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Research Article

Unicity of Meromorphic Function Sharing One Small Function with Its Derivative

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We deal with the problem of uniqueness of a meromorphic function sharing one small function with its k's derivative and obtain some results.

1. Introduction and Main Results

In this article, a meromorphic function means meromorphic in the open complex plane. We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and the standard notations such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, and so on.

Let f and g be two nonconstant meromorphic functions; a meromorphic function $a(z)(\not\equiv\infty)$ is called a small functions with respect to f provided that $T(r,a)=S(r,f)$. Note that the set of all small function of f is a field. Let $b(z)$ be a small function with respect to f and g. We say that f and g share $b(z)$ CM(IM) provided that $f-b$ and $g-b$ have same zeros counting multiplicities (ignoring multiplicities).

Moreover, we use the following notations.

Let k be a positive integer. We denote by $N_{k)}(r,1/(f-a))$ the counting function for the zeros of $f-a$ with multiplicity $\leq k$ and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}(r,1/(f-a))$ be the counting function for the zeros of $f-a$ with multiplicity $\geq k$, and let $\overline{N}_{(k}(r,1/(f-a))$ be the corresponding one for which the multiplicity is not counted. Set $N_k(r,1/(f-a))=\overline{N}(r\cdot 1/(f-a))+\overline{N}_{(2}(r,1/(f-a))+\cdots+\overline{N}_{(k}(r,1/(f-a))$. And we define

$$\delta_p(a,f)=1-\limsup_{r\longrightarrow+\infty}\frac{N_p(r,1/(f-a))}{T(r,f)}. \tag{1.1}$$

Obviously, $1 \geq \Theta(a,f) \geq \delta_p(a,f) \geq \delta(a,f) \geq 0$. For more details, reader can see [1, 2].

Brück (see [3]) considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. *Let f be nonconstant entire function. If f and f' share the value 1 CM and if $N(r,1/f') = S(r,f)$, then $(f'-1)/(f-1) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.*

Yang [4], Zhang [5], and Yu [6] extended Theorem A and obtained many excellent results.

Theorem B (see[5]). *Let f be a nonconstant meromorphic function and, let k be a positive integer. Suppose that f and $f^{(k)}$ share 1 CM and*

$$2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f'}\right) + N\left(r,\frac{1}{f^{(k)}}\right) < (\lambda + o(1))T\left(r,f^{(k)}\right), \tag{1.2}$$

for $r \in I$, where I is a set of infinite linear measure and λ satisfies $0 < \lambda < 1$, then $(f^{(k)}-1)/(f-1) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem C (see[6]). *Let f be a nonconstant, nonentire meromorphic function and $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If*

(1) *f and $a(z)$ have no common poles,*

(2) *$f-a$ and $f^{(k)}-a$ share the value 0 CM,*

(3) *$4\delta(0,f) + 2(k+8)\Theta(\infty,f) > 2k+19$, then $f \equiv f^{(k)}$, where k is a positive integer.*

In the same paper, Yu [6] posed four open questions. Lahiri and Sarkar [7] and Zhang [8] studied the problem of a meromorphic or an entire function sharing one small function with its derivative with weighted shared method and obtained the following result, which answered the open questions posed by Yu [6].

Theorem D (see[8]). *Let f be a non-constant meromorphic function and, let k be a positive integer. Also let $a(z)(\not\equiv 0,\ \infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f-a$ and $f^{(k)}-a$ share 0 IM and*

$$4\overline{N}(r,f) + 3N_2\left(r,\frac{1}{f^{(k)}}\right) + 2\overline{N}\left(r,\frac{1}{(f/a)'}\right) < (\lambda + o(1))T\left(r,f^{(k)}\right), \tag{1.3}$$

for $0 < \lambda < 1, r \in I$, and I is a set of infinite linear measure. Then $(f^{(k)}-a) \setminus (f-a) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

In this article, we will pay our attention to the value sharing of f and $[f^n]^{(k)}$ that share a small function and obtain the following results, which are the improvements and complements of the above theorems.

