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Research Article

Statistical Convergence and Ideal Convergence of Sequences of Functions in 2-Normed Spaces

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We present various kinds of statistical convergence and \mathcal{D} -convergence for sequences of functions with values in 2-normed spaces and obtain a criterion for \mathcal{D} -convergence of sequences of functions in 2-normed spaces. We also define the notion of \mathcal{D} -equistatistically convergence and study \mathcal{D} -equistatistically convergence of sequences of functions.

1. Introduction

The concept of ideal convergence was introduced first by Kostyrko et al. [1] as an interesting generalization of statistical convergence [2–5].

Throughout this paper \mathbb{N} will denote the set of positive integers. Let $(X, \|\cdot\|)$ be a normed space. Let K be a subset of positive integers \mathbb{N} and $j \in \mathbb{N}$. The quotient $d_j(K) = \operatorname{card}(K \cap \{1, \ldots, j\})/j$ is called the j'th *partial density* of K and d_j is a probability measure on $\mathcal{D}(\mathbb{N})$, with support $\{1, \ldots, j\}$ [2, 3].

The limit $d(K) = \lim_{j\to\infty} d_j(K)$ (if exists) is called the *natural density* of K. Clearly, finite subsets have natural density zero and $d(K^c) = 1 - d(K)$ where $K^c = K \setminus \mathbb{N}$, that is, the complement of K. If $K_1 \subseteq K_2$ and K_1, K_2 have natural densities then $d(K_1) \leq d(K_2)$. Furthermore, if $d(K_1) = d(K_2) = 1$, then $d(K_1 \cap K_2) = 1$ [6].

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called to be statistically convergent to $x\in X$ if the set $A(\epsilon)=\{n\in\mathbb{N}:\|x_n-x\|\geq\epsilon\}$ has natural density zero for each $\epsilon>0$. In this case we write st- $\lim_{n\to\infty}x_n=x$ [2–4].

A family $\mathcal{O} \subseteq \mathcal{D}(Y)$ of subsets a nonempty set Y is said to be an ideal in Y if

- (i) $\emptyset \in \mathcal{I}$,
- (ii) $A, B \in \mathcal{D}$ implies $A \cup B \in \mathcal{D}$,
- (iii) $A \in \mathcal{O}$, $B \subseteq A$ implies $B \in \mathcal{O}$,

while an admissible ideal \mathcal{D} of Y further satisfies $\{x\} \in \mathcal{D}$ for each $x \in Y$ [7, 8]. Let $\mathcal{D} \subseteq \mathcal{D}(\mathbb{N})$ be a nontrivial ideal in \mathbb{N} . The sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be \mathcal{D} -convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : ||x_n - x|| \ge \epsilon\}$ belongs to \mathcal{D} [1, 9].

2. Preliminaries

The notion of linear 2-normed spaces has been investigated by Gähler in the 60's [10, 11] and this has been developed extensively in different subjects by others [12–14]. Let X be a real linear space of dimension greater than 1, and $\|\cdot,\cdot\|$ be a nonnegative real-valued function on $X \times X$ satisfying the following conditions:

- (G1) ||x, y|| = 0 if and only if x and y are linearly dependent vectors;
- (G2) ||x, y|| = ||y, x|| for all x, y in X;
- (G3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ where α is real,
- (G4) $||x + y, z|| \le ||x, z|| + ||y, z||$ for all x, y, z in X

 $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X,\|\cdot,\cdot\|)$ is called a linear 2-normed space. In addition, for all scalars α and all x,y,z in X, we have the following properties:

- (1) $\|\cdot,\cdot\|$ is nonnegative;
- (2) $||x, y|| = ||x, y + \alpha x||$;
- (3) ||x y, y z|| = ||x y, x z||.

