{for some } x \in S_\alpha. \tag{6.3}$$

Since $x \in S_\alpha$, we have $x \in (S_\alpha a]$ and $x \leq ya$ for some $y \in S_\alpha$. Thus we have

$$a \leq xa \leq (ya)a = ya^2 \tag{6.4}$$

since $y \in S$, we have $a \in (Sa^2]$.

(B) Let $a, b \in S$. Then by (A), we have

$$f(ab) = f\left((ab)^2\right) = f(a(ba)b) \geq f(ba). \tag{6.5}$$

By symmetry we can prove that $f(ba) \geq f(ab)$. Hence $f(ab) = f(ba)$.

$\Leftarrow$ Assume that for every fuzzy left ideal f of S, we have

$$f\left(a^2\right) = f(a), \quad f(ab) = f(ba) \quad \forall a, b \in S, \tag{6.6}$$

by condition (1) and Theorem 4.6, we have that S is left regular. Let A be a left ideal of S and let $a \in A$. Then $a \in S$, since S is left regular there exists $x \in S$ such that

$$a \leq xa^2 = (xa)a \in (SA)A \subseteq AA = A^2, \tag{6.7}$$

then $a \in (A^2]$ and $A \subseteq (A]$. On the other hand, since A is a left ideal of S, we have $A^2 \subseteq SA \subseteq A$, then $(A^2] \subseteq (A] = A$. Let A and B be left ideals of S and let $x \in (BA]$ then $x \leq ba$ for some $a \in A$ and $b \in B$. We consider the left ideal $L(ab)$ generated by ab, that is, the set $L(ab) = (ab \cup Sab]$. Then by Theorem 3.5, the characteristic $\mathcal{N}$-function $f_{L(ab)}$ of $L(ab)$ defined by

$$f_{L(ab)} : S \longrightarrow [0,1] \mid x \longmapsto f_{L(ab)}(x) := \begin{cases} 1 & \text{if } x \in L(ab), \\ 0 & \text{if } x \notin L(ab) \end{cases} \tag{6.8}$$

is a fuzzy left ideal of S. By hypothesis, we have $f_{L(ab)}(ab) = f_{L(ab)}(ba)$. Since $ab \in L(ab)$, we have $f_{L(ab)}(ab) = 1$ and $f_{L(ab)}(ba) = 1$ and hence $ba \in L(ab) = (ab \cup Sab]$. Then $ba \leq ab$ or $ba \leq yab$ for some $y \in S$. If $ba \leq ab$, then $x \leq ab \in AB$ and $x \in (AB]$. If $ba \leq yab$, then $x \leq yab \in (SA)B \subseteq AB$ and $x \in (AB]$. Thus $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Therefore $(AB] = (BA]$, and by Lemma 6.2, it follows that S is a semilattice of left simple semigroups. $\square$

From left-right dual of Theorem 6.3, we have the following.

Theorem 6.4. *An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of right simple semigroups if and only if for every $\mathcal{N}$-fuzzy right ideal f of S, one has*

$$f\left(a^2\right) = f(a), \quad f(ab) = f(ba) \quad \forall a, b \in S. \tag{6.9}$$

Lemma 6.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup and f an $\mathcal{N}$-fuzzy left (resp., right) ideal of S, and $a \in S$ such that $a \leq a^2$. Then $f(a) = f(a^2)$.*

Proof. Since $a \leq a^2$ and f is an $\mathcal{N}$-fuzzy left ideal of S, we have

$$f(a) \leq f\left(a^2\right) = f(aa) \leq f(a), \tag{6.10}$$

and so $f(a) = f(a^2)$. □

7. Conclusion

Here we provided the concept of an $\mathcal{N}$-fuzzy ideal in ordered semigroups and characterized some classes in terms of $\mathcal{N}$-fuzzy left (resp., right) ideals of ordered semigroups. In this regard, we provided the characterizations of left (resp., right) regular, left (resp., right) simple, and completely regular ordered semigroups in terms of $\mathcal{N}$-fuzzy left (resp., right) ideals.

In our future work we will consider $\mathcal{N}$-fuzzy prime ideals and $\mathcal{N}$-fuzzy filters in ordered semigroups and will establish the relations between them. We will also try to discuss the quotient structure of ordered semigroup in terms of $\mathcal{N}$-fuzzy ideals.

