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Research Article

On Double Summability of Double Conjugate Fourier Series

H. K. Nigam and Kusum Sharma

Department of Mathematics, Faculty of Engineering and Technology, Mody Institute of Technology and Science, Deemed University, Laxmangarh 332311, Sikar, Rajasthan, India

Correspondence should be addressed to Kusum Sharma, kusum31sharma@rediffmail.com

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For the first time, a theorem on double matrix summability of double conjugate Fourier series is established.

1. Introduction

The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.1}$$

Conjugate to the series (1.1) is given by

$$\sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx) \tag{1.2}$$

and is known as conjugate Fourier series. It is well known that the corresponding conjugate function of (1.2) is defined as

$$\overline{f}(x) = \frac{1}{\pi}\int_0^{\pi} \frac{f(x+t) - f(x-t)}{2\tan(1/2)t}\,dt. \tag{1.3}$$

Let $f(x, y)$ is integrable (L) over the square $Q(-\pi, -\pi; \pi, \pi)$ and is periodic with period 2π in each variable.

The double Fourier series of a function $f(x,y)$ which is analogue for two variables of the series (1.1), is given by

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}\left[\alpha_{m,n}\cos mx\cos ny + \beta_{m,n}\sin mx\cos ny + \gamma_{m,n}\cos mx\sin ny + \delta_{m,n}\sin mx\sin ny\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}A_{m,n}(x,y), \tag{1.4}$$

$$\alpha_{m,n} = \frac{1}{\pi^2}\iint_Q f(x,y)\cos mx\cos ny\,dx\,dy \tag{1.5}$$

with three similar expressions for $m,n = 0,1,2,\ldots$ and for $\beta_{m,n}$, $\gamma_{m,n}$and $\delta_{m,n}$ where Q represents the fundamental square $(-\pi,-\pi;\pi,\pi)$.

One can associate three conjugate series to the double Fourier series (1.4) in the following way:

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}\left[-\beta_{m,n}\cos mx\cos ny+\alpha_{m,n}\sin mx\cos ny-\delta_{m,n}\cos mx\sin ny+\gamma_{m,n}\sin mx\sin ny\right], \tag{1.6}$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{m,n}\left[-\gamma_{m,n}\cos mx\cos ny-\delta_{m,n}\sin mx\cos ny+\alpha_{m,n}\cos mx\sin ny+\beta_{m,n}\sin mx\sin ny\right], \tag{1.7}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\delta_{m,n}\cos mx\cos ny-\gamma_{m,n}\sin mx\cos ny-\beta_{m,n}\cos mx\sin ny+\alpha_{m,n}\sin mx\sin ny\right], \tag{1.8}$$

where $\lambda_{m,n}=1,\ \lambda_{m,0}=\lambda_{0,n}=1/2,\ m,n\geq 1,\ \lambda_{0,0}=1/4$.

The conjugate functions $\overline{f}^{(1)}(x,y)$, $\overline{f}^{(2)}(x,y)$ and $\overline{f}^{(3)}(x,y)$ corresponding to (1.7), and (1.8) are defined as

$$\overline{f}^{(1)}(x,y) = -\frac{1}{\pi}\int_0^{\pi}\frac{f(x+s,y)-f(x-s,y)}{2\tan(1/2)s}ds,$$

$$\overline{f}^{(2)}(x,y) = -\frac{1}{\pi}\int_0^{\pi}\frac{f(x,y+t)-f(x,y-t)}{2\tan(1/2)t}dt,$$

$$\overline{f}(x,y) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\left[\frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)-f(x-s,y-t)}{2\tan(1/2)s\ 2\tan(1/2)t}\right]ds\,dt. \tag{1.9}$$

We will consider the symmetric square partial sum of series (1.8).

Let $T = (a_{m,j})$ and $S = (b_{n,k})$ be two infinite triangular matrices. Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n}$ be a double series with $s_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} p_{j,k}$ as its (m,n)th partial sums. The double matrix mean $t_{m,n}$ is given by

$$t_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j} b_{n,k} s_{j,k}, \tag{1.10}$$

The double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n}$ with the sequence of (m,n)th partial sums $(s_{m,n})$ is said to summable by double matrix summability method or summable (T,S) if $t_{m,n}$ tends to a limit s as $m \to \infty$ and $n \to \infty$.

