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Research Article

Generalizing Benford's Law Using Power Laws: Application to Integer Sequences

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Many distributions for first digits of integer sequences are not Benford. A simple method to derive parametric analytical extensions of Benford's law for first digits of numerical data is proposed. Two generalized Benford distributions are considered, namely, the two-sided power Benford (TSPB) distribution, which has been introduced in Hürlimann (2003), and the new Pareto Benford (PB) distribution. Based on the minimum chi-square estimators, the fitting capabilities of these generalized Benford distributions are illustrated and compared at some interesting and important integer sequences. In particular, it is significant that much of the analyzed integer sequences follow with a high P-value the generalized Benford distributions. While the sequences of prime numbers less than 1000, respectively, 10 000 are not at all Benford or TSPB distributed, they are approximately PB distributed with high P-values of 93.3% and 99.9% and reveal after a further deeper analysis of longer sequences a new interesting property. On the other side, Benford's law of a mixing of data sets is rejected at the 5% significance level while the PB law is accepted with a 93.6% P-value, which improves the P-value of 25.2%, which has been obtained previously for the TSPB law.

1. Introduction

Since Newcomb [1] and Benford [2] it is known that many numerical data sets follow Benford's law or are closely approximated by it. To be specific, if the random variable X, which describes the first significant digit in a numerical table, is Benford distributed, then

$$P(X = d) = \log\left(1 + d^{-1}\right), \quad d \in \{1, \dots, 9\}. \tag{1.1}$$

Mathematical explanations of this law have been proposed by Pinkham [3], Cohen [4], Hill [5–9], Allart [10], Janvresse and de la Rue [11], and Kossovsky [12]. The latter author has raised some conjectures, which have been proved in some special cases by Jang et al. [13]. Other explanations of the prevalence of Benford's law exist. For example, Miller and Nigrini [14] obtain it through the study of products of random variables and Kafri [15] through the maximum entropy principle. In the recent years an upsurge of applications of Benford's law has appeared, as can be seen from the compiled bibliography by Hürlimann [16] and the recent online bibliography by Berg and Hill [17]. Among them one might mention Judge and Schechter [18], Judge et al. [19], and Nigrini and Miller [20]. As in the present paper, the latter authors also consider power laws.

Hill [7] also suggested to switch the attention to probability distributions that follow or closely approximate Benford's law. Papers along this path include Leemis et al. [21] and Engel and Leuenberger [22]. Some survival distributions, which satisfy exactly Benford's law, are known. However, there are not many simple analytical distributions, which include as special case Benford's law. Combining facts from Leemis et al. [21] and Dorp and Kotz [23] such a simple one-parameter family of distributions has been considered in Hürlimann [24]. In a sequel to this, a further generalization of Benford's law is considered.

It is important to note that many distributions for first digits of integer sequences are not Benford but are power laws or something close. Thus there is a need for statistical tests for analyzing such hypotheses. In this respect the interest of enlarged Benford laws is twofold. First, parametric extensions may provide a better fit of the data than Benford's law itself. Second, they yield a simple statistical procedure to validate Benford's law. If Benford's model is sufficiently "close" to the one-parameter extended model, then it will be retained. These points will be illustrated through our application to integer sequences.

2. Generalizing Benford's Distribution

If T denotes a random lifetime with survival distribution $S(t) = P(T \geq t)$, then the value Y of the first significant digit in the lifetime T has the probability distribution

$$P(Y = y) = \sum_{i=-\infty}^{\infty} \left\{ S\left(y \cdot 10^i\right) - S\left((y+1) \cdot 10^i\right) \right\}, \quad y \in \{1, \ldots, 9\}. \tag{2.1}$$