Theorem 1.1. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and let f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. If f and $[f^n]^{(k)}$ share $a(z)$ IM and*

$$4\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{(f/a)'}\right)+2N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right) \leq (\lambda+o(1))T\left(r,(f^n)^{(k)}\right), \tag{1.4}$$

or f and $[f^n]^{(k)}$ share $a(z)$ CM and

$$2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{(f/a)'}\right)+N_2\left(r,\frac{1}{(f^n)^{(k)}}\right) \leq (\lambda+o(1))T\left(r,(f^n)^{(k)}\right), \tag{1.5}$$

for $0<\lambda<1$, $r\in I$, and I is a set of infinite linear measure, then $(f-a)\setminus([f^n]^{(k)}-a)\equiv c$, for some constant $c\in\mathbb{C}\setminus\{0\}$.

Theorem 1.2. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. If f and $[f^n]^{(k)}$ share $a(z)$ IM and*

$$(2k+6)\Theta(\infty,f)+3\Theta(0,f)+2\delta_{k+2}(0,f)>2k+10, \tag{1.6}$$

or f and $[f^n]^{(k)}$ share $a(z)$ CM and

$$(k+3)\Theta(\infty,f)+\delta_2(0,f)+\delta_{k+2}(0,f)>k+4, \tag{1.7}$$

then $f\equiv(f^n)^{(k)}$.

Clearly, Theorem 1.1 improves and extends Theorems B and D, while 1.2 improves and extends Theorem C.

2. Some Lemmas

In this section, first of all, we give some definitions which will be used in the whole paper.

Definition 2.1. Let F and G be two meromorphic functions defined in $\mathbb{C}$; assume, that F and G share 1 IM; let z_0 be a zero of $F-1$ with multiplicity p and a zero of $G-1$ with multiplicity q. We denote by$N_E^{1)}(r,1/F-1)$the counting function of the zeros of $F-1$ where $p=q=1$ and by$N_E^{(2}(r,1/F-1)$the counting function of zeros of $F-1$ where $p=q\geq 2$. We denotes by $N_L(r,1/F-1)$ the counting function of the zeros of $F-1$ where $p>q\geq 1$; each point is counted according to its multiplicity, and $\overline{N}_L(r,1/F-1)$ denote its reduced form. In the same way, we can define$N_E^{1)}(r,1/G-1)$, $N_E^{(2}(r,1/G-1)$, $\overline{N}_L(r,1/G-1)$, and so on.

Definition 2.2. In this paper $N_0(r,1/F')$ denotes the counting function of the zeros ofF' which are not the zeros of F and $F-1$, and $\overline{N}_0(r,1/F')$ denotes its reduced form. In the same way, we can define $N_0(r,1/G')$ and $\overline{N}_0(r,1/G')$.

Next we present some lemmas which will be needed in the sequel. Let F,G be two nonconstant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H=\left(\frac{F''}{F'}-2\frac{F'}{F-1}\right)-\left(\frac{G''}{G'}-2\frac{G'}{G-1}\right). \tag{2.1}$$

Lemma 2.3 (see[2]). *Let F,G be two nonconstant meromorphic functions defined in $\mathbb{C}$. If F and G are sharing 1 IM, then*

$$\begin{aligned}N(r,H)\leq{}&\overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned} \tag{2.2}$$

If F and G are sharing 1 CM, then

$$N(r,H)\leq\overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f). \tag{2.3}$$

Lemma 2.4 (see[1]). *Let f be a meromorphic function and a is a finite complex number. Then*

(i) $T(r,1/(f-a))=T(r,f)+O(1)$,

(ii) $m(r,f^{(k)}/f^{(l)})=S(r,f)$ *for* $k>l\geq 0$,

(iii) $T(r,f)\leq\overline{N}(r,f)+\overline{N}(r,1/(f-a_1(z)))+\overline{N}(r,1/(f-a_2(z)))+S(r,f)$,

where $a_1(z)$ $a_2(z)$ are two meromorphic functions such that $T(r,a_i)=S(r,f),(i=1,2)$.