The regularity conditions of double matrix summability means are given by

$$\begin{gathered}\sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j} b_{n,k} \longrightarrow 1 \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty,\\ \lim_{m,n} \sum_{k=0}^{n} \left|a_{m,j} b_{n,k}\right| = 0 \quad \text{for each } j = 1,2,3,\dots,\\ \lim_{m,n} \sum_{j=0}^{m} \left|a_{m,j} b_{n,k}\right| = 0 \quad \text{for each } k = 1,2,3,\dots\end{gathered} \tag{1.11}$$

We write

$$\begin{gathered}\varphi(x,y) = \varphi(x,y;s,t) = f(x+s,y+t) - f(x-s,y+t) - f(x+s,y-t) + f(x-s,y-t),\\ \Psi(x,y) = \int_0^x\int_0^y |\varphi(s,t)|\,ds\,dt,\\ \Psi_1(x,t) = \int_0^x |\varphi(s,t)|\,ds,\\ \Psi_2(s,y) = \int_0^y |\varphi(s,t)|\,dt,\\ \tau = \left(\frac{1}{t}\right) = \text{integral part of } \frac{1}{t},\\ \sigma = \left(\frac{1}{s}\right) = \text{integral part of } \frac{1}{s},\\ \overline{K}_m(s) = \frac{1}{2\pi}\sum_{j=0}^{m} a_{m,j}\frac{\cos(j+(1/2))s}{\sin(s/2)},\\ \overline{K}_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} b_{n,k}\frac{\cos(k+(1/2))t}{\sin(t/2)},\end{gathered} \tag{1.12}$$

Important particular cases of the double matrix summability method are

(i) $(C,1,1)$ summability mean ([1]) if $a_{m,j} = 1/(m+1)$for all m and $b_{n,k} = (1/(n+1))$ for all n;

(ii) $(H,1,1)$ summability mean ([5]) if $a_{m,j} = 1/(m-j+1)\log m$ and $b_{n,k} = 1/(n-k+1)\log n$;

(iii) (N,p_m,q_n) summability mean ([2]) if $a_{m,j} = p_{m-j}/P_m$ and $b_{n,k} = q_{n-k}/Q_n$, provided $P_m = \sum_{j=0}^{m} p_j \neq 0$ and $Q_n = \sum_{k=0}^{n} q_k \neq 0$.

Double matrix summability method (T,S) is assumed to be regular throughout this paper.

2. Main Theorem

Rajagopal [1] previously proved a theorem on the Nörlund summability of Fourier series. Result of Rajagopal [1] contained various results due to Hardy [2], Hirokowa [3], Hirokowa and Kayashima [4], Pati [5], Siddiqui [6], and Singh [7]. Thereafter Sharma [8] proved a theorem dealing with the harmonic summability of double Fourier series. The result of Sharma [8] is a generalization of the theorem due to Hille and Tamarkin [9] for double Fourier series and also is analogous to the theorem of Chow [10] for summability $(C,1,1)$ of the double Fourier series. The theorem of Sharma [8] was generalized by Mishra [11] for double Nörlund summability. The result Mishra [11] was generalized by Okuyama and Miyamoto [12]. But nothing seems to have been done so far to study double matrix summability of conjugate Fourier series. Therefore, the purpose of this paper is to establish the following theorem.

Theorem 2.1. *Let $(a_{m,j})_{j=0}^{m}$ and $(b_{n,k})_{k=0}^{n}$ be two real nonnegative and nondecreasing sequences with $j \le m$ and $k \le n$, respectively.*

Let $T = (a_{m,j})$ and $S = (b_{n,k})$ be two infinite triangular matrices with

$$\begin{gathered} a_{m,j} \ge 0, \quad a_{m,j} = 0, \quad j > m; \\ b_{n,k} \ge 0, \quad b_{n,k} = 0, \quad k > n; \\ A_{m,\sigma} = \sum_{j=0}^{\sigma} a_{m,j}; \qquad B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,k}; \\ A_{m,m} = 1 \quad \forall m \ge 0; \qquad B_{n,n} = 1 \quad \forall n \ge 0. \end{gathered} \tag{2.1}$$

If the conditions

$$\begin{aligned} \Psi(x,y) &= \int_0^x \int_0^y |\psi(s,t)|\,ds\,dt \\ &= O\left\{ \frac{x}{\xi(1/x)} \cdot \frac{y}{\chi(1/y)} \right\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \int_0^\pi \psi_1(x,t)dt &= O\left\{ \frac{x}{\xi(1/x)} \right\}, \\ \int_0^\pi \psi_2(s,y)ds &= O\left\{ \frac{y}{\chi(1/y)} \right\} \end{aligned} \tag{2.3}$$

hold, then the double conjugate Fourier series (1.8) *is double matrix* (T,S) *summable to,* $\overline{f}(x,y)$, *where*

