

Research Article

Equitable Coloring on Total Graph of Bigraphs and Central Graph of Cycles and Paths

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Received 2 December 2010; Accepted 9 February 2011

Academic Editor: Marco Squassina

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The notion of equitable coloring was introduced by Meyer in 1973. In this paper we obtain interesting results regarding the equitable chromatic number $\chi_=(G)$ for the total graph of complete bigraphs $T(K_{m,n})$, the central graph of cycles $C(C_n)$ and the central graph of paths $C(P_n)$.

1. Introduction

The central graph [1, 2] $C(G)$ of a graph G is formed by adding an extra vertex on each edge of G , and then joining each pair of vertices of the original graph which were previously nonadjacent.

The total graph [3, 4] of G has vertex set $V(G) \cup E(G)$ and edges joining all elements of this vertex set which are adjacent or incident in G .

If the set of vertices of a graph G can be partitioned into k classes V_1, V_2, \dots, V_k such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j) , then G is said to be *equitably k -colorable*. The smallest integer k for which G is equitable k -colorable is known as the *equitable chromatic number* [5–10] of G and denoted by $\chi_=(G)$. Additional graph theory terminology used in this paper can be found in [3, 4].

2. Equitable Coloring on Total Graph of Complete Bigraphs

Theorem 2.1. *If $m \leq n$, the equitable chromatic number of total graph of complete bigraphs $K_{m,n}$,*

$$\chi_=(T(K_{m,n})) = \begin{cases} n+1 & \text{if } m < n, \\ n+2 & \text{if } m = n. \end{cases} \quad (2.1)$$

Proof. Let (X, Y) be the bipartition of $K_{m,n}$, where $X = \{v_i : 1 \leq i \leq m\}$ and $Y = \{v'_j : 1 \leq j \leq n\}$. Let u_{ij} ($1 \leq i \leq m; 1 \leq j \leq n$) be the edges of $v_i v'_j$. By the definition of total graph, $T(K_{m,n})$ has the vertex set $\{v_i : 1 \leq i \leq m\} \cup \{v'_j : 1 \leq j \leq n\} \cup \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the vertices $\{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ induce n disjoint cliques of order n in $T(K_{m,n})$. Also v_i ($1 \leq i \leq m$) is adjacent to v'_j ($1 \leq j \leq n$).

Case 1 (if $m = n$, $\chi_=(T(K_{m,n})) = n+2$). Now we partition the vertex set $V(T(K_{m,n}))$ as follows:

$$\begin{aligned} V_1 &= \{u_{11}, u_{2n}, u_{3(n-1)}, u_{4(n-2)}, \dots, u_{(n-1)3}, u_{n2}\}, \\ V_2 &= \{u_{12}, u_{21}, u_{3n}, u_{4(n-1)}, \dots, u_{(n-1)4}, u_{n3}\}, \\ &\vdots \\ V_n &= \{u_{1n}, u_{2(n-1)}, u_{3(n-2)}, u_{4(n-3)}, \dots, u_{(n-1)3}, u_{n1}\}, \\ V_{n+1} &= \{v_1, v_2, \dots, v_n\}, \\ V_{n+2} &= \{v'_1, v'_2, \dots, v'_n\}. \end{aligned} \quad (2.2)$$

Clearly V_1, V_2, \dots, V_{n+2} are independent sets and $|V_i| = n$ ($1 \leq i \leq n+2$) satisfying the condition $||V_i| - |V_j|| = 0$, for any $i \neq j$, $\chi_=(T(K_{m,n})) \leq n+2$. Since there exists a clique of order $n+1$ in $T(K_{m,n})$. $\chi(T(K_{m,n})) \geq n+1$, also each v_i of $T(K_{m,n})$ receives one color different from the color class assigned to the clique induced by $\{u_{ij} : 1 \leq i \leq m; 1 \leq j \leq n\}$. By the definition of total graph, each v_i is adjacent with v'_j ($1 \leq j \leq n$). Therefore, $\{v_1, v_2, \dots, v_m\}$ and $\{v'_1, v'_2, \dots, v'_n\}$ are independent sets and hence $\chi(T(K_{m,n})) \geq n+2$. That is, $\chi_=(T(K_{m,n})) \geq \chi(T(K_{m,n})) \geq n+2$; therefore $\chi_=(T(K_{m,n})) \geq n+2$. Hence $\chi_=(T(K_{m,n})) = n+2$.

