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Research Article

Krammer's Representation of the Pure Braid Group, P_3

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We consider Krammer's representation of the pure braid group on three strings: $P_3 \to GL(3, Z[t^{\pm 1}, q^{\pm 1}])$, where t and q are indeterminates. As it was done in the case of the braid group, B_3, we specialize the indeterminates t and q to nonzero complex numbers. Then we present our main theorem that gives us a necessary and sufficient condition that guarantees the irreducibility of the complex specialization of Krammer's representation of the pure braid group, P_3.

1. Introduction

Let B_n be the braid group on n strings. There are a lot of linear representations of B_n. The earliest was the Artin representation, which is an embedding $B_n \to Aut(F_n)$, the automorphism group of a free group on n generators. Applying the free differential calculus to elements of $Aut(F_n)$ sometimes gives rise to linear representations of B_n and its normal subgroup, the pure braid group denoted by P_n [1]. The Burau, Gassner, and Krammer's representations arise this way. In a previous paper, we considered Krammer's representation of the braid group on three strings and we specialized the indeterminates to nonzero complex numbers. We then found a necessary and sufficient condition that guarantees the irreducibility of such a representation. For more details, see [2].

In Section 2, we introduce some definitions of the pure braid group and Krammer's representation. In Sections 3 and 4, we present our work that leads to our main theorem, Theorem 4.2, which gives a necessary and sufficient condition for the specialization of Krammer's representation of P_3 to be irreducible.

2. Definitions

Definition 2.1 (see [1]). The braid group on n strings, B_n, is the abstract group with presentation $B_n = \{\sigma_1, \ldots, \sigma_{n-1} / \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots n-2, \sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1\}$.

The generators $\sigma_1, \ldots, \sigma_{n-1}$ are called the standard generators of B_n.

Definition 2.2. The kernel of the group homomorphism $B_n \to S_n$ is called the pure braid group on n strands and is denoted by P_n. It consists of those braids which connect the ith item of the left set to the ith item of the right set, for all i. The generators of P_n are $A_{i,j}$, $1 \le i < j \le n$, where $A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$.

Let us recall the Lawrence-Krammer representation of braid groups. This is a representation of B_n in $GL(m, Z[t^{\pm 1}, q^{\pm 1}]) = Aut(V_0)$, where $m = n(n-1)/2$ and V_0 is the free module of rank m over $Z[t^{\pm 1}, q^{\pm 1}]$. The representation is denoted by $K(q,t)$. For simplicity we write K instead of $K(q,t)$. What distinguishes this representation from others is that Krammer's representation defined on the braid group, B_n, is a faithful representation for all $n \ge 3$ [3]. The question of whether or not a specific linear representation of an abstract group is irreducible has always been a significant question to answer, especially those representations of the braid group and its normal subgroups. In a previous result, we determined a necessary and sufficient condition for the specialization of Krammer's representation of B_3 to be irreducible [2]. In our current work, we apply Krammer's representation on the normal subgroup of B_3, namely, the pure braid group, P_3. Having done some computations, we succeed in establishing a necessary and sufficient condition for the complex specialization of Krammer's representation of P_3 to be irreducible.

Definition 2.3 (see [3]). With respect to $\{x_{i,j}\}_{1\le i<j\le n}$, the free basis of V_0, the image of each Artin generator under Krammer's representation is written as

$$K(\sigma_k)(x_{i,j}) = \begin{cases} tq^2 x_{k,k+1}, & i = k,\ j = k+1; \\ (1-q)x_{i,k} + q x_{i,k+1}, & j = k,\ i < k; \\ x_{i,k} + tq^{k-i+1}(q-1)x_{k,k+1}, & j = k+1,\ i < k; \\ tq(q-1)x_{k,k+1} + q x_{k+1,j}, & i = k,\ k+1 < j; \\ x_{k,j} + (1-q)x_{k+1,j}, & i = k+1,\ k+1 < j; \\ x_{i,j}, & i < j < k \text{ or } k+1 < i < j; \\ x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1}, & i < k < k+1 < j. \end{cases} \tag{2.1}$$

