Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 290403, 11 pages
doi:10.1155/2011/290403

Research Article

Exponentially Convex Functions on Hypercomplex Systems

Buthinah A. Bin Dehaish

Department of Mathematics, Science College of Girls, King Abdulaziz University, P.O. Box 53909, Jeddah 21593, Saudi Arabia

Correspondence should be addressed to Buthinah A. Bin Dehaish, bbindehaish@yahoo.com

Received 27 June 2010; Accepted 20 April 2011

Academic Editor: Zayid Abdulhadi

A hypercomplex system (h.c.s.) $L_1(Q,m)$ is, roughly speaking, a space which is defined by a structure measure $(c(A,B,r),\ (A,B \in \mathcal{B}(Q)))$, such space has been studied by Berezanskii and Krein. Our main result is to define the exponentially convex functions (e.c.f.) on (h.c.s.), and we will study their properties. The definition of such functions is a natural generalization of that defined on semigroup.

1. Introduction

Harmonic Analysis theory and its relation with positive definite kernels is one of the most important subjects in functional analysis, which has different applications in mathematics and physics branches.

Mercer (1909) defines a continuous and symmetric real-valued function Φ on $[a,b] \times [a,b] \subseteq R^2$ to be positive type if and only if

$$\iint_a^b C(x)C(y)\Phi(x,y)\,dx\,dy \geq 0, \tag{1.1}$$

where $C(x), C(y) \in C[a,b]$.

Positive definite kernels generate a different kinds of functions, for example, positive, negative, and e.c.f. For more details you can see the work done by Stewart [1] in 1976 who gave a survey of these functions.

Harmonic analysis of these functions on finite and infinite spaces or groups, semigroups, and hypergroups have a long history and many applications in probability theory, operator theory, and moment problem (see [2–10]).

Many studies were done on e.c.f. on different structures (see [10–18]).

Our aim in this study is to carry over the harmonic analysis of the e.c.f to the case of the h.c.s. These functions were first introduced by Berg et al., cf. [2]. The continuous functions $f\ :]a,b[\to\ R$ is e.c.f. if and only if the kernel $(x,y)\mapsto f(x+y)$ is positive definite on the region $](1/2)a,(1/2)b[\times](1/2)a,(1/2)b[$.

Now, I will give a short summary of the h.c.f.

Let Q be a complete separable locally compact metric space of points $p,q,r,\ldots,\beta(Q)$ be the σ-algebra of Borel subsets, and $\beta_0(Q)$ be the subring of $\beta(Q)$, which consists of sets with compact closure. We will consider the Borel measures; that is, positive regular measures on $\beta(Q)$, finite on compact sets. The spaces of continuous functions of finite continuous function, and of bounded functions are denoted by $C(Q)$, $C_0(Q)$, and, $C_b(Q)$, respectively.

An h.c.s. with the basis Q is defined by its structure measure $c(A,B,r)$ $(A,B\in\beta(Q);\ r\in Q)$. A structure measure $c(A,B,r)$ is a Borel measure in A (resp. B) if we fix B,r (resp. A,r) which satisfies the following properties:

(H1) For all $A,B\in\beta_0(Q)$, the function $c(A,B,r)\in C_0(Q)$.

(H2) For all $A,B\in\beta_0(Q)$ and $s,r\in Q$, the following associativity relation holds

$$\int_Q c(A,B,r)d_rc(E_r,C,s)=\int_Q c(B,C,r)d_rc(A,E_r,s),\quad C\in\beta(Q). \tag{1.2}$$

(H3) The structure measure is said to be commutative if

$$c(A,B,r)=c(B,A,r),\quad \big(A,B\in\beta_0(Q)\big) \tag{1.3}$$

A measure m is said to be a multiplicative measure if

$$\int_Q c(A,B,r)dm(r)=m(A)m(B);\quad A,B\in\beta_0(Q). \tag{1.4}$$

(H4) We will suppose the existence of a multiplicative measure.

$$\text{For any } f,g\in L_1(Q,m),\quad \text{the convolution } \big(f*g\big)(r)=\iint_Q f(p)g(q)dm_r(p,q). \tag{1.5}$$

is well defined (see [19]).