Some of the basic properties of 2-norm are introduced in [14]. Given a 2-normed space $(X, \|\cdot, \cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x_n z\| = 0$ for every $z \in X$. This can be written by the formula

$$(\forall z \in Y) \ (\forall \epsilon > 0) \ (\exists n_0 \in \mathbb{N}) \ (\forall n \ge n_0) \quad ||x_n - x_n z|| < \epsilon. \tag{2.1}$$

We write it as

$$x_n \xrightarrow{\|\cdot\cdot\|_{X}} x. \tag{2.2}$$

Lemma 2.1 (see [13]). Let $v = \{v_1, \dots, v_k\}$ be a basis of X. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n \to \infty} ||x_n - x, v_i|| = 0$ for every $i = 1, \dots, k$. We can define the norm $||\cdot||_{\infty}$ on X by

$$||x||_{\infty} := \max\{||x, v_i|| : i = 1, \dots, d = k\}.$$
 (2.3)

Lemma 2.2 (see [13]). A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n\to\infty}||x_n-x||_{\infty}=0$.

Example 2.3. Let $X = \mathbb{R}^2$ be equipped with the 2-norm ||x, y||:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$||x,y|| = |x_1y_2 - x_2y_1|, \quad x = (x_1, x_2), \ y = (y_1, y_2).$$
 (2.4)

Take the standard basis $\{i, j\}$ for \mathbb{R}^2 .

Then, $||x, i|| = |x_2|$ and $||x, j|| = |x_1|$, and so the derived norm $||\cdot||_{\infty}$ with respect to $\{i, j\}$ is

$$||x||_{\infty} = \max\{|x_1|, |x_1|\}, \quad x = (x_1, x_2).$$
 (2.5)

Thus, here the derived norm $\|\cdot\|_{\infty}$ is exactly the same as the uniform norm on R^2 . Since the derived norm is a norm, it is equivalent to the Euclidean norm on R^2 .

Definition 2.4. Let $\mathcal{O} \subset 2^N$ be a nontrivial ideal in \mathbb{N} . The sequence $(x_n)_{n \in \mathbb{N}}$ of X is said to be \mathcal{O} -convergent to x, if for each $\varepsilon > 0$ and nonzero z in X the set $A(\varepsilon) = \{n \in \mathbb{N} : ||x_n - x, z|| \ge \varepsilon\}$ belongs to \mathcal{O} [9].

If $(x_n)_{n\in\mathbb{N}}$ is \mathcal{I} -convergent to x then we write it as

$$\mathcal{O} - \lim_{n \to \infty} \|x_n - x, z\| = 0 \quad \text{or} \quad \mathcal{O} - \lim_{n \to \infty} \|x_n, z\| = \|x, z\|.$$
 (2.6)

The element x is \mathcal{I} -limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

Remark 2.5. If $(x_n)_{n\in\mathbb{N}}$ is any sequence in X and x is any element of X, then the set

$$\{n \in \mathbb{N} : ||x_n - x, z|| \ge \epsilon, \forall z \in X\} = \emptyset$$
(2.7)

since if z = 0, $||x_n - x, z|| = 0 < \epsilon$ so the above set is empty.

Further we will give some examples of ideals and corresponding \mathcal{D} -convergences. Now we give an example of \mathcal{D} -convergence in 2-normed spaces.

Example 2.6. (i) Let \mathcal{O}_f be the family of all finite subsets of \mathbb{N} . Then \mathcal{O}_f is an admissible ideal in \mathbb{N} and \mathcal{O}_f -convergence coincides with usual convergence [11].

(ii) Put $\mathcal{O}_d = \{A \in \mathbb{N} : d(A) = 0\}$. Then \mathcal{O}_d is an admissible ideal in \mathbb{N} and \mathcal{O}_d -convergence coincides with the statistical convergence [15].

Example 2.7. Let $\mathcal{O} = \mathcal{O}_d$. Define the $(x_n)_{n \in \mathbb{N}}$ in 2-normed space $(X, \|\cdot, \cdot\|)$ by

$$x_n = \begin{cases} (0, n), & n = k^2, \ k \in \mathbb{N}, \\ (0, 0), & \text{otherwise} \end{cases}$$
 (2.8)

and let x = (0,0) and $z = (z_1, z_2)$. Then for every $\epsilon > 0$ and $z \in X$

$$\{n \in \mathbb{N} : ||x_n - x, z|| > \epsilon\} \subseteq \{1, 4, 9, 16, \dots, n^2, \dots\}$$
 (2.9)

we have that

$$d(\{n \in \mathbb{N} : ||x_n - x, z|| > \epsilon\}) = 0 \quad \text{for every } \epsilon > 0 \text{ and nonzero } z \in X.$$
 (2.10)

This implies that $\mathcal{O}_d - \lim_{n \to \infty} ||x_n, z|| = ||x, z||$. But, the sequence $(x_n)_{n \in \mathbb{N}}$ is not convergent to x.