Lemma 2.5 (see[7]). *Let f be a non-constant meromorphic function, and k,p are two positive integers. Then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f). \tag{2.4}$$

Lemma 2.6 (see[9]). *Let f be a non-constant meromorphic function and let n be a positive integer. $P(f)=a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f$ where a_i are meromorphic functions such that $T(r,a_i)=S(r,f)(i=1,2,\ldots,n)$, and $a_n\not\equiv 0$. Then*

$$T(r,P(f))=nT(r,f)+S(r,f). \tag{2.5}$$

3. Proof of Theorem 1.1

Let $F = f(z)/a(z), G = (f^n(z))^{(k)}/a(z)$, then

$$F - 1 = \frac{f(z) - a(z)}{a(z)},$$
$$G - 1 = \frac{(f^n(z))^{(k)} - a(z)}{a(z)}. \tag{3.1}$$

From the definitions of F, G and recalling that F and G share value 1 IM(CM), we get

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) = N_E^{1)}\left(r, \frac{1}{G-1}\right) + S(r, f),$$
$$\overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) = \overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right) + S(r, f), \tag{3.2}$$

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, F) + S(r, F), \tag{3.3}$$

$$\overline{N}\left(r, \frac{1}{F-1}\right) = \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, F) = N_E^{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right)$$
$$+ \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r, f). \tag{3.4}$$

We will distinguish two cases below.

Case 1 ($H \not\equiv 0$). From (2.1) it is easy to see that $m(r, H) = S(r, f)$.

Subcase 1.1. Suppose that f and $(f^n)^{(k)}$ share $a(z)$ IM. According to (3.1), F and G share 1 IM except the zeros and poles of $a(z)$. By (3.1), we have

$$\overline{N}(r, F) = \overline{N}(r, f) + S(r, f), \qquad \overline{N}(r, G) = \overline{N}(r, f) + S(r, f). \tag{3.5}$$

Let z_0 be a simple zero of $F-1$ and $G-1$, but $a(z_0) \ne 0, \infty$. Through a simple calculation we know that z_0 is a zero of H, so

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \le N\left(r, \frac{1}{H}\right) + S(r, f) \le T(r, H) + S(r, f) \le N(r, H) + S(r, f). \tag{3.6}$$

From (3.4)–(3.6) and Lemma 2.3, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{G-1}\right) &\le \overline{N}(r,F)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)\\ &\quad+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f)\\ &\le \overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F'}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned}\tag{3.7}$$

It follows by the second fundamental theorem, (3.5), and (3.7) that

$$\begin{aligned}T(r,G) &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,G)\\ &\le 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F'}\right)+2\overline{N}\left(r,\frac{1}{G'}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f).\end{aligned}\tag{3.8}$$

By Lemma 2.5, we have

$$T\left(r,(f^n)^{(k)}\right) \le 4\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{(f/a)'}\right)+2N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)+S(r,f),\tag{3.9}$$

which contradicts (1.4).

Subcase 1.2. Suppose that f and $(f^n)^{(k)}$ share $a(z)$ CM.

Let z_0 be a simple zero of $F-1$ and $G-1$, but $a(z_0)\neq 0,\infty$. By a simple calculation, we can still get $H(z_0)=0$. Therefore

$$N_{1)}\left(r,\frac{1}{F-1}\right) \le N\left(r,\frac{1}{H}\right)+S(r,f)\le N(r,H)+S(r,f).\tag{3.10}$$

Noting that $N_{1)}(r,1/(F-1)) = N_{1)}(r,1/(G-1))+S(r,f)$, by (3.4) and Lemma 2.3, we can deduce

$$\begin{aligned}\overline{N}\left(r,\frac{1}{G-1}\right) &\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)\\ &\quad+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+S(r,f).\end{aligned}\tag{3.11}$$

By the second fundamental theorem, (3.5), and (3.11), we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,G) \\ &\leq 2\overline{N}(r,f)+N_2\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F'}\right)+S(r,f). \end{aligned} \tag{3.12}$$

Taking into account (3.1), we have

$$T\left(r,(f^n)^{(k)}\right) \leq 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{(f/a)'}\right)+N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)+S(r,f). \tag{3.13}$$

This contradicts (1.5).

Case 2 ($H \equiv 0$). Integration yields

$$\frac{1}{F-1} \equiv \frac{A}{G-1}+B, \tag{3.14}$$

where $A,\ B$ are constants and $A \neq 0$. It is easy to see that F and G share 1 CM. Now we claim that $B=0$.

