

## Research Article

# On the Absolute Sum of Chromatic Polynomial Coefficient of Graphs

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The absolute sum of chromatic polynomial coefficient of forest,  $q$ -tree, unicyclic graphs, and quasiwheel graphs, are determined in this paper.

## 1. Introduction

For a century ago, one of the most famous problems in mathematics was to prove the *Four-Color Problem*. During the period that the Four-Color Problem was unsolved, which spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution of this famous problem. In 1913, Birkhoff [1] defined a function  $P(M, \lambda)$  that gives the number of proper  $\lambda$ -colorings of a map  $M$  for a positive integer  $\lambda$ . As we will see,  $P(M, \lambda)$  is a polynomial in  $\lambda$  for every map  $M$  and is called the chromatic polynomial of  $M$ . Consequently, if it could be verified that  $P(M, 4) > 0$  for every map  $M$ , then this would have established the truth of the Four-Color Conjecture. In 1932, Whitney [2] expanded the study of chromatic polynomials from maps to graphs. While Whitney obtained a number of results on chromatic polynomials of graphs and others obtained results on the roots of chromatic polynomials of planar graphs, this did not contribute to a proof of the Four-Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred in 1968 when Read [3] wrote a survey paper on chromatic polynomials. Let  $G$  be a graph and  $\lambda \in \mathbb{N}$ . A mapping  $f : V(G) \rightarrow \{1, 2, \dots, \lambda\}$  is called a  $\lambda$ -coloring of  $G$  if  $f(u) \neq f(v)$  whenever the vertices  $u$  and  $v$  are adjacent in  $G$ . For a positive integer  $\lambda$ , the number of different proper  $\lambda$ -colorings of  $G$  is denoted by  $P(G, \lambda)$  and is called the chromatic polynomial of  $G$ . By convention,  $P(G, 0) = 0$ , and  $P(G, \lambda) \geq 1$  if and only if  $G$  is  $\lambda$ -colorable. More precisely, we have

$$\chi(G) = \min\{\lambda \in \mathbb{N} : P(G, \lambda) \geq 1\}. \quad (1.1)$$

There are many results about the coefficient of the chromatic polynomials of graphs, see related references [4–8]. In [5], the authors characterized the absolute sum of chromatic polynomial of trees, 2-trees, cycles, wheels and completed graphs, and obtained the sharp upper and lower bounds for the absolute sum of chromatic polynomial coefficients of any connected graphs. In this paper, we investigate absolute sum of chromatic polynomial coefficients of forest,  $q$ -tree, unicyclic graphs and quasiwheel graphs.

## 2. Preliminaries

Let  $G = (V, E)$  be a graph whose sets of vertices and edges are  $V(G)$  and  $E(G)$ , respectively,  $n = |V(G)|$ ,  $m = |E(G)|$  are the number of vertices and edges of  $G$ .  $E' \subseteq E(G)$ , we denote by  $G - E'$  ( $G + E'$ ) the subgraph of  $G$  obtained by deleting (adding) the edges of  $E'$ .  $W \subseteq V(G)$ ,  $G - W$  ( $G + W$ ) denote the subgraph of  $G$  obtained by deleting (adding) the vertices of  $W$  and the edges incident with them.  $G \cdot xy$  is the graph obtained from  $G$  by contracting  $x$  and  $y$  and removing any loop and all but one of the multiple edges, if they arise, where  $x, y \in V(G)$ . Terminologies and notations not defined here can be found in [9]. The following basic results will be used and can be found in the references cited.

**Lemma 2.1** (see [9]). (i) Let  $x$  and  $y$  be two nonadjacent vertices in a graph  $G$ . Then  $P(G, \lambda) = P(G + xy, \lambda) + P(G \cdot xy, \lambda)$ .

(ii) Let  $e \in E(G)$ , then  $P(G, \lambda) = P(G - e, \lambda) - P(G \cdot e, \lambda)$ .

(iii) For the empty graph  $O_n$  of order  $n$ , it is clear that  $P(O_n, \lambda) = \lambda^n$ . More generally, if  $G = \bigcup_{i=1}^k G_i$ , then

$$P(G, \lambda) = \prod_{i=1}^k P(G_i, \lambda). \quad (2.1)$$

**Lemma 2.2** (see [5]). (i) Let  $G$  be an arbitrary graph with  $n$  vertices, then the sum of chromatic polynomial of  $P(G, \lambda) = \sum_{i=1}^n a_i \lambda^i$  is

$$\sum_{i=1}^n a_i = \begin{cases} 0, & m \neq 0, \\ 1, & m = 0. \end{cases} \quad (2.2)$$

(ii) Let  $T$  be an arbitrary tree with  $n$  vertices and  $P(T, \lambda) = \sum_{i=1}^n a_i \lambda^i = \lambda(\lambda - 1)^{n-1}$ . Then,  $\sum_{i=1}^n |a_i| = 2^{n-1}$ .