$$\overline{f}(x,y) = -\frac{1}{4\pi^2}\iint_0^{\pi} \frac{1}{\tan(1/2)s\ \tan(1/2)t} ds\, dt \tag{2.4}$$

at every point where these integrals exist provided $\xi(x)$ *and* $\chi(y)$ *are two positive monotonic increasing functions of* x *and* y *such that* $\xi(m) \to \infty$, *as* $m \to \infty$ *and* $\chi(n) \to \infty$, *as* $n \to \infty$,

$$\begin{aligned} &\int_1^m \frac{A_{m,s}}{s\xi(s)} ds = O(1), \quad m \longrightarrow \infty, \\ &\int_1^n \frac{B_{n,t}}{t\chi(t)} dt = O(1), \quad n \longrightarrow \infty. \end{aligned} \tag{2.5}$$

3. Lemmas

Lemma 3.1. *One has*

$$\overline{K}_m(s) = O\left(\frac{1}{s}\right) \quad \text{for } 0 \le s \le \frac{1}{m}. \tag{3.1}$$

Proof. For $0 \le s \le 1/m$, $\sin(t/2) \ge t/\pi$ and $|\cos mt| \le 1$,

$$\begin{aligned} \left|\overline{K}_m(s)\right| &\le \frac{1}{2}\left|\sum_{j=0}^{m} a_{m,j} \frac{\cos(j+(1/2))s}{\sin(s/2)}\right| \\ &\le \frac{1}{2}\left\{\sum_{j=0}^{m} a_{m,j} \frac{\left|\cos(j+(1/2))s\right|}{|\sin(s/2)|}\right\} \\ &\le \frac{\pi}{s}\sum_{j=0}^{m} a_{m,j} \\ &= O\left(\frac{1}{s}\right). \end{aligned} \tag{3.2}$$

□

Lemma 3.2. *One has*

$$\overline{K}_n(t) = O\left(\frac{1}{t}\right) \quad \text{for } 0 \le t \le \frac{1}{n}. \tag{3.3}$$

Proof. This can be proved similar to Lemma 3.1. □

Lemma 3.3 (see [2]). *If* $(a_{m,u})$ *is non-negative and non-decreasing with* μ, *then, for* $0 \le a \le b \le \infty$, $0 \le s \le \pi$ *and any* m,

$$\left|\sum_{\mu=a}^{b} a_{m,m-\mu} e^{i(m-\mu)s}\right| = O(A_{m,\sigma}). \tag{3.4}$$

Lemma 3.4. *If $(b_{n,v})$ is non-negative and non-decreasing with ν, then, for $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n,*

$$\left|\sum_{\nu=a}^{b} b_{n,n-\nu} e^{i(n-\nu)t}\right| = O(B_{n,\tau}). \tag{3.5}$$

Proof. This is similar to Lemma 3.3. □

Lemma 3.5. *One has*

$$\overline{K}_m(s) = O\left(\frac{A_{m,\sigma}}{s}\right) \quad \text{for } 0 < \frac{1}{m} < s \le \pi. \tag{3.6}$$

Proof. Since $1/m < s \le \pi, \sin(s/2) \ge s/\pi$,

$$\begin{aligned}
\left|\overline{K}_m(s)\right| &\le \frac{1}{2\pi}\left|\sum_{j=1}^{m} a_{m,j} \frac{\cos(j+(1/2))s}{\sin(s/2)}\right| \\
&\le \frac{1}{2s}\left|\sum_{j=1}^{m} a_{m,m-j} \operatorname{Re}\left\{e^{i(m-j+(1/2))s}\right\}\right| \\
&\le \frac{1}{2s}\left|\sum_{j=1}^{m} a_{m,m-j} \operatorname{Re}\ e^{i(m-j)s}\right|\left|e^{i(s/2)}\right| \\
&= \frac{1}{2s}\left|\operatorname{Re}\sum_{j=1}^{m} a_{m,m-j}\ e^{i(m-j)s}\right| \\
&= O\left(\frac{1}{s}\right)\cdot O(A_{m,\sigma}) \quad \text{by Lemma 3.3} \\
&= O\left(\frac{A_{m,\sigma}}{s}\right).
\end{aligned} \tag{3.7}$$

□

Lemma 3.6. *One has*

$$\overline{K}_n(t) = O\left(\frac{B_{n,\tau}}{t}\right), \quad \text{for } 0 < \frac{1}{n} < \pi. \tag{3.8}$$