Using the Magnus representation of subgroups of the automorphisms group of free group with $n(n-1)/2$ generators, we determine Krammer's representation $K(q,t) : P_3 \to GL(3, Z[t^{\pm 1}, q^{\pm 1}])$. Here $Z[t^{\pm 1}, q^{\pm 1}]$ is the ring of Laurent polynomials on two variables. The images of the generators under Krammer's representation are given by

$$K(A_{1,2}) = \begin{pmatrix} t^2q^4 & 0 & 0 \\ t^2q^3(q-1) & q & q(1-q) \\ tq(q-1) & 1-q & 1-q+q^2 \end{pmatrix},$$

$$K(A_{2,3}) = \begin{pmatrix} 1-q+q^2 & q(1-q) & tq^3(q-1) \\ 1-q & q & t^2q^4(q-1) \\ 0 & 0 & t^2q^4 \end{pmatrix}, \tag{2.2}$$

$$K(A_{1,3}) = \begin{pmatrix} q & q(q-1) & \dfrac{1-q-tq(q-1)^2}{t} \\ -tq(q-1)^2 & tq\left[tq^2(q^2-q+1)-(q-1)^3\right] & m \\ tq(1-q) & tq(q-1)(1-q+tq^2) & n \end{pmatrix},$$

where

$$m = -1 + q\left[2 - 2q + q^2 + t(q-1)^4 + q^2(1-q)(1+q(q-1))t^2\right],$$
$$n = 1 + q(q-1)\left[1 + t(q-1)\left(-1+q-tq^2\right)\right]. \tag{2.3}$$

Specializing t and q to non zero complex numbers, we consider the complex linear representation $K(q,t) : P_3 \to GL(3,C)$. We show that the only non zero invariant subspace under the action of specialization of Krammer's representation of P_3 coincides with the vectorspace C^3. Here, we regard $M_3(C)$ as acting from the left on column vectors so that eigenvectors and invariant subspaces lie in C^3.

3. Sufficient Condition for Irreducibility

In this section, we find a sufficient condition for the irreducibility of Krammer's representation of the pure braid group on three strings P_3.

Theorem 3.1. *For $(q,t) \in (C^*)^2$, Krammer's representation $K(q,t) : P_3 \to GL(3,C)$ is irreducible if $t^2q^3 \neq 1, tq^3 \neq 1, t \neq -1, q \neq 1, tq \neq 1$, and $tq^2 \neq -1$.*

Proof. For simplicity, we write $K(\alpha)$ instead of $K(q,t)(\alpha)$, where $\alpha \in P_3$. Suppose, to get contradiction, that $K(q,t) : P_3 \to GL(3,C)$ is reducible; then there exists a proper nonzero invariant subspace S, where the dimension of S is either 1 or 2. We will show that a contradiction is obtained in each of these cases.

Assume that dimension of S is 1:

The subspace S has to be one of the following subspaces: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_3\rangle$, $\langle e_1+ue_2\rangle$, $\langle e_2+ue_3\rangle$, $\langle e_1+ue_3\rangle$, $\langle e_1+ue_2+ve_3\rangle$, where u, v are non zero complex numbers.

Case 1 ($S = \langle e_1\rangle$). Since $e_1 \in S$, it follows that $A_{1,2}(e_1) \in S$ which implies that $t^2q^3(q-1) = 0$, a contradiction.

Case 2 ($S = \langle e_2 \rangle$). Since $e_2 \in S$, it follows that $A_{1,2}(e_2) \in S$ which implies that $1 - q = 0$, a contradiction.

Case 3 ($S = \langle e_3 \rangle$). Since $e_3 \in S$, it follows that $A_{1,2}(e_3) \in S$ which implies that $q(1-q) = 0$, a contradiction.

Case 4 ($S = \langle e_1 + ue_2 \rangle,\ u \neq 0$). Since $e_1 + ue_2 \in S$, it follows that $A_{1,2}(e_1 + ue_2) \in S$. This implies that

$$\begin{pmatrix} t^2q^4 \\ t^2q^3(q-1) + qu \\ tq(q-1) + (1-q)u \end{pmatrix} = m \begin{pmatrix} 1 \\ u \\ 0 \end{pmatrix}, \tag{3.1}$$

where m is a complex number. Solving this system of equations implies that $(tq-1)(tq^2+1) = 0$, which is a contradiction to the hypothesis.

Case 5 ($S = \langle e_2 + ue_3 \rangle,\ u \neq 0$). Since $e_2 + ue_3 \in S$, it follows that $A_{2,3}(e_2 + ue_3) \in S$. This implies that

$$\begin{pmatrix} q(1-q) + tq^3(q-1)u \\ q + t^2q^4(q-1)u \\ t^2q^4u \end{pmatrix} = m \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}, \tag{3.2}$$

where m is a complex number. By solving this system of equations, we get that $(tq-1)(tq^2+1) = 0$, which is a contradiction.