Case 2 (if $m < n$, $\chi_=(T(K_{m,n})) = n+1$). Now we partition the vertex set $V(T(K_{m,n}))$ as follows:

$$\begin{aligned} V_1 &= \{u_{11}, u_{22}, u_{33}, u_{44}, \dots, u_{mm}\} \cup \{v'_n\}, \\ V_2 &= \{u_{12}, u_{23}, u_{34}, \dots, u_{m(m-1)}\} \cup \{u_{m1}\} \cup \{v'_1\}, \\ V_3 &= \{u_{13}, u_{24}, u_{35}, \dots, u_{m(m-2)}\} \cup \{u_{(m-1)3}, u_{m2}\} \cup \{v'_2\}, \\ &\vdots \\ V_{n-1} &= \{u_{1(n-1)}, u_{2n}\} \cup \{u_{31}, u_{32}, \dots, u_{m(m-2)}\} \cup \{v'_{n-2}\}, \\ V_n &= \{u_{1n}\} \cup \{u_{21}, u_{32}, \dots, u_{m(m-1)}\} \cup \{v'_{n-1}\}, \\ V_{n+1} &= \{v_1, v_2, v_3, \dots, v_m\}. \end{aligned} \quad (2.3)$$

Clearly V_1, V_2, \dots, V_{n+1} are independent sets of $T(K_{m,n})$. Also $|V_1| = |V_2| = \dots = |V_n| = m + 1$ and $|V_{n+1}| = m$ satisfy the condition $||V_i| - |V_j|| \leq 1$, for any $i \neq j$, $\chi_=(T(K_{m,n})) \leq n + 1$. Since there exists a clique of order $n + 1$ in $T(K_{m,n})$, $\chi(T(K_{m,n})) \geq n + 1$, that is, $\chi_=(T(K_{m,n})) \geq \chi(T(K_{m,n})) \geq n + 1$, therefore $\chi_=(T(K_{m,n})) \geq n + 1$. Hence $\chi_=(T(K_{m,n})) = n + 1$. \square

3. Equitable Coloring on Central Graph of Cycles and Paths

Theorem 3.1. *If $n \geq 5$, the equitable chromatic number of central graph of cycles C_n ,*

$$\chi_=(C(C_n)) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \quad (3.1)$$

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{e_1, e_2, \dots, e_n\}$ be the vertices and edges of C_n taken in the cyclic order. By the definition of central graph, $C(C_n)$ has the vertex set $V(C_n) \cup \{u_i : 1 \leq i \leq n\}$, where u_i is the vertex of subdivision of the edge e_i and joining all the nonadjacent vertices of C_n in $C(C_n)$.

Case 1 (n is odd). We partition the vertex set $V(C(C_n))$ as

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-2}, u_{n-1}\}, \\ V_2 &= \{v_3, v_4, u_n\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{(n-1)/2} &= \{v_{n-2}, v_{n-1}, u_{n-6}, u_{n-5}\}, \\ V_{(n+1)/2} &= \{v_n, u_{n-4}, u_{n-3}\}. \end{aligned} \quad (3.2)$$

Clearly $V_1, V_2, \dots, V_{(n-1)/2}, V_{(n+1)/2}$ are independent sets of $C(C_n)$. Also $|V_1| = |V_3| = |V_4| = \dots = |V_{(n-1)/2}| = 4$ and $|V_2| = |V_{(n+1)/2}| = 3$. The inequality $||V_i| - |V_j|| \leq 1$ holds, for any $i \neq j$, $\chi_=(C(C_n)) \leq (n + 1)/2$. For each i , v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(C_n)) \geq (n + 1)/2$. That is, $\chi_=(C(C_n)) \geq \chi(C(C_n)) \geq (n + 1)/2$, $\chi_=(C(C_n)) \geq (n + 1)/2$. Therefore, $\chi_=(C(C_n)) = (n + 1)/2$.

Case 2 (n is even). Now we partition the vertex set $V(C(C_n))$ as follows:

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-3}, u_{n-2}\}, \\ V_2 &= \{v_3, v_4, u_{n-1}, u_n\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{n/2} &= \{v_{n-1}, v_n, u_{n-5}, u_{n-4}\}. \end{aligned} \quad (3.3)$$

Clearly $V_1, V_2, \dots, V_{n/2}$ are independent sets of $C(C_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = \dots = |V_{n/2}| = 4$. The inequality $\|V_i| - |V_j|\| = 0$ holds, for any $i \neq j$, $\chi_=(C(C_n)) \leq n/2$. For each i , v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(C_n)) \geq n/2$. That is, $\chi_=(C(C_n)) \geq \chi(C(C_n)) \geq n/2$, $\chi_=(C(C_n)) \geq n/2$. Therefore, $\chi_=(C(C_n)) = n/2$. \square

Remark 3.2. If $n = 3, 4$, then $\chi_=(C(C_n)) = 2, 3$, respectively.