The space $L_1(Q,m)$ with the convolution (1.5) is a Banach algebra which is commutative if (H3) holds. This Banach algebra is called the h.c.s. with the basis Q.

A nonzero measurable and bounded almost everywhere function $Q \ni r \to x(r) \in \mathbb{C}$ is said to be a character of the h.c.s. L_1, if for all $A, B \in \beta_0(Q)$

$$\int_Q c(A,B,r)x(r)dm(r) = x(A)x(B),$$
$$\int_C x(r)dm(r) = x(C), \quad C \in \beta_0(Q). \tag{1.6}$$

(H5) An h.c.s. is said to be normal, if there exists an involution homomorphism $Q \ni r \mapsto r^* \in Q$, such that $m(A) = m(A^*)$, and

$$c(A,B,C) = c(C,B^*,A), \quad c(A,B,C) = c(A^*,C,B), \quad \big(A,B \in \beta_0(Q)\big), \tag{1.7}$$

where

$$c(A,B,C) = \int_C c(A,B,r)dm(r). \tag{1.8}$$

(H6) A normal h.c.s. possesses a basis unity if there exists a point $e \in Q$ such that $e^* = e$ and

$$c(A,B,e) = m(A^* \cap B), \quad A,B \in \beta(Q). \tag{1.9}$$

If $r^* = r$ for all $r \in Q$, then the normal h.c.s. is called Hermitian which is commutative.

We should remark that, for a normal h.c.s., the mapping

$$L_1(Q,m) \ni f(r) \longrightarrow f^*(r) \in L_1(Q,m) \tag{1.10}$$

is an involution in the Banach algebra L_1, the multiplicative measure is unique and characters of such a system are continuous. A character x of a normal h.c.s. is said to be Hermitian if

$$x(r^*) = \overline{x(r)}, \quad (r \in Q). \tag{1.11}$$

Let X and X_h be the sets of characters and Hermitian characters, respectively.

A Hermitian character of a Hermitian h.c.s. are real valued $\overline{x(p)} = x(p)$ $(p \in Q)$.

Let $L_1(Q,m)$ be an h.c.s. with a basis Q and Φ a space of complex valued functions on Q. Assume that an operator valued function $Q \ni p \mapsto R_p : \Phi \to \Phi$ is given such that the function $g(p) = (R_p f)(q)$ belongs to Φ for any $f \in \Phi$ and any fixed $q \in Q$. The operators R_p $(p \in Q)$ are called right generalized translation operators, provided that the following axioms are satisfied.

(T1) Associativity axiom: the equality

$$\left(R_p^q(R_q f)\right)(r) = \left(R_q^r(R_p f)\right)(r) \tag{1.12}$$

holds for any elements $p, q \in Q$.

(T2) There exists an element $e \in Q$ such that R_e is the identity in Φ.

By the bilinear form

$$(L_p f, g) = (g^*, R_{p^*} f^*) = (R_r f, g), \quad (f, g \in L_2;\ r, p \in Q), \tag{1.13}$$

we define the left generalized translation operators L_p, such that $L_p f(r) = R_r f(p)$ for almost all p and q with respect to the measure $m \times m$. L_p and R_p have the same properties, so that will call them generalized translation operators.

A one-to-one correspondence exists between normal h.c.s. $L_1(Q, m)$ with basis unity e and weakly continuous families of bounded involutive generalized translation operators L_p satisfying the finiteness condition, preserving positivity in the space $L_2(Q, m)$ with unimodular strongly invariant measure m, and preserving the unit element. Convolution in the hypercomplex system $L_1(Q, m)$ and the corresponding family of generalized translation operators L_p satisfy the relation

$$(f * g)(p) = \int_Q (L_p f)(q) g(q^*) dq = (L_p f, g^*)_2, \quad (f, g \in L_2). \tag{1.14}$$

Moreover, the h.c.s. $L_1(Q, m)$ is commutative if and only if the generalized translation operators L_p $(p \in Q)$ are commutative (see [20]).

