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Research Article

$\mathcal{N}$-Subalgebras in BCK/BCI-Algebras Based on Point $\mathcal{N}$-Structures

Young Bae Jun,[1] Min Su Kang,[2] and Chul Hwan Park[3]

[1] *Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Republic of Korea*
[2] *Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea*
[3] *Department of Mathematics, University of Ulsan, Ulsan 680-749, Republic of Korea*

Correspondence should be addressed to Min Su Kang, sinchangmyun@hanmail.net

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The notion of $\mathcal{N}$-subalgebras of several types is introduced, and related properties are investigated. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \bigvee q)$ are provided, and a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \bigvee q)$ is considered.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far, most of the generalization of the crisp set have been conducted on the unit interval $[0,1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [1] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [2] considered closed ideals in BCH-algebras based on $\mathcal{N}$-structures. To obtain more general form of an $\mathcal{N}$-subalgebra in BCK/BCI-algebras, we define the notions of $\mathcal{N}$-subalgebras of types $(\in,\in), (\in, q), (\in, \in \bigvee q), (q, \in), (q, q)$, and $(q, \in \bigvee q)$, and investigate related properties. We provide a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \bigvee q)$. We give conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \bigvee q)$.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2,0)$. By a *BCI-algebra* we mean a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

(i) $((x * y) * (x * z)) * (z * y) = \theta$,

(ii) $(x * (x * y)) * y = \theta$,

(iii) $x * x = \theta$,

(iv) $x * y = y * x = \theta \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X) \quad (x \leq y \iff x * y = \theta). \tag{2.1}$$

In a BCK/BCI-algebra X, the following hold:

(a1) (for all $x \in X)(x * \theta = x)$,

(a2) (for all $x, y, z \in X)((x * y) * z = (x * z) * y)$

for all $x, y, z \in X$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X.

We refer the reader to the books [3, 4] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\}, & \text{if } \Lambda \text{ is finite}, \\ \sup\{a_i \mid i \in \Lambda\}, & \text{otherwise}, \end{cases}$$
$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\}, & \text{if } \Lambda \text{ is finite}, \\ \inf\{a_i \mid i \in \Lambda\}, & \text{otherwise}. \end{cases} \tag{2.2}$$

Denote by $\mathcal{F}(X, [-1,0])$ the collection of functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X, [-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote a BCK/BCI-algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

Definition 2.1 (see [1]). By a *subalgebra* of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-subalgebra* of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following assertion:

$$(\forall x, y \in X) \quad \left(f(x * y) \leq \bigvee\{f(x), f(y)\}\right). \tag{2.3}$$

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0)$, the set

$$C(f; t) := \{x \in X \mid f(x) \leq t\} \tag{2.4}$$

is called a *closed t-support* of (X, f), and the set

$$O(f;t) := \{x \in X \mid f(x) < t\} \tag{2.5}$$

is called an *open t-support* of (X, f).

Using the similar method to the transfer principle in fuzzy theory (see [5, 6]), Jun et al. [2] considered transfer principle in $\mathcal{N}$-structures as follows.

Theorem 2.2 ($\mathcal{N}$-transfer principle [2]). *An $\mathcal{N}$-structure (X, f) satisfies the property $\overline{\mathcal{P}}$ if and only if for all $\alpha \in [-1, 0]$,*

$$C(f;\alpha) \neq \emptyset \Longrightarrow C(f;\alpha) \text{ satisfies the property } \mathcal{P}. \tag{2.6}$$

Lemma 2.3 (see [1]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of X if and only if every open t-support of (X, f) is a subalgebra of X for all $t \in [-1, 0)$.*

3. Generalized $\mathcal{N}$-Subalgebras

Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(y) = \begin{cases} 0, & \text{if } y \neq x, \\ \alpha, & \text{if } y = x, \end{cases} \tag{3.1}$$

where $\alpha \in [-1, 0)$. In this case, f is denoted by x_α and we call (X, x_α) a *point $\mathcal{N}$-structure*. For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_α) is an *$\mathcal{N}_\in$-subset* (resp., *$\mathcal{N}_q$-subset*) of (X, g) if $g(x) \leq \alpha$ (resp., $g(x) + \alpha + 1 < 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_\in$-subset of (X, g) or an $\mathcal{N}_q$-subset of (X, g), we say (X, x_α) is an *$\mathcal{N}_{\in \vee q}$-subset* of (X, g).