Proof. It can be proved similar to Lemma 3.5 but using Lemma 3.4. □

4. Proof of The Theorem

The (j,k)th partial sums $\overline{s}_{j,k}(x,y)$ of the series (1.8) is given by

$$\overline{s}_{j,k}(x,y) - \overline{f}(x,y) = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} \psi(s,t)\frac{\cos(j+(1/2))s}{\sin(s/2)}\frac{\cos(k+(1/2))t}{\sin(t/2)}ds\,dt. \tag{4.1}$$

Then,

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j} b_{n,k}\left\{\overline{s}_{j,k}(x,y)-\overline{f}(x,y)\right\} \\
&\quad = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(s,t)\sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j} b_{n,k}\frac{\cos(j+(1/2))s}{\sin(s/2)}\cdot\frac{\cos(k+(1/2))t}{\sin(t/2)}ds\,dt
\end{aligned} \tag{4.2}$$

or

$$\begin{aligned}
\overline{t}_{m,n}(x,y)-\overline{f}(x,y) &= \iint_0^{\pi}\psi(s,t)\overline{K}_m(s)\overline{K}_n(t)ds\,dt \\
&= \left(\int_0^{\nu}\int_0^{\eta}+\int_0^{\nu}\int_{\eta}^{\pi}+\int_{\nu}^{\pi}\int_0^{\eta}+\int_{\nu}^{\pi}\int_{\eta}^{\pi}\right)\cdot\psi(s,t)\overline{K}_m(s)\overline{K}_n(t)ds\,dt \\
&= I_1+I_2+I_3+I_4 \quad (\text{say}).
\end{aligned} \tag{4.3}$$

We consider

$$\begin{aligned}
I_1 &= \left(\int_0^{1/m}\int_0^{1/n}+\int_{1/m}^{\nu}\int_0^{1/n}+\int_0^{1/m}\int_{1/n}^{\eta}+\int_{1/m}^{\nu}\int_{1/n}^{\eta}\right)\cdot\psi(s,t)\overline{K}_m(s)\overline{K}_n(t)ds\,dt \\
&= I_{1.1}+I_{1.2}+I_{1.3}+I_{1.4} \quad (\text{say}),
\end{aligned} \tag{4.4}$$

where $s \le \nu$, $t \le \eta$ and (2.2) holds.

Now consider

$$\begin{aligned}
I_{1.1} &= \int_0^{1/m}\int_0^{1/n}\psi(s,t)\overline{K}_m(s)\overline{K}_n(t)ds\,dt \\
&= O\left[\int_0^{1/m}\int_0^{1/n}\left(\frac{1}{s}\right)\left(\frac{1}{t}\right)\psi(s,t)ds\,dt\right] \text{ by Lemma 3.1 and 3.2} \\
&= O(mn)\left[\int_0^{1/m}\int_0^{1/n}\psi(s,t)ds\,dt\right] \\
&= O(mn)\psi\left(\frac{1}{m},\frac{1}{n}\right) \\
&= O(mn)o\left\{\frac{1}{mn\xi(m)\chi(n)}\right\} \text{ by (2.2)} \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned} \tag{4.5}$$

Now,

$$\begin{aligned}
|I_{1.2}| &\le \int_{1/m}^{v}\int_{0}^{1/n} O\left|\psi(s,t)\right|\left|\overline{K}_m(s)\right|\left|\overline{K}_n(t)\right| ds\,dt \\
&\le \int_0^{1/n} |K_n(t)| dt \int_{1/m}^{v} \left|\psi(s,t)\right| \frac{A_{m,[1/s]}}{s} ds \quad \text{by Lemma 3.5} \\
&= \left\{ \int_0^{1/n} \left(\frac{1}{t}\right) dt \int_{1/m}^{v} \left|\psi(s,t)\right| \frac{A_{m,[1/s]}}{s} ds \right\} \quad \text{by Lemma 3.2} \\
&= O(n)\left[\int_0^{1/n} \frac{A_{m,[1/v]}}{v} \psi_1(v,t) dt\right] + O(mn)(A_{m,m})\left[\int_0^{1/n} \psi_1\left(\frac{1}{m},t\right) dt\right] \\
&\quad + O(n)\left[\int_0^{1/n} dt \int_{1/m}^{v} \frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)\left|\psi(s,t)\right| ds\right] \\
&= I_{1.2.1} + I_{1.2.2} + I_{1.2.3} \quad \text{(say)}.
\end{aligned} \tag{4.6}$$