Acknowledgments

The authors would like to thank the learned referees for their valuable comments and suggestions which improved the presentation of the paper and for their interest in their work. We would also like to thank Professor Dost Muhammad for his excellent guidance and valuable suggestions during this work.

References

[1] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, pp. 338–353, 1965.
[2] A. Rosenfeld, “Fuzzy groups,” *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.
[3] N. Kuroki, “Fuzzy bi-ideals in semigroups,” *Commentarii Mathematici Universitatis Sancti Pauli*, vol. 28, no. 1, pp. 17–21, 1979.
[4] N. Kuroki, “On fuzzy ideals and fuzzy bi-ideals in semigroups,” *Fuzzy Sets and Systems*, vol. 5, no. 2, pp. 203–215, 1981.
[5] N. Kuroki, “On fuzzy semiprime ideals in semigroups,” *Fuzzy Sets and Systems*, vol. 8, no. 1, pp. 71–79, 1982.
[6] N. Kuroki, “On fuzzy semigroups,” *Information Sciences*, vol. 53, no. 3, pp. 203–236, 1991.
[7] N. Kuroki, “Fuzzy generalized bi-ideals in semigroups,” *Information Sciences*, vol. 66, no. 3, pp. 235–243, 1992.
[8] N. Kuroki, “On fuzzy semiprime quasi-ideals in semigroups,” *Information Sciences*, vol. 75, no. 3, pp. 201–211, 1993.
[9] R. G. McLean and H. Kummer, “Fuzzy ideals in semigroups,” *Fuzzy Sets and Systems*, vol. 48, no. 1, pp. 137–140, 1992.
[10] K. A. Dib and N. Galham, “Fuzzy ideals and fuzzy bi-ideals in fuzzy semi-groups,” *Fuzzy Sets and System*, vol. 92, pp. 203–215, 1997.
[11] J. Ahsan, K. Saifullah, and M. F. Khan, “Semigroups characterized by their fuzzy ideals,” *Fuzzy Systems and Mathematics*, vol. 9, no. 1, pp. 29–32, 1995.
[12] J. N. Mordeson, D. S. Malik, and N. Kuroki, *Fuzzy Semigroups*, vol. 131 of *Studies in Fuzziness and Soft Computing*, Springer, Berlin, Germany, 2003.
[13] J. N. Mordeson and D. S. Malik, *Fuzzy Automata and Languages*, Theory and Application Computational Mathematics Series, Chapman and Hall/CRC, Boca Raton, Fla, USA, 2002.

[14] N. Kehayopulu and M. Tsingelis, "Fuzzy sets in ordered groupoids," *Semigroup Forum*, vol. 65, no. 1, pp. 128–132, 2002.

[15] N. Kehayopulu and M. Tsingelis, "Fuzzy bi-ideals in ordered semigroups," *Information Sciences*, vol. 171, no. 1–3, pp. 13–28, 2005.

[16] M. Shabir and A. Khan, "Characterizations of ordered semigroups by the properties of their fuzzy generalized bi-ideals," *New Mathematics and Natural Computation*, vol. 4, no. 2, pp. 237–250, 2008.

[17] M. Shabir and A. Khan, "Fuzzy filters in ordered semigroups," *Lobachevskii Journal of Mathematics*, vol. 29, no. 2, pp. 82–89, 2008.

[18] N. Kehayopulu and M. Tsingelis, "On left regular ordered semigroups," *Semigroups Forum*, vol. 25, no. 4, pp. 609–615, 2002.

[19] N. Kehayopulu, "On regular duo ordered semigroups," *Mathematica Japonica*, vol. 37, no. 3, pp. 535–540, 1992.

[20] N. Kehayopulu and M. Tsingelis, "Regular ordered semigroups in terms of fuzzy subsets," *Information Sciences*, vol. 176, no. 24, pp. 3675–3693, 2006.

[21] N. Kehayopulu, "On completely regular ordered semigroups," *Scientiae Mathematicae*, vol. 1, no. 1, pp. 27–32, 1998.

[22] A. Khan, Y. B. Jun, and M. Shabir, "$\mathcal{N}$-fuzzy Bi-ideals in ordered semigroups," submitted.