Theorem 3.1. *For any $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) *(X, f) is an $\mathcal{N}$-subalgebra of X;*

(2) *for any $x, y \in X$ and $t_1, t_2 \in [-1, 0)$, if two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_\in$-subsets of (X, f), then the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_\in$-subset of (X, f).*

Proof. (1) $\Rightarrow$ (2). Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_\in$-subsets of (X, f). Then $f(x) \leq t_1$ and $f(y) \leq t_2$. It follows from (2.3) that

$$f(x * y) \leq \bigvee\{f(x), f(y)\} \leq \bigvee\{t_1, t_2\} \tag{3.2}$$

so that the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_\in$-subset of (X, f).

(2) $\Rightarrow$ (1). For any $x, y \in X$, note that $(X, x_{f(x)})$ and $(X, y_{f(y)})$ are point $\mathcal{N}$-structures which are $\mathcal{N}_\in$-subsets of (X, f). Using (2), we know that the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{f(x), f(y)\}})$ is an $\mathcal{N}_\in$-subset of (X, f). Thus $f(x * y) \leq \bigvee\{f(x), f(y)\}$, and so (X, f) is an $\mathcal{N}$-subalgebra of X. □

Table 1: $*$-operation.

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	θ	θ
b	b	a	θ	a	θ
c	c	a	a	θ	θ
d	d	b	a	b	θ

Definition 3.2. An $\mathcal{N}$-structure (X, f) is called an *$\mathcal{N}$-subalgebra* of type

(i) $(\in, \in)$ (resp., $(\in, q)$ and $(\in, \in \bigvee q)$) if whenever two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_{\in}$-subsets of (X, f) then the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in}$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\bigvee q}$-subset) of (X, f);

(ii) $(q, \in)$ (resp., (q, q) and $(q, \in \bigvee q)$) if whenever two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_q$-subsets of (X, f) then the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in}$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\bigvee q}$-subset) of (X, f).

Note that every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is an $\mathcal{N}$-subalgebra of X (see Theorem 3.1). Note also that every $\mathcal{N}$-subalgebra of types $(\in, \in)$ and $(\in, q)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \bigvee q)$.

Example 3.3. Let $X = \{\theta, a, b, c, d\}$ be a set with a $*$-operation table which is given by Table 1. Then $(X; *, \theta)$ is a BCK-algebra (see [4]). Consider an $\mathcal{N}$-structure (X, f) in which f is defined by

$$f = \begin{pmatrix} \theta & a & b & c & d \\ -0.9 & -0.8 & -0.5 & -0.7 & -0.3 \end{pmatrix}. \quad (3.3)$$

It is routine to verify that (X, f) is an $\mathcal{N}$-subalgebra of types $(\in, \in)$ and $(\in, \in \bigvee q)$. But it is not of type $(q, \in \bigvee q)$.

Example 3.4. Let $X = \{\theta, a, b, c\}$ be a BCI-algebra with a $*$-operation table which is given by Table 2. Consider an $\mathcal{N}$-structure (X, f) in which f is defined by

$$f = \begin{pmatrix} \theta & a & b & c \\ -0.5 & -0.8 & -0.3 & -0.3 \end{pmatrix}. \quad (3.4)$$

Then (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \bigvee q)$. But

(1) (X, f) is not of type $(\in, \in)$ since two point $\mathcal{N}$-structures $(X, a_{-0.7})$ and $(X, a_{-0.76})$ are $\mathcal{N}_{\in}$-subsets of (X, f), but the point $\mathcal{N}$-structure

$$\left(X, (a * a)_{\bigvee\{-0.7,-0.76\}}\right) = (X, \theta_{-0.7}) \quad (3.5)$$

is not an $\mathcal{N}_{\in}$-subset of (X, f) since $f(\theta) = -0.5 \nleq -0.7$;

Table 2: $*$-operation.

$*$	θ	a	b	c
θ	θ	a	b	c
a	a	θ	c	b
b	b	c	θ	a
c	c	b	a	θ

Table 3: $*$-operation.