Then,

$$\begin{aligned}
I_{1.2.1} + I_{1.2.2} &= O(n)\left[\int_0^{1/n} \psi_1(v,t) dt\right] + O(mn)\left[\int_0^{1/n} \psi_1\left(\frac{1}{m},t\right) dt\right] \\
&= O(n)\Psi\left(v,\frac{1}{n}\right) + O(mn)\Psi\left(\frac{1}{m},\frac{1}{n}\right) \\
&= O(n)o\left(\frac{v}{\xi(1/v)} \cdot \frac{1/n}{\chi(n)}\right) + O(mn)o\left(\frac{1/mn}{\xi(m)\chi(n)}\right) \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned} \tag{4.7}$$

Next,

$$\begin{aligned}
I_{1.2.3} &= O(n)\int_0^{1/n} dt \int_{1/m}^{v} \Psi_1(s,t) \frac{A_{m,[1/s]}}{s^2} ds + O(n)\int_0^{1/n} dt \int_{1/m}^{v} \frac{\Psi_1(s,t)}{s} \frac{d}{ds}\left(A_{m,[1/s]}\right) ds \\
&= O(n)\left[\int_{1/m}^{v} \left(\int_0^{1/n} \Psi_1(s,t) dt\right) \frac{A_{m,[1/s]}}{s^2} ds\right] \\
&\quad + O(n)\left[\int_{1/m}^{v} \left(\int_0^{1/n} \Psi_1(s,t)\right) \frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right) ds\right] \\
&= O(n)\int_{1/m}^{v} \Psi_1\left(s,\frac{1}{n}\right) \frac{A_{m,[1/s]}}{s^2} ds + O(n)\int_{1/m}^{v} \Psi_1\left(s,\frac{1}{n}\right) \frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right) ds \\
&= O(n)\int_{1/m}^{v} o\left(\frac{s}{\xi(1/s)} \cdot \frac{1}{n\chi(n)}\right) \frac{A_{m,[1/s]}}{s^2} ds \\
&\quad + O(n)\left[\int_{1/m}^{v} o\left(\frac{s}{\xi(1/s)} \cdot \frac{1}{n\chi(n)}\right) \frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right) ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= o\left[\frac{1}{\chi(n)}\right]\left[\int_{1/\nu}^{m}\frac{A_{m,s}}{s\xi(s)}ds\right] + o\left[\frac{1}{\chi(n)}\right]\left[\int_{1/\nu}^{m}\frac{1}{\xi(s)}\frac{d}{ds}(A_{m,s})ds\right]\\
&= o\left[\frac{1}{\chi(n)}\right]o(1) + o\left[\frac{1}{\xi(m)\chi(n)}\right][A_{m,n}]\\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned}
\tag{4.8}
$$

Thus,

$$
I_{1.2} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.9}
$$

Similarly

$$
I_{1.3} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.10}
$$

Now,

$$
\begin{aligned}
|I_{1.4}| &= O\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}|\varphi(s,t)|\frac{A_{m,[1/s]}}{s}\cdot\frac{B_{n,[1/t]}}{t}dt\,ds\right]\\
&= O\left[\frac{B_{n,[1/\eta]}A_{m,[1/\nu]}\Psi(\nu,\eta)}{\nu\eta} - \frac{mB_{n,[1/\eta]}A_{m,m}\Psi(1/m,\eta)}{\eta}\right.\\
&\quad - \frac{B_{n,[1/\eta]}}{\eta}\int_{1/m}^{\nu}\varphi(s,\eta)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds - \frac{nB_{n,n}A_{m,[1/\nu]}\Psi(\nu,1/n)}{\nu}\\
&\quad + mn\ B_{n,n}A_{m,m}\Psi\left(\frac{1}{m},\frac{1}{n}\right) + nB_{n,n}\int_{1/m}^{\nu}\Psi\left(s,\frac{1}{n}\right)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\\
&\quad + \frac{A_{m,[1/\nu]}}{\nu}\int_{1/m}^{\nu}\Psi_1(s,t)\frac{d}{dt}\left(\frac{B_{n,[1/t]}}{t}\right)dt - mA_{m,m}\int_{1/n}^{\xi}\Psi_2\left(\frac{1}{m},t\right)\frac{d}{dt}\left(\frac{B_{n,[1/t]}}{t}\right)dt\\
&\quad \left. - \left\{\int_{1/m}^{\nu}\int_{1/n}^{\eta}\Psi(s,t)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)\frac{d}{dt}\left(\frac{B_{n,[1/t]}}{t}\right)dt\,ds\right\}\right]\\
&= I_{1.4.1} + I_{1.4.2} + I_{1.4.3} + I_{1.4.4} + I_{1.4.5} + I_{1.4.6} + I_{1.4.7} + I_{1.4.8} + I_{1.4.9} \quad \text{(say)}.
\end{aligned}
\tag{4.11}
$$

