

*Research Article*

## **On Double Summability of Double Conjugate Fourier Series**

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For the first time, a theorem on double matrix summability of double conjugate Fourier series is established.

### **1. Introduction**

The Fourier series of  $f(x)$  is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (1.1)$$

Conjugate to the series (1.1) is given by

$$\sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx) \quad (1.2)$$

and is known as conjugate Fourier series. It is well known that the corresponding conjugate function of (1.2) is defined as

$$\bar{f}(x) = \frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan(1/2)t} dt. \quad (1.3)$$

Let  $f(x, y)$  is integrable (L) over the square  $Q(-\pi, -\pi; \pi, \pi)$  and is periodic with period  $2\pi$  in each variable.

The double Fourier series of a function  $f(x, y)$  which is analogue for two variables of the series (1.1), is given by

$$\begin{aligned} f(x, y) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} [\alpha_{m,n} \cos mx \cos ny + \beta_{m,n} \sin mx \cos ny + \gamma_{m,n} \cos mx \sin ny \\ &\quad + \delta_{m,n} \sin mx \sin ny] \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} A_{m,n}(x, y), \end{aligned} \quad (1.4)$$

$$\alpha_{m,n} = \frac{1}{\pi^2} \iint_Q f(x, y) \cos mx \cos ny dx dy \quad (1.5)$$

with three similar expressions for  $m, n = 0, 1, 2, \dots$  and for  $\beta_{m,n}$ ,  $\gamma_{m,n}$  and  $\delta_{m,n}$  where  $Q$  represents the fundamental square  $(-\pi, -\pi; \pi, \pi)$ .

One can associate three conjugate series to the double Fourier series (1.4) in the following way:

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} [\beta_{m,n} \cos mx \cos ny + \alpha_{m,n} \sin mx \cos ny - \delta_{m,n} \cos mx \sin ny + \gamma_{m,n} \sin mx \sin ny], \quad (1.6)$$

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda_{m,n} [\gamma_{m,n} \cos mx \cos ny - \delta_{m,n} \sin mx \cos ny + \alpha_{m,n} \cos mx \sin ny + \beta_{m,n} \sin mx \sin ny], \quad (1.7)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\delta_{m,n} \cos mx \cos ny - \gamma_{m,n} \sin mx \cos ny - \beta_{m,n} \cos mx \sin ny + \alpha_{m,n} \sin mx \sin ny], \quad (1.8)$$

where  $\lambda_{m,n} = 1$ ,  $\lambda_{m,0} = \lambda_{0,n} = 1/2$ ,  $m, n \geq 1$ ,  $\lambda_{0,0} = 1/4$ .

The conjugate functions  $\bar{f}^{(1)}(x, y)$ ,  $\bar{f}^{(2)}(x, y)$  and  $\bar{f}^{(3)}(x, y)$  corresponding to (1.7), and (1.8) are defined as

$$\begin{aligned} \bar{f}^{(1)}(x, y) &= -\frac{1}{\pi} \int_0^\pi \frac{f(x+s, y) - f(x-s, y)}{2 \tan(1/2)s} ds, \\ \bar{f}^{(2)}(x, y) &= -\frac{1}{\pi} \int_0^\pi \frac{f(x, y+t) - f(x, y-t)}{2 \tan(1/2)t} dt, \\ \bar{f}(x, y) &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \left[ \frac{f(x+s, y+t) - f(x-s, y+t) - f(x+s, y-t) - f(x-s, y-t)}{2 \tan(1/2)s 2 \tan(1/2)t} \right] ds dt. \end{aligned} \quad (1.9)$$

We will consider the symmetric square partial sum of series (1.8).