2. Exponentially Convex Functions

Let $L_1(Q, m)$ be a commutative normal h.c.s. with basis unity.

Definition 2.1. An essentially bounded function $\varphi(r)$ $(r \in Q)$ is called e.c.f if

$$\int \varphi(r)(x^* * x)(r) dr \geq 0, \quad \forall x \in L_1. \tag{2.1}$$

Note that we use the identical involution $x^* = x$, we also present another definition of e.c.f.

A continuous bounded function $\varphi(r)$ $(r \in Q)$ is called e.c.f. if the inequality

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \left(R_{r_j} \varphi\right)(r_j) \geq 0 \tag{2.2}$$

holds for all $r_1, \ldots, r_n \in Q$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $(n \in \mathbb{N})$.

Theorem 2.2. *If the generalized translation operators* R_t *extended to* $L_\infty : C_b(G) \to C_b(G \times G)$. *Then the defination* (2.1) *and* (2.2) *are equivalent for the functions* $\varphi(r) \in C_b(Q)$.

Proof.

$$\begin{aligned}
\int \varphi(t)(x * x)(t)dt &= \int\int (L_t x)(s)\varphi(t)x(s)ds\,dt \\
&= \int\int (L_s x)(t)\varphi(t)x(s)ds\,dt \\
&= \int \left(x, L_{s^*}\overline{\varphi}\right)_2 x(s)ds \\
&= \int\int x(t)\overline{\left(L_{s^*}\overline{\varphi}\right)(t)}x(s)ds \\
&= \int\int (L_s\varphi)(t)x(t)x(s)ds\,dt \\
&= \int\int (R_t\varphi)(s)x(s)x(t)ds\,dt \geq 0.
\end{aligned} \tag{2.3}$$

It follows that $(R_t\varphi)(s) \in C_b(Q \times Q)$, then the last inequality clearly implies (2.2).

Let us prove the converse assertion.

Let Q_n be an increasing sequence of compact sets covering the entire Q, that is, $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n$ and $Q = \bigcup_{i=1}^{n} Q_i$.

We consider a function $h(r) \in C_0(Q)$ and set $\lambda_i = h(r_i)$ in (2.2).

This yields

$$\sum_{i,j=1}^{n} (R_{r_i}\varphi)(r_j)h(r_i)h(r_j) \geq 0. \tag{2.4}$$

By integrating this inequality with respect to each $r_1, \ldots, r_n$ over the set Q_k ($k \in \mathbb{N}$) and collecting similar terms, we conclude that

$$nm(Q_k)\int_{Q_k} (R_r\varphi)(r)h^2(r)dr + n(n-1)\iint_{Q_k} (R_r\varphi)(s)h(r)h(s)dr\,ds \geq 0. \tag{2.5}$$

Further, we divide this inequality by n^2 and pass to the limit as $n \to \infty$. We get

$$\iint_{Q_k} (R_r\varphi)(s)h(r)\overline{h(s)}\,dr\,ds \geq 0, \tag{2.6}$$

for each $k \in \mathbb{N}$. By passing to the limit as $k \to \infty$ and applying Lebesgue theorem, we see that (2.1) holds for all functions from $C_0(Q)$. Approximating an arbitrary function from L_1 by finite continuous functions, we arrive at (2.1). □

By $E \cdot C(Q)$, we shall denote the set of all bounded or continuous e.c.f.

The next theorem is an analog of the Bochner theorem for h.c.s.