3. Convergence for Sequences of Functions in 2-Normed Spaces

We discuss various kinds of convergence and \mathcal{D} -convergence for sequences of functions with values in 2-normed spaces.

Let *X*, *Y* be 2-normed spaces and assume that functions

$$f: X \longrightarrow Y, \qquad f_n: X \longrightarrow Y, \ n \in N$$
 (3.1)

are given.

Definition 3.1. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be positive convergent to f (on X) if

$$f_n(x) \xrightarrow{\|\cdot\cdot\|_Y} f(x)$$
 for each $x \in X$. (3.2)

We write

$$f_n \xrightarrow{\|\cdot\,\|_Y} f. \tag{3.3}$$

This can be expressed by the formula

$$(\forall y \in Y) \ (\forall x \in X) \ (\forall \epsilon > 0) \ (\exists n_0 \in \mathbb{N}) \ (\forall n > n_0) \quad \|f_n(x) - f(x), y\|_Y < \epsilon. \tag{3.4}$$

Remark 3.2. If functions f, f_n are given as in Definition 3.1 and dim $Y < \infty$ then (f_n) is pointwise convergent to f (on X) if and only if

$$(\forall x \in X) \ (\forall \epsilon > 0) \ (\exists n_0 \in \mathbb{N}) \ (\forall n > n_0) \quad \|f_n(x) - f(x), y\|_{\infty} < \epsilon. \tag{3.5}$$

We introduce uniform convergent of $(f_n)_{n\in\mathbb{N}}$ to f by the formula

$$\left(\forall y \in Y\right) \; \left(\forall \epsilon > 0\right) \; \left(\exists n_0 \in \mathbb{N}\right) \; \left(\forall n > n_0\right) \; \left(\forall x \in X\right) \quad \left\|f_n(x) - f(x), y\right\|_Y < \epsilon \tag{3.6}$$

and we write it as

$$f_n \xrightarrow{\|\cdot,\cdot\|_Y} f. \tag{3.7}$$

Example 3.3. If $X = Y = \mathbb{R}^2$ is introduced in Lemma 2.1 then define

$$f(x_1, x_2) = \begin{cases} (0,0) & \text{if } |x_2| < 1, \\ \left(0, \frac{1}{2}\right) & \text{if } |x_2| = 1, \quad f_n(x) = \left(0, \frac{x_2^{2n}}{1 + x_2^{2n}}\right), \\ (0,1) & \text{if } |x_2| > 1 \end{cases}$$
(3.8)

then

$$f_n \xrightarrow{\|y\|_Y} f, \qquad f_n \xrightarrow{\|y\|_Y} f.$$
 (3.9)

Example 3.4. Let $X = Y = [0,1] \times (0,1) \subseteq \mathbb{R}^2$ and define

$$f_n(x_1, x_2) = \left(0, \frac{1}{nx_2 + 1}\right), \qquad f(x_1, x_2) = (0, 0).$$
 (3.10)

Then obviously $f_n \xrightarrow{\|\cdot,\cdot\|_Y} f$. But we show that f_n does not converge uniformly to f in Y. Fix $\varepsilon = 1/2$ and for all $n_0 \in \mathbb{N}$ put $n_0 = n + 1$, $x_n = (0, 1/2n)$ then

$$||f_n(x_1, x_2) - 0||_{\infty} = \left|\frac{1}{nx_2 + 1}\right| = \frac{2}{3} > \varepsilon.$$
 (3.11)