(iii) Let  $T_n^2$  be an arbitrary 2-tree with  $n$  vertices and  $P(T_n^2, \lambda) = \sum_{i=1}^n a_i \lambda^i = \lambda(\lambda - 1)(\lambda - 2)^{n-2}$ . Then,  $\sum_{i=1}^n |a_i| = 2 \cdot 3^{n-2}$ .

(iv) Let  $C_n$  be an arbitrary graph with  $n$  vertices and  $P(C_n, \lambda) = \sum_{i=1}^n a_i \lambda^i = (\lambda - 1)^n + (-1)^n(\lambda - 1)$ . Then,  $\sum_{i=1}^n |a_i| = 2^n - 2$ .

(v) Let  $W_n$  be an arbitrary graph with  $n$  vertices and  $P(W_n, \lambda) = \sum_{i=1}^n a_i \lambda^i = \lambda(\lambda - 2)^{n-1} + (-1)^{n-1}\lambda(\lambda - 2)$ . Then,  $\sum_{i=1}^n |a_i| = 3^{n-1} - 3$ .

(vi) Let  $K_n$  be an arbitrary graph with  $n$  vertices and  $P(K_n, \lambda) = \sum_{i=1}^n a_i \lambda^i = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$ . Then,  $\sum_{i=1}^n |a_i| = \prod_{i=1}^n i$ .

(vii) Let  $G$  be an arbitrary connected graph with  $n$  ( $n > 1$ ) vertices and  $P(G, \lambda) = \sum_{i=1}^n a_i \lambda^i$ , then

$$2^{n-1} \leq \sum_{i=1}^n |a_i| \leq \prod_{i=1}^n i. \tag{2.3}$$

The left equality holds if and only if  $G \cong T_n$  and the right equality holds if and only if  $G \cong K_n$ .

Given  $q \in \mathbb{N}$ , the class of  $q$ -trees  $T_n^q$  is defined recursively as follows: any complete graph  $K_q$  is a  $q$ -tree, and any  $q$ -tree of order  $n + 1$  is a graph obtained from a  $q$ -tree  $G$  of order  $n$ , where  $n \geq q$ , by adding a new vertex and joining it to each vertex of a  $K_q$  in  $G$ . As an example, a 3-tree is depicted in Figure 1(a).

**Lemma 2.3** (see [9]). Let  $T_n^q$  is a  $q$ -tree with  $n$  vertices, then

$$P(T_n^q, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - q + 1). \tag{2.4}$$

**Lemma 2.4** (see [6]). Let  $\mathcal{U}_n(m)$  is a unicyclic graph order  $n$  and girth  $m$ , then

$$P(\mathcal{U}_n(m), \lambda) = (\lambda - 1)^n + (-1)^m (\lambda - 1)^{n-m+1}. \tag{2.5}$$

For  $1 \leq s \leq n - 2$ , we denote by  $W(n, s)$  the graph obtained from  $W_n$  by deleting all but  $s$  consecutive spokes. For convenience, we call  $W(n, s)$  as the quasiwheel graphs, as shown in Figure 1(b).

**Lemma 2.5** (see [9]). Let  $W(n, s)$  is a quasiwheel graph with  $n$  vertices, then

$$P(W(n, s), \lambda) = (\lambda - 2)^{s-1} [(\lambda - 1)^{n-s+1} + (-1)^{n-s}] + (-1)^{n-1} \lambda(\lambda - 2). \tag{2.6}$$

### 3. Main Results

In this section, we investigate the absolute sum of chromatic polynomial coefficients of forests,  $q$ -trees, unicyclic graphs, and quasiwheel graphs.

**Theorem 3.1.** Let  $F$  be the forest of order  $n$  and  $c$  components and  $P(F, \lambda) = \sum_{i=1}^n a_i \lambda^i$ . Then  $\sum_{i=1}^n |a_i| = 2^{n-c}$ .

*Proof.* It is a direct result of the combination of Lemma 2.1(iii) and Lemma 2.2(ii). □

**Theorem 3.2.** Let  $T_n^q$  is a  $q$ -tree of order  $n$  ( $n \geq q$ ) and  $P(T_n^q, \lambda) = \sum_{i=1}^n a_i \lambda^i$ . Then  $\sum_{i=1}^n |a_i| = q! \times (q + 1)^{n-q}$ .