Theorem 2.3. *Every function $\varphi \in E \cdot C(Q)$ admits a unique representation in the form of an integral*

$$\varphi(r) = \int_{X_h} x(r) d\mu(x), \quad (r \in Q), \tag{2.7}$$

where μ is a nonnegative finite regular measure on the space X_h. Conversely, each function of the form (2.7) *belongs to $E \cdot C(Q)$.*

Proof. The proof is similar to that given for Theorem 3.1 of [20], so we omit it. □

Corollary 2.4. *If the product of any two Hermitian characters is e.c., then the product of any two continuous e.c.f. is also e.c.*

Proof. It follows directly from Theorem 2.3. □

Corollary 2.5. *Assume that $L_1(Q, m)$ is a commutative h.c.s. with basis unity. Then a continuous bounded function $\varphi(r)$ is e.c. in the sense of* (2.1) *if and only if it is e.c. in the sense of* (2.2). *Moreover, it has the following properties:*

(i) $\varphi(e) \geq 0$;

(ii) $\varphi(r) = \overline{\varphi(r)}$;

(iii) $|\varphi(r)| \leq \varphi(e)$;

(iv) $|(R_s\varphi)(t)|^2 \leq (R_s\varphi)(s)(R_t\varphi)(t)$;

(v) $|\varphi(s) - \varphi(t)|^2 \leq 2\varphi(e)[\varphi(e) - (R_s\varphi)(t)]$.

Proof. Let $\varphi(r) \in C_b(Q)$ is e.c.f. in the sense of (2.1) and let $r_1, \ldots, r_n \in Q$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Relation (2.7) and the fact that the generalized translation operators are continuous in $L_\infty(Q, m)$ imply that

$$\begin{aligned} \sum_{i,j=1}^{n} \lambda_i \lambda_j (R_{r_i}\varphi)(r_j) &= \sum_{i,j=1}^{n} \lambda_i \lambda_j \int_{X_h} (R_{r_i}x)(r_j) d\mu(x) \\ &= \int_{X_h} \sum_{i,j=1}^{n} \lambda_i \lambda_j x(r_i) x(r_j) d\mu(x) \\ &= \int_{X_h} \left(\sum_{k=1}^{n} \lambda_k x(r_k) \right)^2 d\mu(x) \geq 0. \end{aligned} \tag{2.8}$$

It is also follows from relation (2.7) that

(i) and (ii) are trivial.

(iii)

$$|\varphi(r)| \leq \int_{X_h} |x(r)| d\mu(x) \leq \mu(X) = \varphi(e). \tag{2.9}$$

(iv)

$$\begin{aligned}
|(R_s\varphi)(t)|^2 &= \left|\int_{X_h} x(s)x(t)d\mu(x)\right|^2 \\
&\le \int_{X_h} |x(s)|^2 d\mu(x) \int_{X_h} |x(t)|^2 d\mu(x) \\
&= \int_{X_h} x(s)x(s^*)d\mu(x) \int_{X_h} x(t)x(t^*)d\mu(x) \\
&= (R_s\varphi)(s)(R_t\varphi)(t).
\end{aligned} \tag{2.10}$$

Finally,

(v)

$$\begin{aligned}
|\varphi(s) - \varphi(t)|^2 &\le \left|\int_{X_h} (x(s) - x(t))d\mu(x)\right|^2 \\
&\le \varphi(e) \int_{X_h} \left(|x(s)|^2 + |x(t)|^2 - 2x(s)x(t)\right)d\mu(x) \\
&\le 2\varphi(e) \int_{X_h} (1 - R_s x)(t)d\mu(x) = 2\varphi(e)\left[\varphi(e) - (R_s\varphi)(t)\right].
\end{aligned} \tag{2.11}$$

□

3. Exponentially Convex Functions and Kernals

Inequality (2.2) means that the kernel $K(t,s) = (R_t\varphi)(s)$ is positive definite function. Therefore, this kernel possesses the following properties:

$$\begin{gathered}
K(t,t) \ge 0, \quad K(t,s) = \overline{K(s,t)}, \\
|K(t,s)|^2 \le K(t,t)K(s,s), \\
|K(t,r) - K(s,r)|^2 \le K(r,r)(K(t,t) - 2\operatorname{Re} K(t,s) + K(s,s)).
\end{gathered} \tag{3.1}$$

Now, we can use the properties of the kernel to prove the properties of the e.c.f.