Case 6 ($S = \langle e_1 + ue_3 \rangle,\ u \neq 0$). Since $e_1 + ue_3 \in S$, it follows that $A_{1,2}(e_1 + ue_3) \in S$. This implies that

$$\begin{pmatrix} t^2q^4 \\ t^2q^3(q-1) + q(1-q)u \\ tq(q-1) + (1-q+q^2)u \end{pmatrix} = m \begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix}, \tag{3.3}$$

where m is a complex number. By solving this system of equations, we get that $(tq-1)(tq^2+1)(tq^2+q-1) = 0$.

By our hypothesis, $(tq-1)(tq^2+1) \neq 0$. This implies that $tq^2+q-1 = 0$. That is, $tq^2 = 1-q$. Also, we have that $A_{2,3}(e_1 + ue_3) \in S$. This implies that

$$\begin{pmatrix} 1-q+q^2+tq^3(q-1)u \\ 1-q+t^2q^4(q-1)u \\ t^2q^4u \end{pmatrix} = n \begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix}, \tag{3.4}$$

where n is a complex number. By solving this system of equations, we get that $t^2q^3 = -1$. This means that

$$t^2q^3 = tq\left(tq^2\right) = tq(1-q) = tq - tq^2 = tq - 1 + q. \tag{3.5}$$

This implies that $q(t+1) = 0$, which contradicts the hypothesis.

Case 7 ($S = \langle e_1+ue_2+ve_3\rangle,\ u,v \neq 0$). Since $e_1+ue_2+ve_3 \in S$, it follows that $A_{1,2}(e_1+ue_2+ve_3) \in S$. This implies that

$$\begin{pmatrix} t^2q^4 \\ t^2q^3(q-1) + qu + q(1-q)v \\ tq(q-1) + (1-q)u + (1-q+q^2)v \end{pmatrix} = m\begin{pmatrix} 1 \\ u \\ v \end{pmatrix}, \tag{3.6}$$

where m is a complex number. Since $A_{2,3}(e_1 + ue_2 + ve_3) \in S$, it follows that

$$\begin{pmatrix} 1-q+q^2+q(1-q)u + tq^3(q-1)v \\ 1-q+qu+t^2q^4(q-1)v \\ t^2q^4v \end{pmatrix} = n\begin{pmatrix} 1 \\ u \\ v \end{pmatrix}, \tag{3.7}$$

where n is a complex number. Solving these two system of equations, we get that $m = n = t^2q^4$. Also, we have that

$$q\left(t^2q^3 - 1\right)u + q(q-1)v = t^2q^3(q-1), \tag{3.8}$$

$$(q-1)u + \left(t^2q^4 - q^2 + q - 1\right)v = tq(q-1), \tag{3.9}$$

$$q(1-q)u + tq^3(q-1)v = t^2q^4 - q^2 + q - 1, \tag{3.10}$$

$$q\left(t^2q^3 - 1\right)u - t^2q^4(q-1)v = 1 - q. \tag{3.11}$$

Substracting (3.11) from (3.8), we get that $q(1+t^2q^3)v = 1+t^2q^3$. Here, we have 2 cases whether or not $(1+t^2q^3)$ is zero.

If $1 + t^2q^3 = 0$, then we rewrite (3.8), (3.9), (3.10), and (3.11) to become as follows:

$$2qu - q(q-1)v = q - 1, \tag{3.12}$$

$$(q-1)u - \left(q^2+1\right)v = tq(q-1), \tag{3.13}$$

$$q(1-q)u + tq^3(q-1)v = -\left(q^2+1\right). \tag{3.14}$$

Multiplying (3.13) by q and adding it to (3.14) we get that

$$q\left(tq^3 - tq^2 - q^2 - 1\right)v = tq^3 - tq^2 - q^2 - 1. \tag{3.15}$$

A simple computation shows that $tq^3 - tq^2 - q^2 - 1 \neq 0$. Thus $v = 1/q$. Substituting $v = 1/q$ in (3.12), we get that $u = (q-1)/q$. Substituting u and v in (3.14), we get that $tq^2 = tq - 2$. Having that $t^2q^3 = -1$ implies that $t^2q^3 = tq(tq^2) = tq(tq-2)$. This implies that $(tq-1)^2 = 0$ which contradicts the hypothesis.

This means that $1 + t^2q^3 \neq 0$. Then $v = 1/q$ and $u = (q-1)/q$ by (3.8). Substituting u and v in (3.9), we get that $(tq-1)(tq^2+1) = 0$, which contradicts the hypothesis.

Assume that dimension of S is 2:

Easy computations show that the subspace S cannot be in the form $S = \langle e_i, e_j \rangle$ or $S = \langle e_i + ue_j, e_k \rangle$ for $i \neq j \neq k$.