Let  $T = (a_{m,j})$  and  $S = (b_{n,k})$  be two infinite triangular matrices. Let  $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n}$  be a double series with  $s_{m,n} = \sum_{j=0}^m \sum_{k=0}^n p_{j,k}$  as its  $(m,n)$ th partial sums. The double matrix mean  $t_{m,n}$  is given by

$$t_{m,n} = \sum_{j=0}^m \sum_{k=0}^n a_{m,j} b_{n,k} s_{j,k}, \quad (1.10)$$

The double series  $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n}$  with the sequence of  $(m,n)$ th partial sums  $(s_{m,n})$  is said to summable by double matrix summability method or summable  $(T,S)$  if  $t_{m,n}$  tends to a limit  $s$  as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ .

The regularity conditions of double matrix summability means are given by

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^n a_{m,j} b_{n,k} \rightarrow 1 \quad \text{as } m \rightarrow \infty, n \rightarrow \infty, \\ & \lim_{m,n} \sum_{k=0}^n |a_{m,j} b_{n,k}| = 0 \quad \text{for each } j = 1, 2, 3, \dots, \\ & \lim_{m,n} \sum_{j=0}^m |a_{m,j} b_{n,k}| = 0 \quad \text{for each } k = 1, 2, 3, \dots \end{aligned} \quad (1.11)$$

We write

$$\begin{aligned} \varphi(x, y) &= \varphi(x, y; s, t) = f(x+s, y+t) - f(x-s, y+t) - f(x+s, y-t) + f(x-s, y-t), \\ \Psi(x, y) &= \int_0^x \int_0^y |\varphi(s, t)| ds dt, \\ \Psi_1(x, t) &= \int_0^x |\varphi(s, t)| ds, \\ \Psi_2(s, y) &= \int_0^y |\varphi(s, t)| dt, \\ \tau &= \left( \frac{1}{t} \right) = \text{integral part of } \frac{1}{t}, \\ \sigma &= \left( \frac{1}{s} \right) = \text{integral part of } \frac{1}{s}, \\ \bar{K}_m(s) &= \frac{1}{2\pi} \sum_{j=0}^m a_{m,j} \frac{\cos(j + (1/2))s}{\sin(s/2)}, \\ \bar{K}_n(t) &= \frac{1}{2\pi} \sum_{k=0}^n b_{n,k} \frac{\cos(k + (1/2))t}{\sin(t/2)}, \end{aligned} \quad (1.12)$$

Important particular cases of the double matrix summability method are

- (i)  $(C, 1, 1)$  summability mean ([1]) if  $a_{m,j} = 1/(m+1)$  for all  $m$  and  $b_{n,k} = (1/(n+1))$  for all  $n$ ;

- (ii)  $(H, 1, 1)$  summability mean ([5]) if  $a_{m,j} = 1/(m-j+1) \log m$  and  $b_{n,k} = 1/(n-k+1) \log n$ ;
- (iii)  $(N, p_m, q_n)$  summability mean ([2]) if  $a_{m,j} = p_{m-j}/P_m$  and  $b_{n,k} = q_{n-k}/Q_n$ , provided  $P_m = \sum_{j=0}^m p_j \neq 0$  and  $Q_n = \sum_{k=0}^n q_k \neq 0$ .

Double matrix summability method  $(T, S)$  is assumed to be regular throughout this paper.

## 2. Main Theorem

Rajagopal [1] previously proved a theorem on the Nörlund summability of Fourier series. Result of Rajagopal [1] contained various results due to Hardy [2], Hirokawa [3], Hirokawa and Kayashima [4], Pati [5], Siddiqui [6], and Singh [7]. Thereafter Sharma [8] proved a theorem dealing with the harmonic summability of double Fourier series. The result of Sharma [8] is a generalization of the theorem due to Hille and Tamarkin [9] for double Fourier series and also is analogous to the theorem of Chow [10] for summability  $(C, 1, 1)$  of the double Fourier series. The theorem of Sharma [8] was generalized by Mishra [11] for double Nörlund summability. The result Mishra [11] was generalized by Okuyama and Miyamoto [12]. But nothing seems to have been done so far to study double matrix summability of conjugate Fourier series. Therefore, the purpose of this paper is to establish the following theorem.