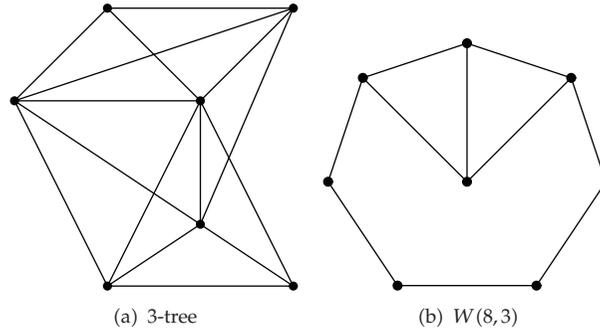


Figure 1: The graphs 3-tree and  $W(8,3)$ .

*Proof.* We prove the result inductively on  $n$ .

(i)  $n = q$ . In this case  $T_n^q \cong K_q$ .

$$P(T_q^q, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - q + 1) = \sum_{i=1}^q a_i \lambda^i, \quad (3.1)$$

then  $\sum_{i=1}^q |a_i| = \prod_{i=1}^q i = q!$ .

(ii) Assume that  $n = k$  ( $k \geq q$ ), let  $P(T_k^q, \lambda) = \sum_{i=1}^k b_i \lambda^i$ , we have  $\sum_{i=1}^k |b_i| = q! \times (q + 1)^{k-q}$ .

(iii) When  $n = k + 1$ , let  $P(T_{k+1}^q, \lambda) = \sum_{i=1}^{k+1} c_i \lambda^i$ .  $u$  is a vertex with degree  $q$ , and the edges connecting  $u$  are  $e_1, e_2, \dots, e_q$ , respectively. By Lemma 2.1(ii), we have

$$\begin{aligned} P(T_{k+1}^q, \lambda) &= P(T_{k+1}^q - e_1, \lambda) - P(T_{k+1}^q \cdot e_1, \lambda) \\ &= P((T_{k+1}^q - e_1) - e_2, \lambda) - P((T_{k+1}^q - e_1) \cdot e_2, \lambda) - P(T_{k+1}^q \cdot e_1, \lambda) \\ &= \dots \\ &= \lambda P(T_k^q, \lambda) - q P(T_k^q, \lambda) \\ &= \lambda \sum_{i=1}^k b_i \lambda^i - q \sum_{i=1}^k b_i \lambda^i. \end{aligned} \quad (3.2)$$

Then,  $\sum_{i=1}^{k+1} |c_i| = \sum_{i=1}^k |b_i| + q \sum_{i=1}^k |b_i| = (q + 1) \times q! \times (q + 1)^{k-q} = q! \times (q + 1)^{k+1-q}$ .  $\square$

Combining above, the results follows.

**Theorem 3.3.** Let  $\mathcal{U}_n(m)$  is a unicyclic graph order  $n$  and girth  $m$  and  $P(\mathcal{U}_n(m), \lambda) = \sum_{i=1}^n a_i \lambda^i$ . Then,  $\sum_{i=1}^n |a_i| = 2^n - 2^{n-m+1}$ .

*Proof.* From Lemma 2.3, we have

$$\begin{aligned}
 P(\mathcal{W}_n(m), \lambda) &= (\lambda - 1)^n + (-1)^m (\lambda - 1)^{n-m+1} \\
 &= \sum_{i=0}^n C_n^i \lambda^{n-i} (-1)^i + (-1)^m \sum_{i=0}^{n-m+1} C_{n-m+1}^i \lambda^{n-m+1-i} (-1)^i \\
 &= \sum_{i=0}^{m-2} C_n^i \lambda^{n-i} (-1)^i + \sum_{i=m-1}^n \left[ C_n^i (-1)^i + C_{n-m+1}^{i+1-m} (-1)^{i+1} \right] \lambda^{n-i}.
 \end{aligned} \tag{3.3}$$

Let  $(\lambda - 1)^n + (-1)^m (\lambda - 1)^{n-m+1} = \sum_{i=1}^n a_i \lambda^i$ , then

$$\begin{aligned}
 \sum_{i=0}^n |a_i| &= \sum_{i=0}^{m-2} |C_n^i| + \sum_{i=m-1}^n |C_n^i - C_{n-m+1}^{i+1-m}| \\
 &= \sum_{i=0}^{m-2} C_n^i + \sum_{i=m-1}^n |C_n^{n-i} - C_{n-m+1}^{n-i}| \\
 &= \sum_{i=0}^{m-2} C_n^i + \sum_{i=m-1}^n C_n^{n-i} - \sum_{i=m-1}^n C_{n-m+1}^{n-i} \\
 &= 2^n - 2^{n-m+1}. \quad \square
 \end{aligned} \tag{3.4}$$