It suffices to consider only the case $S = \langle e_1 + ue_2, e_1 + ve_3 \rangle$, where $u, v \neq 0$.

Since $e_1 + ue_2 \in S$, it follows that $A_{1,2}(e_1 + ue_2) \in S$ and so

$$\begin{pmatrix} t^2q^4 \\ t^2q^3(q-1) + qu \\ tq(q-1) + (1-q)u \end{pmatrix} \in S. \tag{3.16}$$

Also, we have that $e_1 + ve_3 \in S$,then $A_{1,2}(e_1 + ve_3) \in S$, and so

$$\begin{pmatrix} t^2q^4 \\ t^2q^3(q-1) + q(1-q)v \\ tq(q-1) + (1-q+q^2)v \end{pmatrix} \in S. \tag{3.17}$$

This implies that $((q-q^2)v - qu)e_2 + ((1-q+q^2)v + (q-1)u)e_3 \in S$. Note that $((q-q^2)v - qu)$ and $((1-q+q^2)v + (q-1)u)$ cannot be both zeros. Assume then that $(q-q^2)v - qu \neq 0$.

Having that $ue_2 - ve_3 \in S$, we get that $\{u((1-q+q^2)v + (q-1)u) + v((q-q^2)v - qu)\}e_3 \in S$ and so

$$(u + qv)(u - v)e_3 \in S. \tag{3.18}$$

If $(u+qv)(u-v) \neq 0$, then $e_3 \in S$ and thus S is the whole space. Now if $(u+qv)(u-v) = 0$, then we have 2 cases: $u = -qv$ and $u = v$:

$$\text{let } \mathbf{u} = -\mathbf{qv}. \text{ Since} \begin{pmatrix} 0 \\ q \\ 1 \end{pmatrix} \in S, \text{it follows that} \begin{pmatrix} 0 \\ t^2q^3 \\ t^2q^2 \end{pmatrix} \in S. \tag{3.19}$$

On the other hand, we have that

$$q^{-2}A_{2,3}\begin{pmatrix}0\\ q\\ 1\end{pmatrix} = \begin{pmatrix}(q-1)(tq-1)\\ 1+t^2q^2(q-1)\\ t^2q^2\end{pmatrix} \in S. \tag{3.20}$$

Substracting (3.19) from (3.20) we get that $\begin{pmatrix}(q-1)(tq-1)\\ 1-t^2q^2\\ 0\end{pmatrix} \in S$.

This means that

$$(q-1)(tq-1)e_1 + \left(1-t^2q^2\right)e_2 \in S. \tag{3.21}$$

We also have that

$$e_1 - qve_2 \in S. \tag{3.22}$$

Solving (3.21) and (3.22), we get that $((1+tq)+q(1-q)v)e_2 \in S$.

If $(1+tq)+q(1-q)v \neq 0$, we are done. Otherwise, we have that $v = (tq+1)/q(q-1)$ and $u = -qv = (tq+1)/(1-q)$. On the other hand, we have that

$$\begin{pmatrix}1\\ u\\ 0\end{pmatrix} \in S \quad \text{then} \quad \begin{pmatrix}1-q\\ 1+tq\\ 0\end{pmatrix} \in S. \tag{3.23}$$

Also, we have that

$$A_{2,3}\begin{pmatrix}1-q\\ 1+tq\\ 0\end{pmatrix} = \begin{pmatrix}(1-q)(1+q^2+tq^2)\\ 1+q^2+tq^2-q\\ 0\end{pmatrix} \in S. \tag{3.24}$$

Solving (3.23) and (3.24) implies that $q(1+t)(1+tq^2)e_2 \in S$ and thus $e_2 \in S$. Hence $S = C^3$.

Let $\mathbf{u} = \mathbf{v}$. Since $e_2 - e_3 \in S$, it follows that $A_{2,3}(e_2 - e_3) \in S$. That is, we have that

$$\begin{pmatrix}(q-1)(-1-tq^2)\\ 1-t^2q^3(q-1)\\ -t^2q^3\end{pmatrix} \in S. \tag{3.25}$$

We also have that

$$\begin{pmatrix} 0 \\ t^2q^3 \\ -t^2q^3 \end{pmatrix} \in S. \tag{3.26}$$

Substracting (3.26) from (3.25), we get that

$$(q-1)\left(-1-tq^2\right)e_1 + \left(1-t^2q^4\right)e_2 \in \mathrm{S}. \tag{3.27}$$

Also we have that

$$e_1 + ve_2 \in S. \tag{3.28}$$

Solving (3.27) and (3.28), we get that $\{(1+tq^2)[(1-tq^2)+v(q-1)]\}e_2 \in S$. If $[(1-tq^2)+v(q-1)] = 0$, then we get that $u = v = (tq^2-1)/(q-1)$.