Then,

$$
\begin{aligned}
I_{1.4.1} + I_{1.4.2} &= O\left[\frac{B_{n,[1/\eta]}A_{m,[1/\nu]}\Psi(\nu,\eta)}{\nu\eta} + \frac{mB_{n,[1/\eta]}A_{m,m}\Psi(1/m,\eta)}{\eta}\right]\\
&= o\left[\frac{B_{n,[1/\eta]}A_{m,[1/\nu]}}{\alpha(1/\nu)\beta(1/\eta)}\right] + o\left[mB_{n,[1/\eta]}\left\{\frac{1}{m\xi(m)}\cdot\frac{\eta}{\chi(1/\eta)}\right\}\right]\\
&= o\left[\frac{B_{n,[1/\eta]}A_{m,[1/\nu]}}{\xi(1/\nu)\chi(1/\eta)}\right] + o\left[B_{n,[1/\eta]}\frac{1}{\xi(m)}\cdot\frac{\eta}{\chi(1/\eta)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= o(1) + o(1)\left[\frac{1}{\xi(m)}\right] \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty \text{ by the regularity of } (T,S), \\
I_{1.4.3} &= -\frac{B_{n,[1/\eta]}}{\eta}\int_{1/m}^{\nu}\Psi(s,\eta)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds \\
&= o\left(\frac{B_{n,[1/\eta]}}{\eta}\right)\int_{1/m}^{\nu}\left(\frac{s}{\xi(1/s)}\cdot\frac{\eta}{\chi(1/\eta)}\right)\left(\frac{A_{m,[1/s]}}{s^2}\right)ds \\
&\quad + o\left(\frac{B_{n,[1/\eta]}}{\eta}\right)\int_{1/m}^{\nu}o\left(\frac{s}{\xi(1/s)}\cdot\frac{\eta}{\chi(1/\eta)}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds \\
&= o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right)\int_{1/m}^{\nu}\frac{A_{m,[1/s]}}{s\xi(1/s)}ds + o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right)\int_{1/m}^{\nu}\frac{1}{\xi(1/s)}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds \\
&= o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right)\int_{1/\nu}^{m}\frac{A_{m,s}}{s\xi(s)}ds + o\left(\frac{B_{n,s/\eta}}{\chi(1/\eta)}\right)\int_{1/\nu}^{m}\frac{1}{\xi(s)}d\left(A_{m,s}\right) \\
&= o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right)O(1) + o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right)O\left(A_{m,m}\right) \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned}
\tag{4.12}
$$

Thus, we get

$$
I_{1.4.3} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.13}
$$

Similarly

$$
I_{1.4.4} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.14}
$$

Now,

$$
\begin{aligned}
I_{1.4.5} &= mnB_{n,n}A_{m,m}\Psi\left(\frac{1}{m},\frac{1}{n}\right) \\
&= o\left(mnB_{n,n}A_{m,m}\frac{1}{mn\xi(m)\chi(n)}\right) \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned}
\tag{4.15}
$$

Consider

$$\begin{aligned}
I_{1.4.6} &= nB_{n,n}\int_{1/m}^{\nu}\Psi\left(s,\frac{1}{n}\right)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\\
&= nB_{n,n}\int_{1/m}^{\nu}\Psi\left(s,\frac{1}{n}\right)\frac{A_{m,[1/s]}}{s^2}ds + nB_{n,n}\int_{1/m}^{\nu}\Psi\left(s,\frac{1}{n}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds\\
&= nB_{n,n}\int_{1/m}^{\nu}o\left(\frac{s}{\xi(1/s)}\cdot\frac{1}{n\chi(\eta)}\right)\frac{A_{m,[1/s]}}{s^2}ds\\
&\quad + nB_{n,n}\int_{1/m}^{\nu}\left(\frac{s}{\xi(1/s)n\chi(\eta)}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds\\
&= o\left(\frac{nB_{n,n}}{n\chi(n)}\right)\int_{1/m}^{\nu}\frac{A_{m,[1/s]}}{s\xi(1/s)}ds + o\left(\frac{1}{\chi(n)}\right)\int_{1/m}^{\nu}\frac{1}{\xi(1/s)}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds\\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned} \tag{4.16}$$