Definition 3.5. Let X and Y be 2-normed spaces with dim $Y < \infty$ and let $f: X \to Y$ be a function. The function f is said to be sequentially continuous at $x_0 \in X$ if for any sequence $(x_n)_{n \in \mathbb{N}}$ of X converging to x_0 one has

$$f(x_n) \xrightarrow{\|\cdot\cdot\|_Y} f(x_0). \tag{3.12}$$

Definition 3.6. Let X and Y be two 2-normed spaces, and $\dim Y < \infty$. If $f_n : X \to Y$ is a sequence of functions, we say $(f_n)_{n \in \mathbb{N}}$ is equi-continuous (on X) if

$$(\forall z \in X) \ (\forall \varepsilon > 0) \ (\exists \delta > 0) \ (\forall x, x_0 \in X) \quad \|x - x_0, z\|_X < \delta \Longrightarrow \|f_n(x) - f_n(x_0)\|_{\infty} < \varepsilon. \tag{3.13}$$

Corollary 3.7. *Let* X *and* Y *be two 2-normed spaces,* $x_0 \in X$ *with* dim $Y < \infty$. *If* $f: X \to Y$ *is a function such that satisfying the following formula*

$$(\forall z \in X) \ (\forall \varepsilon > 0) \ (\exists \delta > 0) \ (\forall x \in X) \quad \|x - x_0, z\|_X < \delta \Longrightarrow \|f(x) - f(x_0)\|_{\infty} < \varepsilon \tag{3.14}$$

then f is sequentially continuous at x_0 .

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\|\cdot,\cdot\|_X} x_0$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\|f(x) - f(x_0)\|_{\infty} < \varepsilon$ for every $x \in X$ where $\|x - x_0, z\|_X \le \delta$ for each $z \in X$. On the other hand $x_n \xrightarrow{\|\cdot,\cdot\|_X} x_0$ hence for all $z \in X$ there exist n_0 such that $\|x_n - x_0, z\|_X < \delta$ for all $n \ge n_0$. Therefore $f(x_n) \xrightarrow{\|\cdot,\cdot\|_Y} f(x_0)$ and f is sequentially continuous at x_0 .

4. *O*-Convergence of Functions in 2-Normed Spaces

Let X, Y be 2-normed spaces. Fix an admissible ideal $\mathcal{O} \subseteq \mathcal{P}(\mathbb{N})$ and assume that functions $f: X \to Y, f_n: X \to Y, n \in \mathbb{N}$ are given.

Definition 4.1. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions is said to be \mathcal{D} -pointwise convergent to f (on X) if \mathcal{D} - $\lim_{n\to\infty} \|f_n(x) - f(x), z\|_Y = 0$ (in $(Y, \|\cdot, \cdot\|_Y)$) for each $x \in X$. We Write

$$f_n \xrightarrow{\|\cdot\cdot\|_{Y}} _{\mathcal{I}} f. \tag{4.1}$$

This can be expressed by the formula

$$(\forall z \in Y) \ (\forall x \in X) \ (\forall \varepsilon > 0) \ (\exists M \in \mathcal{D}) \ (\forall n \in \mathbb{N} \setminus M) \quad \|f_n(x) - f(x), z\|_Y < \varepsilon. \tag{4.2}$$

Definition 4.2. A sequence, $(f_n)_{n\in\mathbb{N}}$ is said to be \mathcal{I} -uniformly convergent to f (on X) if and only if

$$(\forall z \in Y) \ (\forall \varepsilon > 0) \ (\exists M \in \mathcal{D}) \ (\forall n \in \mathbb{N} \setminus M)(\forall x \in X) \quad \|f_n(x) - f(x), z\|_{Y} \le \varepsilon. \tag{4.3}$$

We write $f_n \xrightarrow{\|\cdot,\cdot\|_{\Upsilon}} f$.