Alternatively, if D denotes the integer-valued random variable satisfying

$$10^D \leq T < 10^{D+1}, \tag{2.2}$$

then the first significant digit can be written in terms of T , and D as

$$Y = \left[T \cdot 10^{-D}\right] = \left[10^{\log T - D}\right], \tag{2.3}$$

where $[x]$ denotes the greatest integer less than or equal to x. In particular, if the random variable $Z = \log T - D$ is uniformly distributed as $U(0,1)$, then the first significant digit Y is exactly Benford distributed. Starting from the *uniform* random variable $W = U(0,2)$ or the *triangular* random variable $W = \text{Triangular}(0,1,2)$ with probability density function $f_W(w) = w$ if $\text{w} \in (0,1)$ and $f_W(w) = 2 - w$ if $w \in [1,2)$, one shows that the random lifetime $T = 10^W$ generates the first digit Benford distribution (Leemis et al. [21, Examples 1 and 2]).

A simple parametric distribution, which includes as special cases both the above uniform and triangular distributions, is the twosided power random variable $W = \mathrm{TSP}(\alpha, c)$ considered in Dorp and Kotz [23] with probability density function

$$f_W(w) = \begin{cases} \dfrac{c}{2}\left(\dfrac{w}{\alpha}\right)^{c-1}, & 0 < w \le \alpha, \\ \dfrac{c}{2}\left(\dfrac{2-w}{2-\alpha}\right)^{c-1}, & \alpha \le w < 2. \end{cases} \tag{2.4}$$

If $c = 1$ then $W = U(0,2)$, and if $c = 2$, $\alpha = 1$ then $W = \mathrm{Triangular}(0,1,2)$. This observation shows that the random lifetime $T = 10^{\mathrm{TSP}(1,c)}$ will generate first digit distributions closely related to Benford's distribution, at least if c is close to 1 or 2.

Theorem 2.1. *Let $W = \mathrm{TSP}(1, c)$ be the twosided power random variable with probability density function*

$$f_W(w) = \begin{cases} \dfrac{c}{2} w^{c-1}, & 0 < w \le 1, \\ \dfrac{c}{2}(2-w)^{c-1}, & 1 \le w < 2, \end{cases} \tag{2.5}$$

and let the integer-valued random variable D satisfy $D \le W < D+1$. Then the first digit random variable $Y = [10^{W-D}]$ has the one-parameter twosided power Benford (TSPB) probability density function

$$f_Y(y) = \frac{1}{2}\left\{ [\log(1+y)]^c - [\log y]^c - [1-\log(1+y)]^c + [1-\log y]^c \right\}, \quad y \in \{1,\ldots,9\}. \tag{2.6}$$

Proof. This has been shown in Hürlimann [24]. □

3. From the Geometric Brownian Motion to the Pareto Benford Law

Another interesting distribution, which also takes the form of a twosided power law, is the double Pareto random variable $W = DP(s, \alpha, \beta)$ considered in Reed [25] with probability density function

$$f_W(w) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta}\left(\dfrac{w}{s}\right)^{\beta-1}, & w \le s, \\ \dfrac{\alpha\beta}{\alpha+\beta}\left(\dfrac{w}{s}\right)^{-\alpha-1}, & w \ge s. \end{cases} \tag{3.1}$$

Recall the stochastic mechanism and the natural motivation, which generates this distribution. It is often assumed that the time evolution of a stochastic phenomena X_t involves a variable but size independent proportional growth rate and can thus be modeled by a geometric Brownian motion (GBM) described by the stochastic differential equation

$$dX = \mu \cdot X \cdot dt + \sigma \cdot X \cdot dW, \tag{3.2}$$

where dW is the increment of a Wiener process. Since the proportional increment of a GBM in time dt has a systematic component $\mu \cdot dt$ and a random white noise component $\sigma \cdot dW$, GBM can be viewed as a stochastic version of a simple exponential growth model. The GBM has long been used to model the evolution of stock prices (Black-Scholes option pricing model), firm sizes, city sizes, and individual incomes. It is well known that empirical studies on such phenomena often exhibit power-law behavior. However, the state of a GBM after a fixed time T follows a lognormal distribution, which does not exhibit power-law behavior.