As similar to $I_{1.4.3}$,

$$I_{1.4.7} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.17}$$

As similar to $I_{1.4.6}$,

$$I_{1.4.8} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.18}$$

Now,

$$\begin{aligned}
I_{1.4.9} &= O\left[\left\{\int_{1/m}^{\nu}\int_{1/n}^{\eta}\Psi(s,t)\left(\frac{A_{m,[1/s]}}{s^2}\right)+\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)\right\}\cdot\left\{\frac{B_{n,[1/t]}}{t^2}+\frac{1}{t}\frac{d}{dt}\left(B_{n,[1/t]}\right)\right\}dt\,ds\right]\\
&= I_{1.4.9.1} + I_{1.4.9.2} + I_{1.4.9.3} + I_{1.4.9.4} \quad (\text{say}).
\end{aligned} \tag{4.19}$$

Then,

$$\begin{aligned}
I_{1.4.9.1} + I_{1.4.9.2} &= O\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\Psi(s,t)\left\{\frac{A_{m,[1/s]}}{s^2}\cdot\frac{B_{n,[1/t]}}{t^2}\right\}dt\,ds\right]\\
&\quad + O\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\Psi(s,t)\left\{\frac{A_{m,[1/s]}}{s^2}\frac{1}{t}\cdot\frac{d}{dt}\left(B_{n,[1/t]}\right)\right\}dt\,ds\right]\\
&= O\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}o\left\{\frac{s}{\xi(1/s)}\cdot\frac{t}{\chi(1/t)}\right\}\frac{A_{m,[1/s]}}{s^2}\cdot\frac{B_{n,[1/t]}}{t^2}dt\,ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+ O\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta} o\left\{\frac{s}{\xi(1/s)}\cdot\frac{t}{\chi(1/t)}\right\}\frac{A_{m,[1/s]}}{s^2}\frac{1}{t}\cdot\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\,ds\right] \\
&= o\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\frac{A_{m,[1/s]}B_{n,[1/t]}}{s\xi(1/s)t\,\chi(1/t)}dt\,ds\right] \\
&+ o\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\frac{A_{m,[1/s]}}{s\xi(1/s)\chi(1/t)}\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\,ds\right] \\
&= o\left[\int_{1/\nu}^{m}\frac{A_{m,s}}{s\xi(s)}ds\int_{1/\eta}^{n}\frac{B_{n,t}}{t\chi(t)}dt\right] \\
&+ o\left[\int_{1/\nu}^{m}\frac{A_{m,s}}{s\xi(s)}ds\int_{1/\eta}^{n}\frac{1}{\chi(t)}\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\right] \\
&= o(1) + o(1)\int_{1/\eta}^{n} O(1)\frac{d}{dt}\left(B_{n,[1/t]}\right)dt \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned}
\tag{4.20}
$$

Similarly,

$$
I_{1.4.9.3} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.21}
$$

Now,

$$
\begin{aligned}
I_{1.4.9.4} &= O\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\Psi(s,t)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)\frac{1}{t}\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\,ds\right] \\
&= o\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\left(\frac{s}{\xi(1/s)}\cdot\frac{t}{\chi(1/t)}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)\frac{1}{t}\cdot\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\,ds\right] \\
&= o\left[\int_{1/m}^{\nu}\int_{1/n}^{\eta}\left(\frac{1}{\xi(1/s)\cdot\chi(1/t)}\right)\frac{d}{ds}\left(A_{m,[1/s]}\right)\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\,ds\right] \\
&= o\left[\int_{1/m}^{\nu}\frac{1}{\xi(1/s)}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds\int_{1/n}^{\eta}\frac{1}{\chi(1/t)}\frac{d}{dt}\left(B_{n,[1/t]}\right)dt\right] \\
&= o\left[\int_{1/m}^{\nu}O(1)\frac{d}{ds}\left(A_{m,[1/s]}\right)ds\int_{1/n}^{\eta}O(1)\frac{d}{dt}\left(B_{n,[1/t]}\right)d\right] \\
&= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned}
\tag{4.22}
$$

Hence,

$$
I_{1.4} = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.23}
$$

Therefore,

$$I_1 = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.24}$$

Now, $1/m < \nu < \pi,\ 1/n < \eta < \pi$, thus we obtain

$$\begin{aligned} |I_3| &\le \int_\nu^\pi |K_m(s)| ds \int_0^{1/n} \left|\varphi(s,t)\right| |K_n(t)| dt + \int_\nu^\pi |K_m(s)| ds \int_{1/n}^{\nu} \left|\varphi(s,t)\right| |K_n(t)| dt \\ &= I_{3.1} + I_{3.2} \quad (\text{say}). \end{aligned} \tag{4.25}$$