Remark 4.3. If $\mathcal{O} = \mathcal{O}_d$ then $\xrightarrow{\|\cdot,\cdot\|_Y} \mathcal{O}_d$ and $\xrightarrow{\|\cdot,\cdot\|_Y} \mathcal{O}_d$ will be read (respectively) as \mathcal{O} -pointwise and

 \mathcal{D} -uniform statistically convergence. If $f_n \xrightarrow{\|\cdot f\|_Y} \mathcal{D}_d f$, then for all $x \in X f_n(x) \xrightarrow{\|\cdot f\|_Y} \mathcal{D}_d f(x)$ which may be given by the formula

$$(\forall x \in X) \ (\forall \varepsilon > 0) \quad \{ n \in \mathbb{N} : \| f_n(x) - f(x) \|_{\infty} \ge \varepsilon \} \in \mathcal{O}_d$$
 (4.4)

we have by [15]

$$(\forall x \in X) \ (\forall \varepsilon, \delta > 0) \ (\exists n_0 \in \mathbb{N}) \ (\forall n \ge n_0) \quad d_i(\{n \in \mathbb{N} : ||f_n(x) - f(x)||_{\infty} \ge \varepsilon\}) < \delta. \tag{4.5}$$

Remark 4.4. We obviously have

$$f_{n} \xrightarrow{\|\cdot y\|_{Y}} f \Longrightarrow f_{n} \xrightarrow{\|\cdot y\|_{Y}} f,$$

$$f_{n} \xrightarrow{\|\cdot y\|_{Y}} f \Longleftrightarrow \sup_{x \in X} \|f_{n}(x) - f(x), z\|_{Y} \xrightarrow{\|\cdot y\|_{Y}} 0 \quad \forall z \in Y.$$

$$(4.6)$$

Remark 4.5. Let \mathcal{D} be such that \mathcal{D} -convergence of sequences of points in $(Y, \|\cdot, \cdot\|_Y)$ is strictly more general than the usual convergence. Then there is a sequence $(y_n)_{n\in\mathbb{N}}\subseteq Y$, such that

$$y_n \xrightarrow{\|\cdot\cdot\|_Y} y$$
 but $\lim_{n \to \infty} \|y_n - y, z\|_Y \neq 0$ for each $z \in Y$. (4.7)

Putting $f_n(x) = y_n$ and f(x) = y for $x \in X$ and $n \in \mathbb{N}$, we have

$$f_n \xrightarrow{\|\cdot\|_Y} \mathcal{I}_{f_n} f$$
 but $\neg f_n \xrightarrow{\|\cdot\|_Y} f$. (4.8)

Thus, in this situation, \mathcal{D} -uniform convergence of sequences of functions is strictly more general than the usual uniform convergence.

Theorem 4.6. Let $\mathcal{D} \subseteq \mathcal{D}(\mathbb{N})$ be an admissible ideal and X, Y be two 2-normed spaces with $\dim Y < \infty$. Assume that $f_n \xrightarrow{\|\cdot,\cdot\|_Y} f$ (on X) where functions $f_n : X \to Y, n \in \mathbb{N}$ are equi-continuous (on X) and $f : X \to Y$. Then f is sequentially continuous (on X).

Proof. Let $z, x_0 \in X$ and $\varepsilon > 0$. By equi-continuty of f_n 's, there exist $\delta > 0$ such that $\|f_n(x_0) - f_n(x)\|_{\infty} \le \varepsilon$ for every $n \in \mathbb{N}$ whenever $\|x - x_0, z\| < \delta$.

Fix $x \in X$ such that $||x - x_0, z|| < \delta$. Since $f_n \xrightarrow{||y||_Y} f$, the set

$$\left\{ n \in N : \|f_n(x_0) - f(x_0)\|_{\infty} \ge \frac{\varepsilon}{3} \right\} \cup \left\{ n \in N : \|f_n(x) - f(x)\|_{\infty} \ge \frac{\varepsilon}{3} \right\}$$
(4.9)

is in \mathcal{I} and different from \mathbb{N} . Hence there exists $n_0 \in N$ such that

$$||f_{n_0}(x_0) - f(x_0)||_{\infty} < \frac{\varepsilon}{3}, \qquad ||f_{n_0}(x) - f(x)||_{\infty} < \frac{\varepsilon}{3}.$$
 (4.10)

We have

$$||f(x_{0}) - f(x)||_{\infty} \leq ||f(x_{0}) - f_{n_{0}}(x_{0})||_{\infty} + ||f_{n_{0}}(x_{0}) - f_{n_{0}}(x)||_{\infty} + ||f_{n_{0}}(x) - f(x)||_{\infty}$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon$$
(4.11)

and by (Corollary 3.7) f is sequentially continuous at $x_0 \in X$.

