Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 279246, 5 pages
doi:10.1155/2011/279246

Research Article

Equitable Coloring on Total Graph of Bigraphs and Central Graph of Cycles and Paths

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Received 2 December 2010; Accepted 9 February 2011

Academic Editor: Marco Squassina

The notion of equitable coloring was introduced by Meyer in 1973. In this paper we obtain interesting results regarding the equitable chromatic number $\chi_=$ for the total graph of complete bigraphs $T(K_{m,n})$, the central graph of cycles $C(C_n)$ and the central graph of paths $C(P_n)$.

1. Introduction

The central graph [1, 2] $C(G)$ of a graph G is formed by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously nonadjacent.

The total graph [3, 4] of G has vertex set $V(G) \cup E(G)$ and edges joining all elements of this vertex set which are adjacent or incident in G.

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable*. The smallest integer k for which G is equitable k-colorable is known as the *equitable chromatic number* [5–10] of G and denoted by $\chi_=(G)$. Additional graph theory terminology used in this paper can be found in [3, 4].

2. Equitable Coloring on Total Graph of Complete Bigraphs

Theorem 2.1. *If $m \le n$, the equitable chromatic number of total graph of complete bigraphs $K_{m,n}$,*

$$\chi_{=}(T(K_{m,n})) = \begin{cases} n+1 & \text{if } m < n, \\ n+2 & \text{if } m = n. \end{cases} \tag{2.1}$$

Proof. Let (X, Y) be the bipartition of $K_{m,n}$, where $X = \{v_i : 1 \le i \le m\}$ and $Y = \{v'_j : 1 \le j \le n\}$. Let u_{ij} $(1 \le i \le m; 1 \le j \le n)$ be the edges of $v_i v'_j$. By the definition of total graph, $T(K_{m,n})$ has the vertex set $\{v_i : 1 \le i \le m\} \cup \{v'_j : 1 \le j \le n\} \cup \{u_{ij} : 1 \le i \le m, 1 \le j \le n\}$ and the vertices $\{u_{ij} : 1 \le i \le m, 1 \le j \le n\}$ induce n disjoint cliques of order n in $T(K_{m,n})$. Also v_i $(1 \le i \le m)$ is adjacent to v'_j $(1 \le j \le n)$.

Case 1 (if $m = n$, $\chi_{=}(T(K_{m,n})) = n+2$). Now we partition the vertex set $V(T(K_{m,n}))$ as follows:

$$\begin{aligned} V_1 &= \{u_{11}, u_{2n}, u_{3(n-1)}, u_{4(n-2)}, \dots, u_{(n-1)3}, u_{n2}\}, \\ V_2 &= \{u_{12}, u_{21}, u_{3n}, u_{4(n-1)}, \dots, u_{(n-1)4}, u_{n3}\}, \\ &\vdots \\ V_n &= \{u_{1n}, u_{2(n-1)}, u_{3(n-2)}, u_{4(n-3)}, \dots, u_{(n-1)3}, u_{n1}\}, \\ V_{n+1} &= \{v_1, v_2, \dots, v_n\}, \\ V_{n+2} &= \{v'_1, v'_2, \dots, v'_n\}. \end{aligned} \tag{2.2}$$

Clearly $V_1, V_2, \dots, V_{n+2}$ are independent sets and $|V_i| = n$ $(1 \le i \le n+2)$ satisfying the condition $||V_i| - |V_j|| = 0$, for any $i \ne j$, $\chi_{=}(T(K_{m,n})) \le n+2$. Since there exists a clique of order $n+1$ in $T(K_{m,n})$. $\chi(T(K_{m,n})) \ge n+1$, also each v_i of $T(K_{m,n})$ receives one color different from the color class assigned to the clique induced by $\{u_{ij} : 1 \le i \le m; 1 \le j \le n\}$. By the definition of total graph, each v_i is adjacent with v'_j $(1 \le j \le n)$. Therefore, $\{v_1, v_2, \dots, v_m\}$ and $\{v'_1, v'_2, \dots, v'_n\}$ are independent sets and hence $\chi(T(K_{m,n})) \ge n+2$. That is, $\chi_{=}(T(K_{m,n})) \ge \chi(T(K_{m,n})) \ge n+2$; therefore $\chi_{=}(T(K_{m,n})) \ge n+2$. Hence $\chi_{=}(T(K_{m,n})) = n+2$.