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	θ	θ
b	b	b	θ	θ	b
c	c	b	a	θ	b
d	d	d	d	d	θ

(2) (X, f) is not of type $(q, \in \bigvee q)$ since two point $\mathcal{N}$-structures $(X, a_{-0.42})$ and $(X, b_{-0.88})$ are $\mathcal{N}_q$-subsets of (X, f), but the point $\mathcal{N}$-structure

$$\left(X, (a * b)_{\bigvee\{-0.42,-0.88\}}\right) = (X, c_{-0.42}) \tag{3.6}$$

is not an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f);

(3) (X, f) is not of type $(\in \bigvee q, \in \bigvee q)$ since two point $\mathcal{N}$-structures $(X, a_{-0.6})$ and $(X, c_{-0.82})$ are $\mathcal{N}_{\in \bigvee q}$-subsets of (X, f), but the point $\mathcal{N}$-structure

$$\left(X, (a * c)_{\bigvee\{-0.6,-0.82\}}\right) = (X, b_{-0.6}) \tag{3.7}$$

is not an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f).

Example 3.5. Let $X = \{\theta, a, b, c, d\}$ be a set with a $*$-operation table which is given by Table 3. Then $(X; *, \theta)$ is a BCK-algebra (see [4]). Consider an $\mathcal{N}$-structure (X, f) in which f is defined by

$$f = \begin{pmatrix} \theta & a & b & c & d \\ -0.8 & -0.7 & 0 & 0 & -0.6 \end{pmatrix}. \tag{3.8}$$

Then (X, f) is an $\mathcal{N}$-subalgebra of type $(q, \in \bigvee q)$.

Theorem 3.6. *If (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$, then the open 0-support of (X, f) is a subalgebra of X.*

Proof. Let (X, f) be an $\mathcal{N}$-subalgebra of type $(\in, \in)$. If f is zero, that is, $f(x) = 0$ for all $x \in X$, then $O(f; 0) = \emptyset$ which is a subalgebra of X. Assume that f is nonzero and let $x, y \in O(f; 0)$. Then $f(x) < 0$ and $f(y) < 0$. Suppose that $f(x * y) = 0$. Note that $(X, x_{f(x)})$ and $(X, y_{f(y)})$

are point $\mathcal{N}$-structures which are $\mathcal{N}_{\in}$-subsets of (X, f). But the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{f(x), f(y)\}})$ is not an $\mathcal{N}_{\in}$-subset of (X, f) because $f(x * y) = 0 > \bigvee\{f(x), f(y)\}$. This is a contradiction, and so $f(x * y) < 0$, that is, $x * y \in O(f; 0)$. Hence $O(f; 0)$ is a subalgebra of X. □

Theorem 3.7. *If (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, q)$, then the open 0-support of (X, f) is a subalgebra of X.*

Proof. Let $x, y \in O(f; 0)$. Then $f(x) < 0$ and $f(y) < 0$. If $f(x * y) = 0$, then

$$f(x * y) + \bigvee\{f(x), f(y)\} + 1 = \bigvee\{f(x), f(y)\} + 1 \geq 0. \tag{3.9}$$

Thus the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{f(x), f(y)\}})$ is not an $\mathcal{N}_q$-subset of (X, f), which is impossible since $(X, x_{f(x)})$ and $(X, y_{f(y)})$ are point $\mathcal{N}$-structures which are $\mathcal{N}_{\in}$-subsets of (X, f). Therefore, $f(x * y) < 0$, that is, $x * y \in O(f; 0)$. This shows that the open 0-support of (X, f) is a subalgebra of X. □

Theorem 3.8. *If (X, f) is an $\mathcal{N}$-subalgebra of type $(q, \in)$, then the open 0-support of (X, f) is a subalgebra of X.*

Proof. Let $x, y \in O(f; 0)$. Then $f(x) < 0$ and $f(y) < 0$, which imply that (X, x_{-1}) and (X, y_{-1}) are point $\mathcal{N}$-structures which are $\mathcal{N}_q$-subsets of (X, f). If $f(x * y) = 0$, then the point $\mathcal{N}$-structure $(X, (x*y)_{\bigvee\{-1,-1\}})$ is not an $\mathcal{N}_{\in}$-subset of (X, f), a contradiction. Therefore, $f(x*y) < 0$, that is, $x * y \in O(f; 0)$, and so the open 0-support of (X, f) is a subalgebra of X. □