Now we have that $e_1 + ue_2 \in S$ and so

$$\begin{pmatrix} (q-1)(1+q^2-tq^3) \\ (tq^2-1)(1+q^2-tq^3) \\ 0 \end{pmatrix} \in S. \tag{3.29}$$

We also have that

$$A_{2,3}\begin{pmatrix} q-1 \\ tq^2-1 \\ 0 \end{pmatrix} = \begin{pmatrix} (q-1)(1+q^2-tq^3) \\ -q^2+q-1+tq^3 \\ 0 \end{pmatrix} \in S. \tag{3.30}$$

Substracting (3.30) from (3.29), we get that $q(1-tq)(tq^3-1)e_2 \in S$ and so $e_2 \in S$. Thus $S = C^3$. □

Next, we find a necessary condition that guarantees the irreducibility of the complex specialization of Krammer's representation of P_3.

4. Necessary Condition for Irreducibility

We present the following theorem.

Theorem 4.1. *For $(q,t) \in (C^*)^2$, Krammer's representation $K(q,t) : P_3 \to GL(3,C)$ is reducible if one of the following conditions is satisfied:*

(1) $t^2q^3 = 1$,

(2) $tq^3 = 1$,

(3) $t = -1$,

(4) $q = 1$,

(5) $tq = 1$,

(6) $tq^2 = -1$.

Proof. Notice that the first three conditions followed from the reducibility on B_3. Under each of the last three conditions of our hypothesis, we find a proper nonzero invariant subspace under the action of complex specialization of Krammer's representation of P_3. Recall that the matrices $K(A_{1,2})$, $K(A_{2,3})$, and $K(A_{1,3})$ that will be used in the proof are those given in Definition 2.3. □

Proof of 4 ($q = 1$). We have that

$$K(A_{1,2}) = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K(A_{2,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2 \end{pmatrix},$$
$$K(A_{1,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

We take the invariant subspace as the one generated by $e_1 = (1, 0, 0)$. □

Proof of 5 ($tq = 1$). We have that

$$K(A_{1,2}) = \begin{pmatrix} q^2 & 0 & 0 \\ q(q-1) & q & q(1-q) \\ q-1 & 1-q & 1-q+q^2 \end{pmatrix},$$
$$K(A_{2,3}) = \begin{pmatrix} 1-q+q^2 & q(1-q) & q^2(q-1) \\ 1-q & q & q^2(q-1) \\ 0 & 0 & q^2 \end{pmatrix}, \tag{4.2}$$
$$K(A_{1,3}) = \begin{pmatrix} q & q(q-1) & (1-q)q^2 \\ -(q-1)^2 & 1+2q(q-1) & -q(q-1)^2 \\ 1-q & q-1 & q \end{pmatrix}.$$

We take the invariant subspace as the one generated by $m = (0, q, 1)^T$. More precisely, we have that

$$K(A_{1,2})(m) = m, \qquad K(A_{2,3})(m) = q^2 m, \qquad K(A_{1,3})(m) = q^2 m. \tag{4.3}$$

□

Proof of 6 ($tq^2 = -1$). We have that

$$K(A_{1,2}) = \begin{pmatrix} 1 & 0 & 0 \\ 1+tq & q & q(1-q) \\ -1-tq & 1-q & 1-q+q^2 \end{pmatrix},$$

$$K(A_{2,3}) = \begin{pmatrix} 1-q+q^2 & q(1-q) & q(1-q) \\ 1-q & q & q-1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.4)$$

$$K(A_{1,3}) = \begin{pmatrix} q & q(q-1) & q(q-1) \\ q-2-tq & q^2-2q+2 & (q-1)^2 \\ tq+1 & q-1 & q \end{pmatrix}.$$

We take the invariant subspace as the one generated by $m = (-q, 1, 0)^T$. More precisely, we have that

$$K(A_{1,2})(m) = m, \qquad K(A_{2,3})(m) = q^2 m, \qquad K(A_{1,3})(m) = m. \quad (4.5)$$

□

Combining Theorems 3.1 and 4.1, we obtain our main theorem.

Theorem 4.2. *For $(q,t) \in (C^*)^2$, Krammer's representation $K(q,t) : P_3 \to GL(3,C)$ is irreducible if and only if $t^2q^3 \neq 1$, $tq^3 \neq 1$, $t \neq -1$, $q \neq 1$, $tq \neq 1$, and $tq^2 \neq -1$.*

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