Using Lemma 3.2 and Lemma 3.5, we have

$$\begin{aligned} I_{3.1} &= \int_\nu^\pi \frac{A_{m,\sigma}}{s} ds \int_0^{1/n} \left|\varphi(s,t)\right| \frac{1}{t} dt \\ &= O(n) \int_\nu^\pi \frac{A_{m,\sigma}}{s} ds \int_0^{1/n} \left|\varphi(s,t)\right| dt \\ &= O(n) \int_0^\pi \Psi_2\left(s, \frac{1}{n}\right) ds \\ &= O\left(\frac{1}{\chi(n)}\right) \\ &= o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \end{aligned} \tag{4.26}$$

Using Lemma 3.5 and Lemma 3.6,

$$\begin{aligned} I_{3.2} &= O\left[\int_\nu^\pi \frac{A_{m,\sigma}}{s} ds \int_{1/n}^{\eta} \left|\varphi(s,t)\right| \frac{B_{n,[\tau]}}{t} dt\right] \\ &= O\left[\int_\nu^\pi ds \left\{ \left(\Psi_2(s,t) \frac{B_{n,[\tau]}}{t}\right)_{1/n}^{\eta} - \int_{1/n}^{\eta} \Psi(s,t) \frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t}\right) dt \right\}\right] \\ &= O\left[\int_\nu^\pi ds \left\{ \Psi_2(s,\eta) \frac{B_{n,[1/n]}}{\eta} - \Psi_2\left(s, \frac{1}{n}\right) n B_{n,n} \right\}\right] \end{aligned}$$

$$
\begin{aligned}
&+\left[\int_{\nu}^{\pi} ds \int_{1/n}^{\eta} \Psi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t}\right)dt\right] \\
&= O\left[\int_{\nu}^{\pi} \Psi_2(s,\eta)\frac{B_{n,[1/n]}}{\eta}ds\right] + O(n)\left[\int_{0}^{\pi} \Psi_2\left(s,\frac{1}{n}\right)ds\right] \\
&\quad + O\left[\int_{\nu}^{\pi} ds \int_{1/n}^{\eta} \Psi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t}\right)dt\right] \\
&= o(1) + o(1) + o(1)(\text{similar to } I_{1.4.9}) \\
&= o(1), \quad \text{as } m \to \infty,\ n \to \infty.
\end{aligned}
\tag{4.27}
$$

Hence,

$$
I_3 = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.28}
$$

Similarly,

$$
I_2 = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.29}
$$

By the regularity conditions of matrix method and Riemann-Lebesgue theorem,

$$
I_4 = o(1), \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{4.30}
$$

Therefore,

$$
\bar{t}_{m,n}(x,y) - \overline{f}(x,y) = o(1) \tag{4.31}
$$

This completes the proof of the theorem.

References

[1] C. T. Rajagopal, "On the Nörlund summability of Fourier series," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 59, pp. 47–53, 1963.
[2] G. H. Hardy, "On the summability of Fourier series," *Proceedings London Mathematical Society*, vol. 12, pp. 365–372, 1913.
[3] H. Hirokawa, "On the Nörlund summability of Fourier series and its conjugate series," *Proceedings of the Japan Academy*, vol. 44, pp. 449–451, 1968.
[4] H. Hirokawa and I. Kayashima, "On a sequence of Fourier coefficients," *Proceedings of the Japan Academy*, vol. 50, pp. 57–62, 1974.
[5] T. Pati, "A generalization of a theorem of Iyengar on the harmonic summability of Fourier series," *Indian Journal of Mathematics*, vol. 3, pp. 85–90, 1961.
[6] J. A. Siddiqui, "On the harmonic summability of Fourier series," *Proceedings of the National Institute of Sciences of India*, vol. 28, pp. 527–531, 1948.
[7] T. Singh, "On Nörlund summability of Fourier series and its conjugate series," *Proceedings of the National Institute of Sciences of India A*, vol. 29, pp. 65–73, 1963.

[8] P. L. Sharma, "On the harmonic summability of double Fourier series," *Proceedings of the American Mathematical Society*, vol. 9, pp. 979–986, 1958.
[9] E. Hille and J. D. Tamarkin, "On the summability of Fourier series. I," *Transactions of the American Mathematical Society*, vol. 34, no. 4, pp. 757–783, 1932.
[10] Y. S. Chow, "On the Cesàro summability of double Fourier series," *The Tohoku Mathematical Journal*, vol. 5, pp. 277–283, 1954.
[11] K. N. Mishra, "Summability of double Fourier series by double Nörlund method," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 13, no. 3, pp. 289–295, 1985.
[12] Y. Okuyama and I. Miyamoto, "On the double Nörlund summability of double Fourier series," *Tamkang Journal of Mathematics*, vol. 27, no. 2, pp. 133–144, 1996.