Indeed,

$$\varphi(r) = (R_e\varphi)(r) = K(e,r) = \overline{K(r,e)} = \overline{\varphi(r)}. \tag{3.2}$$

Similarly, $(R_r\varphi)(s) = \overline{(R_s\varphi)(r)}$. This implies that

$$|(R_t\varphi)(s)|^2 \le (R_t\varphi)(t)(R_s\varphi)(s), \tag{3.3}$$

that is, (iv).

By setting $s = e$ in (iv), we obtain

$$\begin{aligned}\left|(R_t\varphi)(e)\right|^2 &\le (R_t\varphi)(t)(R_e\varphi)(e),\\ \left|(I\varphi)(t)\right|^2 &\le (R_t\varphi)(t)\varphi(e),\\ \left|\varphi(t)\right|^2 &\le \varphi(e)(R_t\varphi)(t).\end{aligned} \tag{3.4}$$

In view of the relation $|(L_p f)(q)| \le \|f\|_\infty$ $(f \in C_b(Q))$, we have

$$(R_r\varphi)(r) \le \|\varphi\|_\infty. \tag{3.5}$$

Consequently,

$$\left|\varphi(r)\right|^2 \le \varphi(e), \tag{3.6}$$

which implies (iii) and, hence, (i). Finally, (v) follows from the last inequality for $K(t,s)$, where $r = e$.

$$\begin{aligned}\left|\varphi(s) - \varphi(t)\right|^2 &\le \left|(R_t\varphi)(e) - (R_s\varphi)(e)\right|^2\\ &\le (R_e\varphi)(e)\left[(R_t\varphi)(t) - 2R_e(R_t\varphi)(s) + (R_s\varphi)(s)\right]\\ &= \varphi(e)\left[(R_t\varphi)(t) - 2(R_t\varphi)(s) + (R_s\varphi)(s)\right]\\ &\le 2\varphi(e)\left[\varphi(e) - (R_s\varphi)(t)\right].\end{aligned} \tag{3.7}$$

4. Exponentially Convex Functions and Representations of Hypercomplex Systems

In this section, we will give the relation between the h.c.s. and e.c.f.

Let $L_1(Q,m)$ be a normal h.c.s. with basis unity e. The family of bounded operators $U = (U_p)_{p\in Q}$ in a separable Hilbert space $\mathcal{H}$ is called a representation of an h.c.s. if

(1) $U_e = 1$,

(2) $U_p^* = U_{p^*}$ $(p \in Q)$,

(3) for each $\zeta \in \mathcal{H}$, the vector $Q \ni p \mapsto U_p\zeta \in \mathcal{H}$ is weakly continuous,

(4) for all $A, B \in \mathcal{B}_0(Q)$

$$\int c(A,B,r)U_r dr = \int_A U_p dp \int_B U_q dq. \tag{4.1}$$

Condition (4.1) implies that the function $Q \ni p \mapsto \| U_p\|$ is locally bounded.

Example 4.1. The family of generalized translation operators L_p $(p \in Q)$ defines a representation of the h.c.s. $L_1(Q,m)$ in Helbert space $L_2(Q,m)$.

Let $Q \ni p \mapsto U_p$ be a representation of the h.c.s. $L_1(Q, m)$. Below, we consider representation that satisfy conditions (1.5)–(2.2) and the following additional condition:

(5) the function $Q \ni p \mapsto \| U_p\|$ is bounded.

Such representation are called bounded.