Theorem 3.3. *If $n \geq 5$, the equitable chromatic number of central graph of paths P_n ,*

$$\chi_=(C(P_n)) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \quad (3.4)$$

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{e_1, e_2, \dots, e_n\}$ be the vertices and edges of P_n . By the definition of central graph, $C(P_n)$ has the vertex set $V(P_n) \cup \{u_i : 1 \leq i \leq n-1\}$, where u_i is the vertex of subdivision of the edge e_i and joining all nonadjacent vertices of P_n in $C(P_n)$.

Case 1 (n is odd). Now we partition the vertex set $V(C(P_n))$ as follows:

$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-2}\}, \\ V_2 &= \{v_3, v_4, u_{n-1}\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{(n-1)/2} &= \{v_{n-1}, v_{n-2}, u_{n-6}, u_{n-5}\}, \\ V_{(n+1)/2} &= \{v_n, u_{n-4}, u_{n-3}\}. \end{aligned} \quad (3.5)$$

Clearly $V_1, V_2, \dots, V_{(n-1)/2}, V_{(n+1)/2}$ are independent sets of $C(P_n)$. Also $|V_3| = |V_4| = \dots = |V_{(n-1)/2}| = 4$ and $|V_1| = |V_2| = |V_{(n+1)/2}| = 3$. The inequality $\|V_i| - |V_j|\| \leq 1$ holds, for any $i \neq j$, $\chi_=(C(P_n)) \leq (n+1)/2$. For each i , v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(P_n)) \geq (n+1)/2$. That is, $\chi_=(C(P_n)) \geq \chi(C(P_n)) \geq (n+1)/2$, $\chi_=(C(P_n)) \geq (n+1)/2$. Therefore $\chi_=(C(P_n)) = (n+1)/2$.

Case 2 (n is even). Now we partition the vertex set $V(C(P_n))$ as follows:

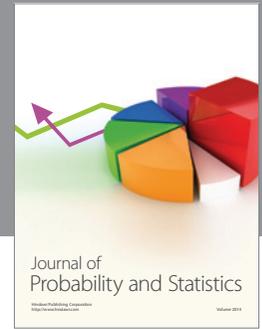
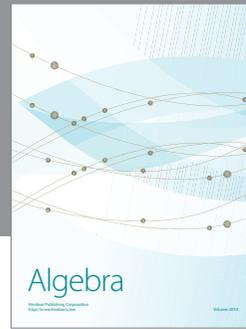
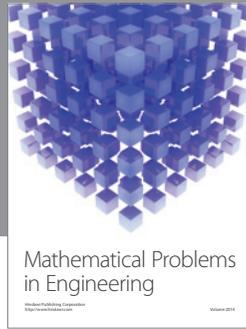
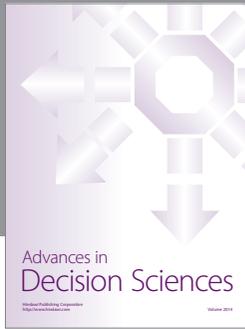
$$\begin{aligned} V_1 &= \{v_1, v_2, u_{n-3}, u_{n-2}\}, \\ V_2 &= \{v_3, v_4, u_{n-1}\}, \\ V_3 &= \{v_5, v_6, u_1, u_2\}, \\ V_4 &= \{v_7, v_8, u_3, u_4\}, \\ &\vdots \\ V_{n/2} &= \{v_{n-1}, v_n, u_{n-5}, u_{n-4}\}. \end{aligned} \quad (3.6)$$

Clearly $V_1, V_2, \dots, V_{n/2}$ are independent sets of $C(P_n)$. Also $|V_1| = |V_3| = |V_4| = \dots = |V_{n/2}| = 4$ and $|V_2| = 3$. The inequality $\|V_i| - |V_j| \leq 1$ holds for any $i \neq j$, $\chi_=(C(P_n)) \leq n/2$. For each i , v_i is nonadjacent with v_{i-1} and v_{i+1} and hence $\chi(C(P_n)) \geq n/2$. That is, $\chi_=(C(P_n)) \geq \chi(C(P_n)) \geq n/2$, $\chi_=(C(P_n)) \geq n/2$. Therefore, $\chi_=(C(P_n)) = n/2$. \square

Remark 3.4. If $n = 1, 2, 3, 4$, then $\chi_=(C(P_n)) = 1, 2, 3, 3$, respectively.

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