If $\overline{N}(r,f) \neq S(r,f)$, then by (3.14) we get $B=0$. So our claim holds. Hence we can assume that

$$\overline{N}(r,f)=S(r,f). \tag{3.15}$$

If $B \neq 0$, then we can rewrite (3.14) as

$$\frac{1}{F-1} \equiv \frac{B(G-1+A/B)}{G-1}. \tag{3.16}$$

So

$$\overline{N}\left(r,\frac{1}{G-1+A/B}\right)=\overline{N}(r,F)=S(r,f). \tag{3.17}$$

If $A \neq B$, then by Lemma 2.4 and (3.17) we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1+A/B}\right)+S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+S(r,f) \leq T(r,G)+S(r,f). \end{aligned} \tag{3.18}$$

Hence

$$T(r,G)=\overline{N}\left(r,\frac{1}{G}\right)+S(r,f),\tag{3.19}$$

that is,

$$T\left(r,(f^n)^{(k)}\right)=\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)+S(r,f).\tag{3.20}$$

This is a contradiction with (1.4) and (1.5). If $A=B$, then from (3.14) we get $1/(F-1)=AG/(G-1)$. We rewrite it as

$$-\frac{a^2}{f^n(Af-a-aA)}\equiv\frac{(f^n)^{(k)}}{f^n}.\tag{3.21}$$

So by Lemmas 2.4 and 2.6 and (3.15), we have

$$\begin{aligned}(n+1)T(r,f)&=T\left(r,\frac{(f^n)^{(k)}}{f^n}\right)+S(r,f)\\&\le nN\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\le nT(r,f)+S(r,f).\end{aligned}\tag{3.22}$$

This implies that $T(r,f)=S(r,f)$, since $n\ge 1$. This is impossible. Hence our claim is right. So $(G-1)/(F-1)=A$. Theorem 1.1 is, thus, completely proved.

4. Proof of Theorem 1.2

The proof is similar to the proof of Theorem 1.1. Let F and G be defined as in Theorem 1.1; hence, we have (3.1)–(3.5). We still distinguish two cases.

Case 1. $H\not\equiv 0$

Subcase 1.1. Suppose that f and $(f^n)^{(k)}$ share $a(z)$ IM, then we can still get (3.6) and (3.7). Then by the second fundamental theorem, Lemma 2.3, and (3.5) we have

$$\begin{aligned}T(r,F)&\le\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)-N_0\left(r,\frac{1}{F'}\right)+S(r,F)\\&\le 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{G'}\right)+2\overline{N}\left(r,\frac{1}{F'}\right)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f).\end{aligned}\tag{4.1}$$

Applying Lemma 2.5 to the above inequality and noticing the definition of F, G, we get

$$\begin{aligned} T(r,f) &\leq (2k+6)\overline{N}(r,f) + 3\overline{N}\left(r,\frac{1}{f}\right) + 2N_{k+2}N\left(r,\frac{1}{f}\right) + S(r,f) \\ &\leq \left[(2k+6)(1-\Theta(\infty,f)) + 3 - 3\Theta(0,f) + 2 - 2\delta_{k+2}(0,f)\right]T(r,f) + S(r,f). \end{aligned} \tag{4.2}$$

This implies that

$$(2k+6)\Theta(\infty,f) + 3\Theta(0,f) + 2\delta_{k+2}(0,f) \leq 2k+10. \tag{4.3}$$

This contradicts (1.6).

Subcase 1.2. Suppose that f and $(f^n)^{(k)}$ share $a(z)$ CM. Similarly as above, we can easily obtain $N_{1)}(r, 1/(F-1)) = N_{1)}(r, 1/(G-1)) + S(r,f)$; by Lemma 2.3, we can deduce

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) \leq \overline{N}(r,F) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}_0\left(r,\frac{1}{F'}\right) \\ + \overline{N}_0\left(r,\frac{1}{G'}\right) + \overline{N}_{(2}\left(r,\frac{1}{G-1}\right) + S(r,f). \end{aligned} \tag{4.4}$$

So by the second fundamental theorem, (4.4), and using Lemma 2.5 again, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) - N_0\left(r,\frac{1}{F'}\right) + S(r,f) \\ &\leq 2\overline{N}(r,f) + N_2\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{G'}\right) + S(r,f) \\ &\leq \left[(k+5) - (k+3)\Theta(\infty,f) - \delta_2(0,f) - \delta_{k+2}(0,f)\right]T(r,f) + S(r,f). \end{aligned} \tag{4.5}$$

This implies that

$$(k+3)\Theta(\infty,f) + \delta_2(0,f) + \delta_{k+2}(0,f) \leq k+4. \tag{4.6}$$

This contradicts (1.7).