**Theorem 2.1.** Let  $(a_{m,j})_{j=0}^m$  and  $(b_{n,k})_{k=0}^n$  be two real nonnegative and nondecreasing sequences with  $j \leq m$  and  $k \leq n$ , respectively.

Let  $T = (a_{m,j})$  and  $S = (b_{n,k})$  be two infinite triangular matrices with

$$\begin{aligned} a_{m,j} &\geq 0, \quad a_{m,j} = 0, \quad j > m; \\ b_{n,k} &\geq 0, \quad b_{n,k} = 0, \quad k > n; \\ A_{m,\sigma} &= \sum_{j=0}^{\sigma} a_{m,j}; \quad B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,k}; \\ A_{m,m} &= 1 \quad \forall m \geq 0; \quad B_{n,n} = 1 \quad \forall n \geq 0. \end{aligned} \tag{2.1}$$

If the conditions

$$\begin{aligned} \Psi(x, y) &= \int_0^x \int_0^y |\psi(s, t)| ds dt \\ &= O\left\{ \frac{x}{\xi(1/x)} \cdot \frac{y}{\chi(1/y)} \right\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \int_0^\pi \psi_1(x, t) dt &= O\left\{ \frac{x}{\xi(1/x)} \right\}, \\ \int_0^\pi \psi_2(s, y) ds &= O\left\{ \frac{y}{\chi(1/y)} \right\} \end{aligned} \tag{2.3}$$

hold, then the double conjugate Fourier series (1.8) is double matrix  $(T, S)$  summable to,  $\bar{f}(x, y)$ , where

$$\bar{f}(x, y) = -\frac{1}{4\pi^2} \int_0^\pi \int_0^\pi \frac{1}{\tan(1/2)s \tan(1/2)t} ds dt \quad (2.4)$$

at every point where these integrals exist provided  $\xi(x)$  and  $\chi(y)$  are two positive monotonic increasing functions of  $x$  and  $y$  such that  $\xi(m) \rightarrow \infty$ , as  $m \rightarrow \infty$  and  $\chi(n) \rightarrow \infty$ , as  $n \rightarrow \infty$ ,

$$\begin{aligned} \int_1^m \frac{A_{m,s}}{s\xi(s)} ds &= O(1), \quad m \rightarrow \infty, \\ \int_1^n \frac{B_{n,t}}{t\chi(t)} dt &= O(1), \quad n \rightarrow \infty. \end{aligned} \quad (2.5)$$

### 3. Lemmas

**Lemma 3.1.** One has

$$\bar{K}_m(s) = O\left(\frac{1}{s}\right) \quad \text{for } 0 \leq s \leq \frac{1}{m}. \quad (3.1)$$

*Proof.* For  $0 \leq s \leq 1/m$ ,  $\sin(t/2) \geq t/\pi$  and  $|\cos mt| \leq 1$ ,

$$\begin{aligned} |\bar{K}_m(s)| &\leq \frac{1}{2} \left| \sum_{j=0}^m a_{m,j} \frac{\cos(j + (1/2))s}{\sin(s/2)} \right| \\ &\leq \frac{1}{2} \left\{ \sum_{j=0}^m a_{m,j} \frac{|\cos(j + (1/2))s|}{|\sin(s/2)|} \right\} \\ &\leq \frac{\pi}{s} \sum_{j=0}^m a_{m,j} \\ &= O\left(\frac{1}{s}\right). \end{aligned} \quad \square \quad (3.2)$$

**Lemma 3.2.** One has

$$\bar{K}_n(t) = O\left(\frac{1}{t}\right) \quad \text{for } 0 \leq t \leq \frac{1}{n}. \quad (3.3)$$