Theorem 3.9. *If (X, f) is an $\mathcal{N}$-subalgebra of type (q, q), then f is constant on the open 0-support of (X, f).*

Proof. Assume that f is not constant on the open 0-support of (X, f). Then there exists $y \in O(f; 0)$ such that $t_y = f(y) \neq f(\theta) = t_0$. Then either $t_y < t_0$ or $t_y > t_0$. Suppose that $t_y > t_0$ and choose $t_1, t_2 \in [-1, 0)$ such that $t_2 < -1 - t_y < t_1 < -1 - t_0$. Then $f(0) + t_1 + 1 = t_0 + t_1 + 1 < 0$ and $f(y) + t_2 + 1 = t_y + t_2 + 1 < 0$, and so (X, θ_{t_1}) and (X, y_{t_2}) are point $\mathcal{N}$-structures which are $\mathcal{N}_q$-subsets of (X, f). Since

$$f(y * \theta) + \bigvee\{t_1, t_2\} + 1 = f(y) + t_1 + 1 = t_y + t_1 + 1 > 0, \tag{3.10}$$

the point $\mathcal{N}$-structure $(X, (y*\theta)_{\bigvee\{t_1, t_2\}})$ is not an $\mathcal{N}_q$-subset of (X, f), which is a contradiction. Next assume that $t_y < t_0$. Then $f(y) + (-1 - t_0) + 1 = t_y - t_0 < 0$, and so (X, y_{-1-t_0}) is an $\mathcal{N}_q$ -subset of (X, f). Note that

$$f(y * y) + (-1 - t_0) + 1 = f(\theta) - t_0 = t_0 - t_0 = 0, \tag{3.11}$$

and thus $(X, (y*y)_{\bigvee\{-1-t_0, -1-t_0\}})$ is not an $\mathcal{N}_q$-subset of (X, f). This is impossible, and therefore f is constant on the open 0-support of (X, f). □

Theorem 3.10. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ if and only if it satisfies*

$$(\forall x, y \in X) \quad \left(f(x * y) \leq \bigvee\{f(x), f(y), -0.5\}\right). \tag{3.12}$$

Proof. Suppose that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$. For any $x, y \in X$, assume that $\bigvee\{f(x), f(y)\} > -0.5$. If $f(a*b) > \bigvee\{f(a), f(b)\}$ for some $a, b \in X$, then there exists $t \in [-1, 0)$ such that $f(a * b) > t \geq \bigvee\{f(a), f(b)\}$. Thus, point $\mathcal{N}$-structures (X, a_t) and (X, b_t) are $\mathcal{N}_\in$-subsets of (X, f), but the point $\mathcal{N}$-structure $(X, (a * b)_{\bigvee\{t,t\}})$ is not an $\mathcal{N}_{\in \vee q}$-subset of (X, f), a contradiction. Hence $f(x*y) \leq \bigvee\{f(x), f(y)\}$ whenever $\bigvee\{f(x), f(y)\} > -0.5$ for all $x, y \in X$. Now suppose that $\bigvee\{f(x), f(y)\} \leq -0.5$. Then point $\mathcal{N}$-structures $(X, x_{-0.5})$ and $(X, y_{-0.5})$ are $\mathcal{N}_\in$-subsets of (X, f), which imply that the point $\mathcal{N}$-structure $(X, (x*y)_{\bigvee\{-0.5,-0.5\}})$ is an $\mathcal{N}_{\in \vee q}$-subset of (X, f). Hence $f(x * y) \leq -0.5$. Otherwise, $f(x * y) - 0.5 + 1 > -0.5 - 0.5 + 1 = 0$, that is, $(X, (x * y)_{-0.5})$ is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction. Consequently, $f(x * y) \leq \bigvee\{f(x), f(y), -0.5\}$ for all $x, y \in X$.