Why does one observe power-law behavior for phenomena apparently evolving like a GBM? A simple mechanism, which generates the power-law behavior in the tails, consists to assume that the time T of observation itself is a random variable, whose distribution is an exponential distribution. The distribution of X_T with fixed initial state s is described by the *double Pareto distributio*n $\mathrm{DP}(s,\alpha,\beta)$ with density function (3.1), where $\alpha,\ \beta > 0$, and $\alpha,\ -\beta$ are the positive roots of the characteristic equation

$$\left(\mu - \frac{1}{2}\sigma^2\right)z + \frac{1}{2}\sigma^2 z^2 = \lambda, \tag{3.3}$$

where λ is the parameter of the exponentially distributed random variable T. Setting $s = 1$ one obtains the following generalized Benford distribution.

Theorem 3.1. *Let $W = DP(1,\alpha,\beta)$ be the double Pareto random variable with probability density function*

$$f_W(w) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta}(w)^{\beta-1}, & w \le 1, \\ \dfrac{\alpha\beta}{\alpha+\beta}(w)^{-\alpha-1}, & w \ge 1. \end{cases} \tag{3.4}$$

Let the integer-valued random variable D satisfy $D \le W < D+1$. Then the first digit random variable $Y = [10^{W-D}]$ has the two-parameter *Pareto Benford* (PB) probability density function

$$\begin{aligned} f_Y(y) = {} & \frac{\alpha}{\alpha+\beta}\left\{ [\log(1+y)]^{\beta} - [\log(y)]^{\beta} \right\} \\ & + \frac{\beta}{\alpha+\beta} \cdot \sum_{k=1}^{\infty}\left\{ [k+\log(y)]^{-\alpha} - [k+\log(1+y)]^{-\alpha} \right\}, \quad y \in \{1,\dots,9\}. \end{aligned} \tag{3.5}$$

Proof. The probability density function of $T = 10^W$ is given by

$$f_T(t) = \frac{1}{t \cdot \ln 10} \cdot f_W\left(\frac{\ln t}{\ln 10}\right) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta}\dfrac{1}{t\cdot\ln 10}\left(\dfrac{\ln t}{\ln 10}\right)^{\beta-1}, & 1 < t \le 10, \\ \dfrac{\alpha\beta}{\alpha+\beta}\dfrac{1}{t\cdot\ln 10}\left(\dfrac{\ln t}{\ln 10}\right)^{-\alpha-1}, & t > 10. \end{cases} \tag{3.6}$$

It follows that the first significant digit of T, namely, $Y = [T \cdot 10^{-D}]$, has probability density

$$f_Y(y) = \sum_{k=0}^{\infty} \int_{10^k y}^{10^k (y+1)} f_T(t) dt. \tag{3.7}$$

Making the change of variable $u = \ln t / \ln 10$, one obtains (3.5) as follows:

$$\begin{aligned} f_Y(y) &= \frac{\alpha\beta}{\alpha+\beta} \left\{ \int_{\log y}^{\log(y+1)} u^{\beta-1} du + \sum_{k=1}^{\infty} \int_{k+\log y}^{k+\log(y+1)} u^{-\alpha-1} du \right\} \\ &= \frac{\alpha\beta}{\alpha+\beta} \left\{ \frac{1}{\beta} u^{\beta} \Big|_{\log(y)}^{\log(1+y)} + \sum_{k=1}^{\infty} \frac{-1}{\alpha} u^{-\alpha} \Big|_{k+\log(y)}^{k+\log(1+y)} \right\} \\ &= \frac{\alpha}{\alpha+\beta} \left\{ [\log(1+y)]^{\beta} - [\log(y)]^{\beta} \right\} + \frac{\beta}{\alpha+\beta} \cdot \sum_{k=1}^{\infty} \left\{ [k+\log(y)]^{-\alpha} - [k+\log(1+y)]^{-\alpha} \right\}. \end{aligned} \tag{3.8}$$

□

One notes that setting $\beta = 1$ and letting α goes to infinity, the Pareto Benford distribution converges to Benford's law. Other important paper, which links Benford's law to GBMs' law on the one side, is Kontorovich and Miller [26] and to Black-Scholes' law on the other side is Schürger [27]. Another law, which includes as a special case the Benford law, is the Planck distribution of photons at a given frequency, as shown recently by Kafri [28, 29].