Case 2 ($H \equiv 0$). Similarly, we can also get (3.14). Next we claim that $B = 0$. If $\overline{N}(r,f) \neq S(r,f)$, then it follows that $B = 0$ from (3.14). Hence, we may assume that (3.15) holds. If $B \neq 0$ and $B \neq -1$, then

$$\frac{A}{G-1} \equiv -\frac{BF-(B+1)}{F-1}, \tag{4.7}$$

and so

$$\overline{N}(r,G) = \overline{N}\left(r,\frac{1}{F-(B+1)/B}\right). \tag{4.8}$$

Again by second fundamental theorem and (4.4) we have

$$T(r,F)=\overline{N}\left(r,\frac{1}{F}\right)+S(r,f), \tag{4.9}$$

that is,

$$T(r,f)\leq\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\leq T(r,f)+S(r,f). \tag{4.10}$$

Then we have $T(r,f)=\overline{N}(r,1/f)$, and it follows that $\Theta(0,f)=0$ and from (3.15) we have $\Theta(\infty,f)=1$; then with (1.6) and (1.7) we may deduce $\delta_{k+2}(0,f)>1$. It is impossible, and we can assume that $B=-1$; thus, we can get

$$\frac{(f^n)^{(k)}}{a}-(A+1)\equiv -A\cdot a\cdot\frac{1}{f}. \tag{4.11}$$

It shows that $T(r,f)=T(r,(f^n)^{(k)})$.

If $A=-1$, by (4.11), then we have $f\cdot(f^n)^{(k)}\equiv a^2$, which with the above equality may lead to $T(r,f)=S(r,f)$, which is impossible. If $A\neq -1$, then by second fundamental theorem, Lemma 2.5, (3.15), and (4.11) we have

$$\begin{aligned}T\left(r,(f^n)^{(k)}\right)&\leq\overline{N}\left(r,\frac{1}{(f^n)^{(k)}-a(A+1)}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)+S(r,f),\\&\leq k\overline{N}(r,f)+N_{k+2}\left(r,\frac{1}{f}\right)+S(r,f)\leq T(r,f)+S(r,f),\end{aligned} \tag{4.12}$$

which with (3.15) may deduce $N_{k+2}(r,1/f)=T(r,f)+S(r,f)$; so $\delta_{k+2}(o,f)=0$, which with $\Theta(\infty,f)=1$ and (1.6) may deduce $\Theta(0,f)>1$, which is impossible. Hence our claim holds.

Next we will prove that $A=1$. From (3.17) we have $G-1\equiv A(F-1)$. Then

$$\overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F+1/A-1}\right). \tag{4.13}$$

If $A\neq 1$, then we have

$$T(r,F)\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f). \tag{4.14}$$

By Lemma 2.5, we get

$$T(r,f)\leq(k+1)\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k+2}\left(r,\frac{1}{f}\right)+S(r,f). \tag{4.15}$$

It implies that

$$(k+1)\Theta(\infty, f) + \Theta(0, f) + \delta_{k+2}(0, f) \le k+2. \tag{4.16}$$

Combining (4.16) with (1.6) yields

$$2(k+2) + \Theta(0, f) \ge 2(k+3)\Theta(\infty, f) + 3\Theta(0, f) + 2\delta_{2+k}(0, f) - 4\Theta(\infty, f) > 2k+6, \tag{4.17}$$

that is, $\Theta(0, f) > 2$. This is a contradiction.

Combining (4.16) with (1.7) yields

$$k+2+2\Theta(\infty, f) \ge (k+3)\Theta(\infty, f) + \Theta(0, f) + \delta_{k+2}(0, f) > k+4, \tag{4.18}$$

that is, $\Theta(\infty, f) > 1$, which is also a contradiction. Hence $A = 1$ and $f \equiv (f^n)^{(k)}$. Now Theorem 1.2 has been completely proved.

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