Let $L_1(Q, m) \ni x \mapsto U_x$ be a representation of the Banach algebra $L_1(Q, m)$ in a separable Hilbert space $\mathscr{H}$.

Two representation of an h.c.s. $L_1(Q, m)$ are unitarity equivalent if and only if the corresponding representations of the algebra $L_1(Q, m)$ are equivalent h.c.s.

We recall that a representation of the Banach algebra $L_1(Q, m)$ in $\mathscr{H}$ is said to be cyclic if there exists a vector $\zeta \in \mathscr{H}$, cyclic vector, such that the linear subspace $\{U_x\zeta : x \in L_1(Q, m)\}$ is dense in $\mathscr{H}$.

Corollary 4.2. *For any bounded representation U_r of a normal h.c.s. with basis unity that satisfies the condition of separate continuity the following relation holds:*

$$R_s(U_r\zeta, \eta)_{\mathscr{H}} = (U_rU_s\zeta, \eta)_{\mathscr{H}},$$
$$L_s(U_r\zeta, \eta)_{\mathscr{H}} = (U_sU_r\zeta, \eta)_{\mathscr{H}}, \quad (r, s \in Q;\ \zeta, \eta \in \mathscr{H}). \tag{4.2}$$

For the proof (see [20]).

Theorem 4.3. *Let $L_1(Q, m)$ be a normal h.c.s. with basis unity satisfying the condition of separate continuity. Then there is a bijection between the collection of continuous bounded function on Q e.c. in the sense of* (2.1) *and the set of classes of unitarily equivalent bounded cyclic representation on the h.c.s. $L_1(Q, m)$. This bijection is given by the relation*

$$\varphi(r) = (U_r\zeta_0, \zeta_0)_{\mathscr{H}}, \quad (r \in Q), \tag{4.3}$$

where $Q \in E \cdot C(Q)$ and U_r is the corresponding representation of the h.c.s. $L_1(Q, m)$ in a Hilbert space $\mathscr{H}$ with cyclic vector ζ_0.

Proof. If U_r is a bounded representation of the h.c.s. $L_1(Q, m)$ with cyclic vector ζ_0. Then the function $\varphi(r) = (U_r\zeta_0, \zeta_0)_{\mathscr{H}}$ is e.c.f. in the sense of (2.1). Indeed, Let $x \in L_1(Q, m)$. Then

$$\begin{aligned} \int \varphi(r)(x * x)(r)dr &= \left(\int (x * x)(r)\ U_r dr\zeta_0, \zeta_0\right)_{\mathscr{H}} \\ &= (U_{x*x}\zeta_0, \zeta_0)_{\mathscr{H}} \\ &= \| U_x\zeta_0\|_2^2 \geq 0. \end{aligned} \tag{4.4}$$

□

Corollary 4.4. *For a normal h.c.s. with basis unity that satisfies the condition of separate continuity, the concepts of e.c. in the sense of* (2.1) *and* (2.2) *are equivalent.*

Proof. It suffices to show that if $\varphi(r)$ is e.c. in the sense of (2.1), then relation (2.2) holds for any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $r_1, \dots, r_n \in Q$. Indeed, by virtue of (4.2) and (4.3), we have

$$\begin{aligned}\sum_{i,j=1}^{n} \lambda_i \lambda_j (R_{r_i}\varphi)(r_j) &= \sum_{i,j}^{n} \lambda_i \lambda_j R_{r_i}\left(U_{r_j}\zeta_0, \zeta_0\right)_{\mathscr{H}} \\ &= \sum_{i,j=1}^{n} \lambda_i \lambda_j \left(U_{r_j} U_{r_i}\zeta_0, \zeta_0\right)_{\mathscr{H}} \\ &= \left\| \sum \lambda_i U_{r_i}\zeta_0 \right\|_{\mathscr{H}}^2 \geq 0.\end{aligned} \tag{4.5}$$

□

Acknowledgments

This Project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. 18-10/429. The author, therefore, acknowledges with thanks DSR support for Scientific Research.