5. O-Equistatistically Convergent

Let X, Y be two 2-normed spaces with dim $Y < \infty$ and $\mathcal{O} = \mathcal{O}_d \subseteq 2^X$ be admissible ideal on X.

Definition 5.1. $A(f_n)_{n\in\mathbb{N}}$ is called \mathcal{O} -equi-statistically convergent to f (we write it as $f_n \overset{\|\cdot,\cdot\|_Y}{\leadsto} \mathcal{O}_d f$) if for every $\varepsilon > 0$ the sequence $(g_{j,\varepsilon})_{j\in\mathbb{N}}$ of functions $g_{j,\varepsilon}: X \to \mathbb{R}$ given by

$$g_{j,\varepsilon}(x) = d_j(\left\{n \in \mathbb{N} : \left\| f_n(x) - f(x) \right\|_{\infty} \ge \varepsilon\right\}), \quad x \in X$$
(5.1)

is uniformly convergent to the zero function (on X). Hence $f_n \stackrel{\|\cdot,\cdot\|_Y}{\leadsto} \mathcal{I}_d f$ if and only if the following formula holds:

$$(\forall \varepsilon, \delta > 0) \ (\exists n_0 \in \mathbb{N}) \ (\forall j \ge n_0) \ (\forall x \in X), \quad d_j(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_{\infty} \ge \varepsilon\}) < \delta. \tag{5.2}$$

Corollary 5.2. *The following properties hold:*

- (i) $f_n \stackrel{\|\cdot,\cdot\|_Y}{\leadsto}_{\mathcal{I}_d} f$ implies $f_n \stackrel{\|\cdot,\cdot\|_Y}{\longleftrightarrow}_{\mathcal{I}_d} f$,
- (ii) $f_n \xrightarrow{\|\cdot,\cdot\|_Y} \gamma_d f$ implies $f_n \overset{\|\cdot,\cdot\|_Y}{\leadsto} \gamma_d f$.

Proof.

- (i) If $f_n \stackrel{\|\cdot\|_{Y}}{\leadsto} _{\mathcal{I}d} f$ by the monotonicity of operator d_j , we take $\varepsilon = \delta$ in Definition 4.2. Thus it is obvious.
- (ii) Assume $f_n \xrightarrow[\text{uniform}]{} \mathcal{I}_d f$ and $\varepsilon > 0$. By Definition 4.2 there exist a set $M \in \mathcal{I}_d$ such that $\|f_n(x) f(x)\|_{\infty} < \varepsilon$ for all $n \in \mathcal{I}_d \setminus M$ and $x \in X$. Since $M \in \mathcal{I}_d$. We can pick $n_0 \in N$ such that $d_j(M) < \varepsilon$ for all $j \geq n_0$. Let $x \in X$ and $n \in \mathbb{N}$. Thus $\|f_n(x) f(x)\|_{\infty} \geq \varepsilon$ implies $n \in M$. Hence for each $j \geq n_0$, we have

$$d_{j}(\{n \in \mathbb{N} : \|f_{n}(x) - f(x)\|_{\infty} \ge \varepsilon\}) \le d_{j}(M) < \varepsilon$$
(5.3)

by Definition 4.2 witnesses that $f_n \stackrel{\|\cdot,\cdot\|_Y}{\leadsto} g_d f$.