**Theorem 3.4.** Let  $W(n, s)$  is a quasiwheel graph with  $n$  vertices and  $P(W(n, s), \lambda) = \sum_{i=1}^n a_i \lambda^i$ . Then,  $\sum_{i=1}^n |a_i| = 3^{s-1} (2^{n-s+1} - 1) - 3$ .

*Proof.* From Lemma 2.5, we have

$$\begin{aligned}
 P(W(n, s), \lambda) &= (\lambda - 2)^{s-1} \left[ (\lambda - 1)^{n-s+1} + (-1)^{n-s} \right] + (-1)^{n-1} \lambda (\lambda - 2) \\
 &= \left[ \lambda^{s-1} + \dots + C_{s-1}^1 \lambda (-2)^{s-2} + (-2)^{s-1} \lambda^{n-s+1} + \dots + C_{n-s+1}^2 \lambda^2 (-1)^{n-s-1} \right. \\
 &\quad \left. + C_{n-s+1}^1 \lambda (-1)^{n-s} \right] + (-1)^{n-1} (\lambda^2 - 2\lambda) \\
 &= \lambda^n + \dots + \left[ C_{s-1}^1 2^{s-2} C_{n-s+1}^1 (-1)^{n-2} + C_{n-s+1}^2 2^{s-1} (-1)^{n-2} + (-1)^{n-1} \right] \lambda^2 \\
 &\quad + \left[ C_{n-s+1}^1 2^{s-1} (-1)^{n-1} + 2(-1)^n \right] \lambda.
 \end{aligned} \tag{3.5}$$

Let  $(\lambda - 2)^{s-1} [(\lambda - 1)^{n-s+1} + (-1)^{n-s}] = \sum_{i=1}^n b_i \lambda^i$ , then  $\sum_{i=1}^n |b_i| = 3^{s-1} (2^{n-s+1} - 1)$ . Moreover,

$$\begin{aligned}
 \sum_{i=1}^n b_i \lambda^i &= \lambda^n + \dots + \left[ C_{s-1}^1 2^{s-2} C_{n-s+1}^1 (-1)^{n-2} + C_{n-s+1}^2 2^{s-1} (-1)^{n-2} \right] \lambda^2 \\
 &\quad + C_{n-s+1}^1 2^{s-1} (-1)^{n-1} \lambda.
 \end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}\sum_{i=1}^n |b_i| &= \sum_{i=3}^n |b_i| + C_{s-1}^1 2^{s-2} C_{n-s+1}^1 + C_{n-s+1}^2 2^{s-1} + C_{n-s+1}^1 2^{s-1}, \\ \sum_{i=1}^n |a_i| &= \sum_{i=3}^n |b_i| + C_{s-1}^1 2^{s-2} C_{n-s+1}^1 - 1 + C_{n-s+1}^2 2^{s-1} + C_{n-s+1}^1 2^{s-1} - 2 \\ &= 3^{s-1} (2^{n-s+1} - 1) - 3.\end{aligned}\tag{3.7}$$

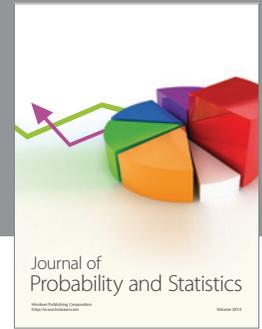
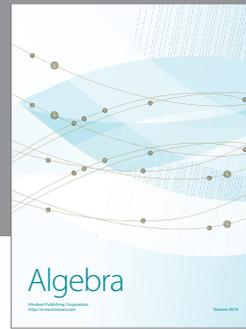
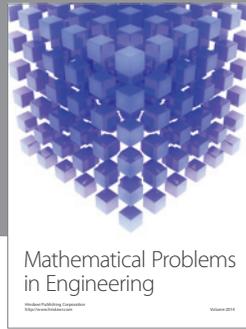
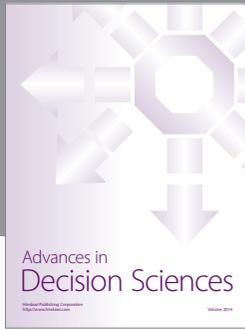
This completes the proof.  $\square$

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