4. Fitting the First Digit Distributions of Integer Sequences

Minimum chi-square estimation of the generalized Benford distributions is straightforward by calculation with modern computer algebra systems. The fitting capabilities of the new distributions are illustrated at some interesting and important integer sequences. The first digit occurrences of the analyzed integer sequences are listed in Table 1. The minimum chi-square estimators of the generalized distributions as well as an assumed summation index m for the infinite series (3.5) are displayed in Table 2. Statistical results are summarized in Table 3. For comparison we list the chi-square values and their corresponding P-values. The obtained results are discussed.

The definition, origin, and comments on the mathematical interest of a great part of these integer sequences have been discussed in Hürlimann [24]. Further details on all sequences can be retrieved from the considerable related literature. The *mixing sequence* represents the aggregate of the integer sequences considered in Hürlimann [24].

All of the 19 considered integer sequences are quite well fitted by the new PB distribution. For 14 sequences the minimum chi-square is the smallest among the three comparative values and in the other 5 cases its value does not differ much from the chi-square of the TSPB distribution (bold cells in Table 3 and Table 5).

A strong numerical evidence for the Benford property for the Fibonacci, Bell, Catalan, and partition numbers is observed (corresponding italic cells in Tables 2 and 3). In particular, the values of the parameters α, β of the BP distribution for the Fibonacci sequence are close to 1 and ∞, which means that the BP distribution is almost Benford as remarked

Table 1: First digit distributions of some integer sequences.

Name of sequence	Sample size	Percentage of first digit occurrences								
		1	2	3	4	5	6	7	8	9
Benford law		30.1	17.6	12.5	9.7	7.9	6.7	5.8	5.1	4.6
Square	100	21.0	14.0	12.0	12.0	9.0	9.0	8.0	7.0	8.0
Cube	500	28.2	14.8	11.4	9.8	8.8	7.8	6.6	6.8	5.8
Cube	1000	22.6	15.9	12.4	10.6	9.4	8.3	7.4	7.1	6.3
Cube	10000	22.5	15.8	12.6	10.6	9.3	8.3	7.5	7.0	6.4
Square root	99	19.2	17.2	15.2	13.1	11.1	9.1	7.1	5.1	3.0
Prime < 100	25	16.0	12.0	12.0	12.0	12.0	8.0	16.0	8.0	4.0
Prime < 1000	168	14.9	11.3	11.3	11.9	10.1	10.7	10.7	10.1	8.9
Prime < 10000	1229	13.0	11.9	11.3	11.3	10.7	11.0	10.2	10.3	10.3
Princeton number	25	28.0	8.0	12.0	12.0	8.0	12.0	8.0	4.0	8.0
Mixing sequence	618	28.3	14.6	11.5	9.9	7.6	7.8	8.1	6.6	5.7
Pentagonal number	100	35.0	12.0	10.0	8.0	10.0	6.0	8.0	5.0	6.0
Keith number	71	32.4	14.1	14.1	7.0	4.2	7.0	12.7	2.8	5.6
Bell number	100	31.0	15.0	10.0	12.0	10.0	8.0	5.0	6.0	3.0
Catalan number	100	33.0	18.0	11.0	11.0	8.0	8.0	4.0	3.0	4.0
Lucky number	45	42.2	17.8	8.9	4.4	2.2	6.7	8.9	2.2	6.7
Ulam number	44	45.5	13.6	6.8	6.8	4.5	6.8	4.5	6.8	4.5
Numeri ideoni	65	30.8	18.5	13.8	10.8	6.2	3.1	7.7	6.2	3.1
Fibonacci number	100	30.0	18.0	13.0	9.0	8.0	6.0	5.0	7.0	4.0
Partition number	94	28.7	17.0	14.9	9.6	7.4	6.4	7.4	5.3	3.2

Table 2: Minimum chi-square estimators.