Conversely, assume that (3.12) is valid. Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_\in$-subsets of (X, f). If $f(x * y) \leq \bigvee\{t_1, t_2\}$, then $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_\in$-subset of (X, f). Suppose that $f(x * y) > \bigvee\{t_1, t_2\}$. Then $\bigvee\{f(x), f(y)\} \leq -0.5$. Otherwise, we have

$$f(x * y) \leq \bigvee\{f(x), f(y), -0.5\} = \bigvee\{f(x), f(y)\} \leq \bigvee\{t_1, t_2\}, \tag{3.13}$$

a contradiction. It follows that

$$f(x * y) + \bigvee\{t_1, t_2\} + 1 < 2f(x * y) + 1 \leq 2\bigvee\{f(x), f(y), -0.5\} + 1 = 0 \tag{3.14}$$

and so $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_q$-subset of (X, f). Consequently, $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in \vee q}$-subset of (X, f), and thus (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$. □

We provide conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q)$.

Theorem 3.11. *Let S be a subalgebra of X and let (X, f) be an $\mathcal{N}$-structure such that*

(1) (*for all* $x \in X$) $(x \in S \Rightarrow f(x) \leq -0.5)$,

(2) (*for all* $x \in X$) $(x \notin S \Rightarrow f(x) = 0)$.

Then (X, f) is an $\mathcal{N}$-subalgebra of type $(q, \in \vee q)$.

Proof. Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_q$-subsets of (X, f). Then $f(x) + t_1 + 1 < 0$ and $f(y) + t_2 + 1 < 0$. Thus $x * y \in S$ because if it is impossible, then $x \notin S$ or $y \notin S$. Thus $f(x) = 0$ or $f(y) = 0$, and so $t_1 < -1$ or $t_2 < -1$. This is a contradiction. Hence $f(x * y) \leq -0.5$. If $\bigvee\{t_1, t_2\} < -0.5$, then $f(x * y) + \bigvee\{t_1, t_2\} + 1 < 0$ and thus the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_q$-subset of (X, f). If $\bigvee\{t_1, t_2\} \geq -0.5$, then $f(x*y) \leq -0.5 \leq \bigvee\{t_1, t_2\}$ and so the point $\mathcal{N}$-structure $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_\in$-subset of (X, f). Therefore, the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in \vee q}$-subset of (X, f). This completes the proof. □

Theorem 3.12. *Let (X, f) be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q)$. If f is not constant on the open 0-support of (X, f), then $f(x) \leq -0.5$ for some $x \in X$. In particular, $f(\theta) \leq -0.5$.*

Proof. Assume that $f(x) > -0.5$ for all $x \in X$. Since f is not constant on the open 0-support of (X, f), there exists $x \in O(f;0)$ such that $t_x = f(x) \neq f(\theta) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. For the case $t_0 < t_x$, choose $r < -0.5$ such that $t_0 + r + 1 < 0 < t_x + r + 1$. Then the point $\mathcal{N}$-structure (X, θ_r) is an $\mathcal{N}_q$-subset of (X, f). Since (X, x_{-1}) is an $\mathcal{N}_q$-subset of (X, f). It follows from (a1) that the point $\mathcal{N}$-structure $(X, (x * \theta)_{\bigvee\{r,-1\}}) = (X, x_r)$ is an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f). But, $f(x) > -0.5 > r$ implies that the point $\mathcal{N}$-structure (X, x_r) is not an $\mathcal{N}_{\in}$-subset of (X, f). Also, $f(x) + r + 1 = t_x + r + 1 > 0$ implies that the point $\mathcal{N}$-structure (X, x_r) is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction. Now, if $t_0 > t_x$ then we can take $r < -0.5$ such that $t_x + r + 1 < 0 < t_0 + r + 1$. Then (X, x_r) is an $\mathcal{N}_q$-subset of (X, f), and $f(x * x) = f(\theta) = t_0 > r = \bigvee\{r, r\}$ induces that $(X, (x * x)_{\bigvee\{r,r\}})$ is not an $\mathcal{N}_{\in}$-subset of (X, f). Since

$$f(x * x) + \bigvee\{r, r\} + 1 = f(\theta) + r + 1 = t_0 + r + 1 > 0, \tag{3.15}$$