References

[1] J. Stewart, "Positive definite functions and generalizations, a historical survey," *The Rocky Mountain Journal of Mathematics*, vol. 6, no. 3, pp. 409–434, 1976.
[2] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups*, Springer, New York, NY, USA, 1984.
[3] W. R. Bloom and P. Ressel, "Positive definite and related functions on hypergroups," *Canadian Journal of Mathematics*, vol. 43, no. 2, pp. 242–254, 1991.
[4] W. R. Bloom and P. Ressel, "Representations of negative definite functions on polynomial hypergroups," *Archiv der Mathematik*, vol. 78, no. 4, pp. 318–328, 2002.
[5] W. R. Bloom and P. Ressel, "Negative definite and Schoenberg functions on commutative hypergroups," *Journal of the Australian Mathematical Society*, vol. 79, no. 1, pp. 25–37, 2005.
[6] W. R. Bloom and H. Heyer, "Negative definite functions and convolution semigroups of probability measures on a commutative hypergroup," *Probability and Mathematical Statistics*, vol. 16, no. 1, pp. 156–176, 1996.
[7] W. R. Bloom and P. Ressel, "Exponentially bounded positive-definite functions on a commutative hypergroup," *Journal of the Australian Mathematical Society*, vol. 61, no. 2, pp. 238–248, 1996.
[8] A. M. Zabel and B. A. Bin Dehaish, "Levy Khinchin formula on commutative hypercomplex system," *Kyungpook Mathematical Journal*, vol. 48, no. 4, pp. 559–575, 2008.
[9] N. I. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, Edinburgh, UK, 1965.
[10] A. Devinatz, "The representation of functions as a Laplace-Stieltjes integrals," *Duke Mathematical Journal*, vol. 22, pp. 185–191, 1955.
[11] I. M. Gali and A. S. Okb El-Bab, "Integral representation of exponentially convex functions of infinite number of variables," *Proceedings of the Mathematical and Physical Society of Egypt*, vol. 50, pp. 15–26, 1980.
[12] A. S. Okb El-Bab, "Exponentially convex functions on certain sequence spaces," in *Proceedings of the 6th Annual Operations Research Conference*, vol. 6, no. 1, pp. 261–277, Zagazig, Egypt, 1980.
[13] A. S. Okb El-Bab, "Exponentially convex function on free groups," in *Proceedings of the International Congress of Mathematicians*, Warsaw, Poland, 1983.
[14] A. S. Okb El-Bab and A. M. Zabel, "Harmonic analysis on linear topological spaces," *Kyungpook Mathematical Journal*, vol. 30, no. 1, pp. 37–44, 1990.
[15] A. S. Okb El-Bab and M. A. Bajnaid, "Harmonic analysis on foundation semigroup," *Association for the Advancement of Modelling and Simulation Techniques in Enterprises (AMSE, France)*, vol. 37, pp. 21–35, 2000.

[16] A. M. Zabel and M. A. Bajnaid, “The extension problem for exponentially convex functions,” *Kyungpook Mathematical Journal*, vol. 44, no. 1, pp. 31–39, 2004.
[17] H. A. Ali, “Exponentially convex function on commutative *-semigroups,” Accepted for publication In A. M. S. E., 2006.
[18] H. A. Ali, “The uniform convergence of a sequence of weighted bounded exponentially convex functions on foundation semigroups,” *Kyungpook Mathematical Journal*, vol. 46, pp. 420–435, 2006.
[19] Y. M. Berezanskii and S. G. Krein, “Hypercomplex systems with a compact basis,” *Ukrainskiĭ Matematicheskiĭ Zhurnal*, vol. 3, pp. 184–204, 1951.
[20] J. M. Berezanskii and A. A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems*, Naukova Dumka, Kyiv, Ukrania, 1992.