*Proof.* This can be proved similar to Lemma 3.1.  $\square$

**Lemma 3.3** (see [2]). If  $(a_{m,\mu})$  is non-negative and non-decreasing with  $\mu$ , then, for  $0 \leq a \leq b \leq \infty$ ,  $0 \leq s \leq \pi$  and any  $m$ ,

$$\left| \sum_{\mu=a}^b a_{m,m-\mu} e^{i(m-\mu)s} \right| = O(A_{m,\sigma}). \quad (3.4)$$

**Lemma 3.4.** If  $(b_{n,\nu})$  is non-negative and non-decreasing with  $\nu$ , then, for  $0 \leq a \leq b \leq \infty$ ,  $0 \leq t \leq \pi$  and any  $n$ ,

$$\left| \sum_{\nu=a}^b b_{n,n-\nu} e^{i(n-\nu)t} \right| = O(B_{n,\tau}). \quad (3.5)$$

*Proof.* This is similar to Lemma 3.3.  $\square$

**Lemma 3.5.** One has

$$\bar{K}_m(s) = O\left(\frac{A_{m,\sigma}}{s}\right) \quad \text{for } 0 < \frac{1}{m} < s \leq \pi. \quad (3.6)$$

*Proof.* Since  $1/m < s \leq \pi$ ,  $\sin(s/2) \geq s/\pi$ ,

$$\begin{aligned} |\bar{K}_m(s)| &\leq \frac{1}{2\pi} \left| \sum_{j=1}^m a_{m,j} \frac{\cos(j + (1/2))s}{\sin(s/2)} \right| \\ &\leq \frac{1}{2s} \left| \sum_{j=1}^m a_{m,m-j} \operatorname{Re} \left\{ e^{i(m-j+(1/2))s} \right\} \right| \\ &\leq \frac{1}{2s} \left| \sum_{j=1}^m a_{m,m-j} \operatorname{Re} \left| e^{i(m-j)s} \right| \right| \left| e^{i(s/2)} \right| \\ &= \frac{1}{2s} \left| \operatorname{Re} \sum_{j=1}^m a_{m,m-j} e^{i(m-j)s} \right| \\ &= O\left(\frac{1}{s}\right) \cdot O(A_{m,\sigma}) \quad \text{by Lemma 3.3} \\ &= O\left(\frac{A_{m,\sigma}}{s}\right). \end{aligned} \quad (3.7)$$

$\square$

**Lemma 3.6.** One has

$$\bar{K}_n(t) = O\left(\frac{B_{n,\tau}}{t}\right), \quad \text{for } 0 < \frac{1}{n} < t \leq \pi. \quad (3.8)$$

*Proof.* It can be proved similar to Lemma 3.5 but using Lemma 3.4.  $\square$

## 4. Proof of Theorem

The  $(j,k)$ th partial sums  $\bar{s}_{j,k}(x,y)$  of the series (1.8) is given by

$$\bar{s}_{j,k}(x,y) - \bar{f}(x,y) = \frac{1}{4\pi^2} \int_0^\pi \int_0^\pi \varphi(s,t) \frac{\cos(j + (1/2))s}{\sin(s/2)} \frac{\cos(k + (1/2))t}{\sin(t/2)} ds dt. \quad (4.1)$$

Then,

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^n a_{m,j} b_{n,k} \left\{ \bar{s}_{j,k}(x, y) - \bar{f}(x, y) \right\} \\ &= \frac{1}{4\pi^2} \int_0^\pi \int_0^\pi \phi(s, t) \sum_{j=0}^m \sum_{k=0}^n a_{m,j} b_{n,k} \frac{\cos(j + (1/2))s}{\sin(s/2)} \cdot \frac{\cos(k + (1/2))t}{\sin(t/2)} ds dt \end{aligned} \quad (4.2)$$

or

$$\begin{aligned} \bar{t}_{m,n}(x, y) - \bar{f}(x, y) &= \iint_0^\pi \psi(s, t) \bar{K}_m(s) \bar{K}_n(t) ds dt \\ &= \left( \int_0^\nu \int_0^\eta + \int_0^\nu \int_\eta^\pi + \int_\nu^\pi \int_0^\eta + \int_\nu^\pi \int_\eta^\pi \right) \cdot \psi(s, t) \bar{K}_m(s) \bar{K}_n(t) ds dt \\ &= I_1 + I_2 + I_3 + I_4 \quad (\text{say}). \end{aligned} \quad (4.3)$$