$(X, (x * x)_{\bigvee\{r,r\}})$ is not an $\mathcal{N}_q$-subset of (X, f). Hence $(X, (x * x)_{\bigvee\{r,r\}})$ is not an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f), which is a contradiction. Therefore $f(x) \leq -0.5$ for some $x \in X$. We now prove that $f(\theta) \leq -0.5$. Assume that $f(\theta) = t_0 > -0.5$. Note that there exists $x \in X$ such that $f(x) = t_x \leq -0.5$ and so $t_x < t_0$. Choose $t_1 < t_0$ such that $t_x + t_1 + 1 < 0 < t_0 + t_1 + 1$. Then $f(x) + t_1 + 1 = t_x + t_1 + 1 < 0$, and thus the point $\mathcal{N}$-structure (X, x_{t_1}) is an $\mathcal{N}_q$-subset of (X, f). Now we have

$$f(x * x) + \bigvee\{t_1, t_1\} + 1 = f(\theta) + t_1 + 1 = t_0 + t_1 + 1 > 0 \tag{3.16}$$

and $f(x * x) = f(\theta) = t_0 > t_1 = \bigvee\{t_1, t_1\}$. Hence $(X, (x * x)_{\bigvee\{t_1,t_1\}})$ is not an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f), a contradiction. Therefore $f(\theta) \leq -0.5$. □

Corollary 3.13. *If (X, f) is an $\mathcal{N}$-subalgebra of types $(q, \in)$ or (q, q) in which f is not constant on the open 0-support of (X, f), then $f(x) \leq -0.5$ for some $x \in X$. In particular, $f(\theta) \leq -0.5$.*

Theorem 3.14. *Let X be a BCK-algebra and let (X, f) be an $\mathcal{N}$-subalgebra of type $(q, \in \bigvee q)$ such that f is not constant on the open 0-support of (X, f). If*

$$f(\theta) = \bigwedge_{x \in X} f(x), \tag{3.17}$$

then $f(x) \leq -0.5$ for all $x \in O(f;0)$.

Proof. Assume that $f(x) > -0.5$ for all $x \in X$. Since f is not constant on the open 0-support of (X, f), there exists $y \in O(f;0)$ such that $t_y = f(y) \neq f(\theta) = t_0$. Then $t_y > t_0$. Choose $t_1 < -0.5$ such that $t_0 + t_1 + 1 < 0 < t_y + t_1 + 1$. Then (X, θ_{t_1}) is an $\mathcal{N}_q$-subset of (X, f). Note that the point $\mathcal{N}$-structure (X, y_{-1}) is an $\mathcal{N}_q$-subset of (X, f). It follows that $(X, (y * \theta)_{\bigvee\{-1,t_1\}}) = (X, y_{t_1})$ is an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f). But $f(y) > -0.5 > t_1$ induces that (X, y_{t_1}) is not an $\mathcal{N}_{\in}$-subset of (X, f), and $f(y) + t_1 + 1 = t_y + t_1 + 1 > 0$ induces that (X, y_{t_1}) is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and so $f(x) \leq -0.5$ for some $x \in X$. Now, if possible, let $t_0 = f(\theta) > -0.5$. Then there exists $x \in X$ such that $t_x = f(x) \leq -0.5$. Thus $t_x < t_0$. Take $t_1 < t_0$ such that $t_x + t_1 + 1 < 0 < t_0 + t_1 + 1$. Then two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, θ_{-1}) are $\mathcal{N}_q$-subsets of (X, f), but $(X, (\theta * x)_{\bigvee\{-1,t_1\}}) = (X, \theta_{t_1})$ is not an $\mathcal{N}_{\in \bigvee q}$-subset of (X, f), a contradiction. Hence $f(\theta) \leq -0.5$. Finally let $t_x = f(x) > -0.5$ for some $x \in O(f;0)$. Taking $t_1 < 0$ such that

$t_x + t_1 > -0.5$, then two point $\mathcal{N}$-structures (X, x_{-1}) and $(X, \theta_{-0.5+t_1})$ are $\mathcal{N}_q$-subsets of (X, f). But

$$f(x) - 0.5 + t_1 + 1 = t_x - 0.5 + t_1 + 1 > -0.5 - 0.5 + 1 = 0 \tag{3.18}$$

implies that the point $\mathcal{N}$-structure $(X, x_{-0.5+t_1})$ is not an $\mathcal{N}_q$-subset of (X, f). Hence the point $\mathcal{N}$-structure $(X, (x * \theta)_{\bigvee\{-1, -0.5+t_1\}}) = (X, x_{-0.5+t_1})$ is not an $\mathcal{N}_{\in \vee q}$-subset of (X, f), a contradiction. Therefore $f(x) \leq -0.5$ for all $x \in O(f; 0)$. □

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