Example 5.3. Define $f:[0,1] \times [0,1] \to \mathbb{R}^2$, $f_n:[0,1] \times [0,1] \to \mathbb{R}^2$, $n \in \mathbb{N}$

$$f_n(x_1, x_2) = \begin{cases} \left(0, \frac{1}{n}\right), & \text{if } x_2 = \frac{1}{n}, \\ (0, 0), & \text{otherwise,} \end{cases}$$
 $f(x_1, x_2) = (0, 0),$ (5.4)

Then $f_n \overset{\|\cdot\cdot\|_Y}{\leadsto} _{\mathcal{I}_d} f$ but $\neg f_n \overset{\|\cdot\cdot\|_Y}{\longleftrightarrow} _{\mathcal{I}_d} f$. Indeed, let $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Then for all $j \ge k$ and $x = (x_1, x_2) \in [0, 1] \times [0, 1]$ we have

$$d_{j}(\left\{n \in \mathbb{N} : \left\|f_{n}(x) - f(x)\right\|_{\infty} > \varepsilon\right\}) \le \frac{1}{j} \le \frac{1}{k} \le \varepsilon.$$
 (5.5)

Hence $f_n \stackrel{\|\cdot,\cdot\|_Y}{\leadsto} g_d f$.

Suppose that $f_n \xrightarrow[\text{uniform}]{\|\cdot\cdot\|_{\Upsilon}} \mathcal{I}_d f$. Thus there is the set $M \in \mathcal{I}_d$ such that for all $n \in \mathcal{I}_d \setminus M$ and $x \in [0,1] \times [0,1]$ we have $\|f_n(x) - f(x)\|_{\infty} < 1$.

Choose $k \in \mathcal{O}_d \setminus M$. Then f_k must be the zero function, a contradiction.

Theorem 5.4. Assume $f: X \to Y$ and $f_n: X \to Y$ for $n \in \mathbb{N}$ fix $x_0 \in X$. If $f_n \overset{\|\cdot,\cdot\|_Y}{\leadsto} g_d$ f and all functions $f_n, n \in \mathbb{N}$, are sequentially continuous at x_0 then f is sequentially continuous at x_0 .

Proof. Let $\varepsilon > 0$. Since $f_n \stackrel{\|\cdot,\cdot\|_Y}{\leadsto} \mathfrak{I}_d f$, we can find $n_0 \in \mathbb{N}$ such that

$$d_{n_0}\left(\left\{n\in\mathbb{N}: \left\|f_n(x)-f(x)\right\|_{\infty}\geq \frac{\varepsilon}{3}\right\}\right)<\frac{1}{2}\quad \forall x\in X.$$
 (5.6)

Put $E(x) = \{n \le K : \|f_n(x) - f(x)\|_{\infty} < \varepsilon/3\}$, $x \in X$. In other word d_{n_0} is a probability measure on $\mathcal{P}(\mathbb{N})$ with the support $\{1, \ldots, n_0\}$, it follows that $d_{n_0}(E(x)) > 1/2$ for all $x \in X$. By the sequentially continuity of f_1, \ldots, f_{n_0} at x_0 , there exist $\delta > 0$ such that $\|f_i(x) - f_i(x_0)\|_{\infty} < \varepsilon/3$ for all $1 \le i \le n_0$ and $x \in X$, $\|x - x_0, z\| < \delta$ for each $z \in X$. Fix x such that $x \in X$, $\|x - x_0, z\| < \delta$ for each $x \in X$.

Since $d_{n_0}(E(x)) > 1/2$ and $d_{n_0}(E(x_0)) > 1/2$, there exists $p \in E(x) \cap E(x_0)$ such that

$$\|f(x) - f(x_0)\|_{\infty} \le \|f(x) - f_p(x)\|_{\infty} + \|f_p(x) - f_p(x_0)\|_{\infty} + \|f_p(x_0) - f(x_0)\|_{\infty}$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon.$$
(5.7)

Thus we show that $||f(x) - f(x_0)||_{\infty} < \varepsilon$ for all $x \in X$, $||x - x_0, z|| < \delta$ for each $z \in X$. \square

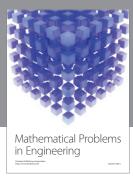
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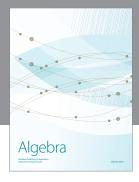
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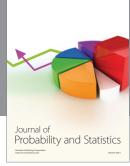
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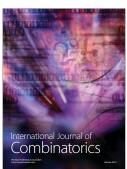








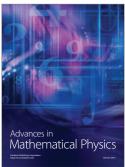




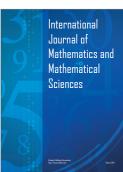


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