Case 2 (if $m < n$, $\chi_{=}(T(K_{m,n})) = n+1$). Now we partition the vertex set $V(T(K_{m,n}))$ as follows:

$$\begin{aligned} V_1 &= \{u_{11}, u_{22}, u_{33}, u_{44}, \dots, u_{mm}\} \cup \{v'_n\}, \\ V_2 &= \{u_{12}, u_{23}, u_{34}, \dots, u_{m(m-1)}\} \cup \{u_{m1}\} \cup \{v'_1\}, \\ V_3 &= \{u_{13}, u_{24}, u_{35}, \dots, u_{m(m-2)}\} \cup \{u_{(m-1)3}, u_{m2}\} \cup \{v'_2\}, \\ &\vdots \\ V_{n-1} &= \{u_{1(n-1)}, u_{2n}\} \cup \{u_{31}, u_{32}, \dots, u_{m(m-2)}\} \cup \{v'_{n-2}\}, \\ V_n &= \{u_{1n}\} \cup \{u_{21}, u_{32}, \dots, u_{m(m-1)}\} \cup \{v'_{n-1}\}, \\ V_{n+1} &= \{v_1, v_2, v_3, \dots, v_m\}. \end{aligned} \tag{2.3}$$

Clearly $V_1, V_2, \ldots, V_{n+1}$ are independent sets of $T(K_{m,n})$. Also $|V_1| = |V_2| = \cdots = |V_n| = m+1$ and $|V_{n+1}| = m$ satisfy the condition $||V_i| - |V_j|| \leq 1$, for any $i \neq j$, $\chi_=(T(K_{m,n})) \leq n+1$. Since there exists a clique of order $n+1$ in $T(K_{m,n})$. $\chi(T(K_{m,n})) \geq n+1$, that is, $\chi_=(T(K_{m,n})) \geq \chi(T(K_{m,n})) \geq n+1$, therefore $\chi_=(T(K_{m,n})) \geq n+1$. Hence $\chi_=(T(K_{m,n})) = n+1$. □

3. Equitable Coloring on Central Graph of Cycles and Paths

Theorem 3.1. *If $n \geq 5$, the equitable chromatic number of central graph of cycles C_n,*

$$\chi_=(C(C_n)) = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd}, \\ \dfrac{n}{2} & \text{if } n \text{ is even}. \end{cases} \tag{3.1}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ be the vertices and edges of C_n taken in the cyclic order. By the definition of central graph, $C(C_n)$ has the vertex set $V(C_n) \cup \{u_i : 1 \leq i \leq n\}$, where u_i is the vertex of subdivision of the edge e_i and joining all the nonadjacent vertices of C_n in $C(C_n)$.

Case 1 (n is odd). We partition the vertex set $V(C(C_n))$ as

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-2}, u_{n-1}\}, \\ V_2 &= \{v_3, v_4, u_n\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{(n-1)/2} &= \{v_{n-2}, v_{n-1}, u_{n-6}, u_{n-5}\}, \\ V_{(n+1)/2} &= \{v_n, u_{n-4}, u_{n-3}\}. \end{aligned} \tag{3.2}$$

Clearly $V_1, V_2, \ldots V_{(n-1)/2}, V_{(n+1)/2}$ are independent sets of $C(C_n)$. Also $|V_1| = |V_3| = |V_4| = \cdots = |V_{(n-1)/2}| = 4$ and $|V_2| = |V_{(n+1)/2}| = 3$. The inequality $||V_i| - |V_j|| \leq 1$ holds, for any $i \neq j$, $\chi_=(C(C_n)) \leq (n+1)/2$. For each i, v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(C_n)) \geq (n+1)/2$. That is, $\chi_= C(C_n) \geq \chi(C(C_n)) \geq (n+1)/2$, $\chi_=(C(C_n)) \geq (n+1)/2$. Therefore, $\chi_=(C(C_n)) = (n+1)/2$.