Name of sequence	Sample size	TSPB	PB		
		Parameter	Parameters		
		c	alpha	beta	m
Square	100	0.79837	15.55957	1.74552	100
Cube	500	2.46519	5.55849	1.69860	100
Cube	1000	2.26798	20.56506	1.47082	100
Cube	10000	2.27054	20.53577	1.475760	100
Square root	99	1.40176	89491723	1.34334	100
Prime < 100	25	2.68581	23.13952	2.14449	100
Prime < 1000	168	2.95216	22.99754	2.28436	100
Prime < 10000	1229	3.03542	29.76729	2.30760	100
Princeton number	25	2.76170	6.94595	2.36119	100
Mixing sequence	618	2.53958	4.78641	1.83119	100
Pentagonal number	100	2.94847	2.06797	3.31268	100
Keith number	71	2.73338	2.16107	2.63720	1000
Bell number	100	*1.08191*	10.14820	*1.24828*	100
Catalan number	100	*1.13522*	0.67095	*1.15377*	5000
Lucky number	45	3.15721	7.56962	0.94576	100
Ulam number	44	3.55375	9.99445	0.81215	100
Numeri ideoni	65	*1.12410*	1297612.16	*0.98591*	100
Fibonacci number	100	*2.05365*	257000.42	*1.00560*	100
Partition number	94	*1.23268*	0.65651	1.71409	1000

Table 3: Fitting integer sequences to the Benford and generalized Benford distributions

Name of sequence	Sample size	Benford		Twosided Power Benford		Pareto Benford	
		chi-square	*P*-value	chi-square	*P*-value	chi-square	*P*-value
Square	100	9.096	33.43	7.837	34.72	**0.362**	*99.91*
Cube	500	9.696	28.70	5.808	56.23	**0.286**	*99.96*
Cube	1000	46.459	0.00	43.725	0.00	**0.48**	*99.81*
Cube	10000	443.745	0.00	472.011	0.00	**3.138**	*79.13*
Square root	99	8.612	37.61	7.002	42.86	**2.778**	*83.61*
Prime < 100	25	7.741	45.91	7.299	39.84	**1.849**	*93.30*
Prime < 1000	168	45.016	0.00	36.651	0.00	**0.333**	*99.93*
Prime < 10000	1229	387.194	0.00	307.322	0.00	**3.297**	*77.07*
Princeton number	25	3.452	90.29	2.762	89.72	**1.302**	*97.16*
Mixing sequence	618	15.550	4.93	9.014	25.17	**1.819**	*93.55*
Pentagonal number	100	5.277	72.76	2.127	*95.24*	**1.968**	92.26
Keith number	71	9.215	32.45	7.688	*36.09*	**7.402**	28.53
Bell number	100	3.069	*93.00*	3.014	88.37	**2.607**	85.63
Catalan number	100	2.404	*96.61*	2.304	94.11	**1.934**	92.57
Lucky number	45	7.693	46.40	**5.165**	*63.98*	5.564	47.37
Ulam number	44	6.350	60.81	**2.520**	*92.56*	2.526	86.56
Numeri ideoni	65	2.594	*95.72*	**2.522**	92.54	2.584	85.89
Fibonacci number	100	1.029	*99.81*	**1.021**	99.45	1.027	98.46
Partition number	94	1.394	*99.43*	**1.132**	99.24	1.513	95.86