We consider

$$\begin{aligned} I_1 &= \left( \int_0^{1/m} \int_0^{1/n} + \int_{1/m}^\nu \int_0^{1/n} + \int_0^{1/m} \int_{1/n}^\eta + \int_{1/m}^\nu \int_{1/n}^\eta \right) \cdot \psi(s, t) \bar{K}_m(s) \bar{K}_n(t) ds dt \\ &= I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4} \quad (\text{say}), \end{aligned} \quad (4.4)$$

where  $s \leq \nu, t \leq \eta$  and (2.2) holds.

Now consider

$$\begin{aligned} I_{1.1} &= \int_0^{1/m} \int_0^{1/n} \psi(s, t) \bar{K}_m(s) \bar{K}_n(t) ds dt \\ &= O \left[ \int_0^{1/m} \int_0^{1/n} \left( \frac{1}{s} \right) \left( \frac{1}{t} \right) \psi(s, t) ds dt \right] \text{ by Lemma 3.1 and 3.2} \\ &= O(mn) \left[ \int_0^{1/m} \int_0^{1/n} \psi(s, t) ds dt \right] \\ &= O(mn) \psi \left( \frac{1}{m}, \frac{1}{n} \right) \\ &= O(mn) o \left\{ \frac{1}{mn\xi(m)\chi(n)} \right\} \text{ by (2.2)} \\ &= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \end{aligned} \quad (4.5)$$

Now,

$$\begin{aligned}
|I_{1.2}| &\leq \int_{1/m}^{\nu} \int_0^{1/n} O|\psi(s, t)| |\bar{K}_m(s)| |\bar{K}_n(t)| ds dt \\
&\leq \int_0^{1/n} |K_n(t)| dt \int_{1/m}^{\nu} |\psi(s, t)| \frac{A_{m,[1/s]}}{s} ds \quad \text{by Lemma 3.5} \\
&= \left\{ \int_0^{1/n} \left( \frac{1}{t} \right) dt \int_{1/m}^{\nu} |\psi(s, t)| \frac{A_{m,[1/s]}}{s} ds \right\} \quad \text{by Lemma 3.2} \\
&= O(n) \left[ \int_0^{1/n} \frac{A_{m,[1/\nu]}}{\nu} \psi_1(\nu, t) dt \right] + O(mn)(A_{m,m}) \left[ \int_0^{1/n} \psi_1\left(\frac{1}{m}, t\right) dt \right] \\
&\quad + O(n) \left[ \int_0^{1/n} dt \int_{1/m}^{\nu} \frac{d}{ds} \left( \frac{A_{m,[1/s]}}{s} \right) |\psi(s, t)| ds \right] \\
&= I_{1.2.1} + I_{1.2.2} + I_{1.2.3} \quad (\text{say}). \tag{4.6}
\end{aligned}$$

Then,

$$\begin{aligned}
I_{1.2.1} + I_{1.2.2} &= O(n) \left[ \int_0^{1/n} \psi_1(\nu, t) dt \right] + O(mn) \left[ \int_0^{1/n} \psi_1\left(\frac{1}{m}, t\right) dt \right] \\
&= O(n)\Psi\left(\nu, \frac{1}{n}\right) + O(mn)\Psi\left(\frac{1}{m}, \frac{1}{n}\right) \\
&= O(n)o\left(\frac{\nu}{\xi(1/\nu)} \cdot \frac{1/n}{\chi(n)}\right) + O(mn)o\left(\frac{1/mn}{\xi(m)\chi(n)}\right) \\
&= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.7}
\end{aligned}$$