Case 2 (n is even). Now we partition the vertex set $V(C(C_n))$ as follows:

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-3}, u_{n-2}\}, \\ V_2 &= \{v_3, v_4, u_{n-1}, u_n\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{n/2} &= \{v_{n-1}, v_n, u_{n-5}, u_{n-4}\}. \end{aligned} \tag{3.3}$$

Clearly $V_1, V_2, \ldots V_{n/2}$ are independent sets of $C(C_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = \cdots = |V_{n/2}| = 4$. The inequality $||V_i| - |V_j|| = 0$ holds, for any $i \neq j$, $\chi_{=}(C(C_n)) \leq n/2$. For each i, v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(C_n)) \geq n/2$. That is, $\chi_{=}C(C_n) \geq \chi(C(C_n)) \geq n/2$, $\chi_{=}(C(C_n)) \geq n/2$. Therefore, $\chi_{=}(C(C_n)) = n/2$. □

Remark 3.2. If $n = 3, 4$, then $\chi_{=}(C(C_n)) = 2, 3$, respectively.

Theorem 3.3. *If $n \geq 5$, the equitable chromatic number of central graph of paths P_n,*

$$\chi_{=}(C(P_n)) = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases} \tag{3.4}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{e_1, e_2, \ldots, e_n\}$ be the vertices and edges of P_n. By the definition of central graph, $C(P_n)$ has the vertex set $V(P_n) \cup \{u_i : 1 \leq i \leq n-1\}$, where u_i is the vertex of subdivision of the edge e_i and joining all nonadjacent vertices of P_n in $C(P_n)$.

Case 1 (n is odd). Now we partition the vertex set $V(C(P_n))$ as follows:

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-2}\}, \\ V_2 &= \{v_3, v_4, u_{n-1}\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{(n-1)/2} &= \{v_{n-1}, v_{n-2}, u_{n-6}, u_{n-5}\}, \\ V_{(n+1)/2} &= \{v_n, u_{n-4}, u_{n-3}\}. \end{aligned} \tag{3.5}$$

Clearly $V_1, V_2, \ldots V_{(n-1)/2}, V_{(n+1)/2}$ are independent sets of $C(P_n)$. Also $|V_3| = |V_4| = \cdots = |V_{(n-1)/2}| = 4$ and $|V_1| = |V_2| = |V_{(n+1)/2}| = 3$. The inequality $||V_i| - |V_j|| \leq 1$ holds, for any $i \neq j$, $\chi_{=}(C(P_n)) \leq (n+1)/2$. For each i, v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(P_n)) \geq (n+1)/2$. That is, $\chi_{=}C(P_n) \geq \chi(C(P_n)) \geq (n+1)/2$, $\chi_{=}(C(P_n)) \geq (n+1)/2$. Therefore $\chi_{=}(C(P_n)) = (n+1)/2$.

Case 2 (n is even). Now we partition the vertex set $V(C(P_n))$ as follows:

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-3}, u_{n-2}\}, \\ V_2 &= \{v_3, v_4, u_{n-1}\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{n/2} &= \{v_{n-1}, v_n, u_{n-5}, u_{n-4}\}. \end{aligned} \tag{3.6}$$

Clearly $V_1, V_2, \ldots V_{n/2}$ are independent sets of $C(P_n)$. Also $|V_1| = |V_3| = |V_4| = \cdots = |V_{n/2}| = 4$ and $|V_2| = 3$. The inequality $\||V_i| - |V_j|\| \leq 1$ holds for any $i \neq j$, $\chi_{=}(C(P_n)) \leq n/2$. For each i, v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(P_n)) \geq n/2$. That is, $\chi_{=}C(P_n) \geq \chi(C(P_n)) \geq n/2$, $\chi_{=}(C(P_n)) \geq n/2$. Therefore, $\chi_{=}(C(P_n)) = n/2$. □

Remark 3.4. If $n = 1, 2, 3, 4$, then $\chi_{=}(C(P_n)) = 1, 2, 3, 3$, respectively.

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