after Theorem 3.1. It is well known that the Fibonacci sequence is Benford distributed (e.g., Brown and Duncan [30], Wlodarski [31], Sentance [32], Webb [33], Raimi (1976), [34] Brady [35] and Kunoff [36]).The same result for Bell numbers has been derived formally in Hürlimann [24, Theorem 4.1]. More generally, a proof that a generic solution of a generic difference equation is Benford is found in Miller and Takloo-Bighash [37] (see also Jolissaint [38, 39]). Results for squares and cubes are also obtained. Recall that the exact probability distribution of the first digit of mth integer powers with at most n digits is known and asymptotically related to Benford's law (e.g., Hürlimann [40]). The fit of the PB distribution is very good when restricted to finite sequences but breaks down for longer sequences. A further remarkable result is that Benford's law of the mixing sequence is rejected at the 5% significance level while the PB law is accepted with a 93.6%P-value, which improves the P-value of 25.2% obtained for the TSPB law in Hürlimann [24].

The sequence of primes merits a deeper analysis. The Benford property for it has long been studied. Diaconis (1977) [41] shows that primes are not Benford distributed. However, it is known that the sequence of primes is Benford distributed with respect to other densities rather than with the usual natural density [42–44]. According to Serre [45, Page76], , Bombieri has noted that the analytical density of primes with first digit 1 is $\log_{10}2$, and this result can be easily generalized to Benford behavior for any first digit. Table 3 shows that the primes less than 1,000 respectively 10 000 are not at all Benford or TSPB distributed, but they are approximately PB distributed with high P-values of 93.3% and 99.9%. Does this statistical result reveal a new property of the prime number sequence? To answer this question it is

Table 4: First digit distributions of prime number sequences with optimal cutoff.

Sample size	First digit occurrences 1	2	3	4	5	6	7	8	9
25	4	3	3	3	3	2	4	2	1
168	25	19	19	20	17	18	18	17	15
1216	160	146	139	139	131	135	125	127	114
9486	1193	1129	1097	1069	1055	1013	1027	1003	900
77736	9585	9142	8960	8747	8615	8458	8435	8326	7468
657934	80020	77025	75290	74114	72951	72257	71564	71038	63675
5701502	686048	664277	651085	641594	633932	628206	622882	618610	554868

Table 5: Best and linear best Pareto Benford fit for prime number sequences.

Sample size	PB Parameters		PB best fit		PB linear best fit	
	alpha	beta	chi-square/sample size	P-value	chi-square/sample size	P-value
25	23.13952	2.14449	7.396%	93.30	**8.407**%	91.01
168	22.99754	2.28436	0.198%	99.93	**0.781**%	97.10
1216	30.15504	2.25800	0.175%	90.76	**0.152**%	93.34
9486	32.59544	2.28442	0.172%	1.20	**0.084**%	23.86
77736	33.26550	2.31262	0.175%	0.00	**0.075**%	0.00
657934	33.82622	2.32908	0.185%	0.00	**0.070**%	0.00
5701502	34.28132	2.34148	0.188%	0.00	**0.065**%	0.00

necessary to take into account longer sequences and look at other cutoffs than 10^k for an integer k. Our calculations show that among those prime sequences below 10^k for fixed k there is exactly one sequence with minimum chi-square value with an optimal cutoff at a prime with first digit 9. Tables 4 and 5 summarize our results for the primes up to 10^8. Besides the PB best fit with minimum chi-square we also list the PB "linear best" fit obtained from the PB best fit by taking a linear decreasing number of primes between those with the same number of primes with first digit 1 and 9 as in the PB best fit. Though the P-value goes to zero very rapidly the ratio of the minimum chi-square value to the sample size is more stable. For the PB linear best fit this goodness-of-fit statistic, which is also considered in Leemis et al. [21], even decreases and indicates therefore that the first digits of the prime number sequence might be distributed this way. For this it remains to test using more powerful computing whether the mentioned property still holds for even longer sequences of primes. One observes that the best fit parameters as the sample size increases to infinity are quite stable and increase only slightly.

Finally, it might be worthwhile to mention another recent intriguing result by Kafri [29], which shows that digits distribution of prime numbers obeys the Planck distribution, which is another generalized Benford law as already mentioned at the end of Section 3.

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