Next,

$$\begin{aligned}
I_{1.2.3} &= O(n) \int_0^{1/n} dt \int_{1/m}^{\nu} \Psi_1(s, t) \frac{A_{m,[1/s]}}{s^2} ds + O(n) \int_0^{1/n} dt \int_{1/m}^{\nu} \frac{\Psi_1(s, t)}{s} \frac{d}{ds}(A_{m,[1/s]}) ds \\
&= O(n) \left[ \int_{1/m}^{\nu} \left( \int_0^{1/n} \Psi_1(s, t) dt \right) \frac{A_{m,[1/s]}}{s^2} ds \right] \\
&\quad + O(n) \left[ \int_{1/m}^{\nu} \left( \int_0^{1/n} \Psi_1(s, t) dt \right) \frac{1}{s} \frac{d}{ds}(A_{m,[1/s]}) ds \right] \\
&= O(n) \int_{1/m}^{\nu} \Psi_1\left(s, \frac{1}{n}\right) \frac{A_{m,[1/s]}}{s^2} ds + O(n) \int_{1/m}^{\nu} \Psi_1\left(s, \frac{1}{n}\right) \frac{1}{s} \frac{d}{ds}(A_{m,[1/s]}) ds \\
&= O(n) \int_{1/m}^{\nu} o\left(\frac{s}{\xi(1/s)} \cdot \frac{1}{n\chi(n)}\right) \frac{A_{m,[1/s]}}{s^2} ds \\
&\quad + O(n) \left[ \int_{1/m}^{\nu} o\left(\frac{s}{\xi(1/s)} \cdot \frac{1}{n\chi(n)}\right) \frac{1}{s} \frac{d}{ds}(A_{m,[1/s]}) ds \right]
\end{aligned}$$

$$\begin{aligned}
&= o\left[\frac{1}{\chi(n)}\right] \left[ \int_{1/\nu}^m \frac{A_{m,s}}{s\xi(s)} ds \right] + o\left[\frac{1}{\chi(n)}\right] \left[ \int_{1/\nu}^m \frac{1}{\xi(s)} \frac{d}{ds}(A_{m,s}) ds \right] \\
&= o\left[\frac{1}{\chi(n)}\right] o(1) + o\left[\frac{1}{\xi(m)\chi(n)}\right] [A_{m,n}] \\
&= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.8}$$

Thus,

$$I_{1.2} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.9}$$

Similarly

$$I_{1.3} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.10}$$

Now,

$$\begin{aligned}
|I_{1.4}| &= O\left[\int_{1/m}^\nu \int_{1/n}^\eta |\psi(s, t)| \frac{A_{m,[1/s]}}{s} \cdot \frac{B_{n,[1/t]}}{t} dt ds\right] \\
&= O\left[\frac{B_{n,[1/\eta]} A_{m,[1/\nu]} \Psi(\nu, \eta)}{\nu \eta} - \frac{m B_{n,[1/\eta]} A_{m,m} \Psi(1/m, \eta)}{\eta}\right] \\
&\quad - \frac{B_{n,[1/\eta]}}{\eta} \int_{1/m}^\nu \psi(s, \eta) \frac{d}{ds} \left( \frac{A_{m,[1/s]}}{s} \right) ds - \frac{n B_{n,n} A_{m,[1/\nu]} \Psi(\nu, 1/n)}{\nu} \\
&\quad + mn B_{n,n} A_{m,m} \Psi\left(\frac{1}{m}, \frac{1}{n}\right) + n B_{n,n} \int_{1/m}^\nu \Psi\left(s, \frac{1}{n}\right) \frac{d}{ds} \left( \frac{A_{m,[1/s]}}{s} \right) ds \\
&\quad + \frac{A_{m,[1/\nu]}}{\nu} \int_{1/m}^\nu \Psi_1(s, t) \frac{d}{dt} \left( \frac{B_{n,[1/t]}}{t} \right) dt - mA_{m,m} \int_{1/n}^\xi \Psi_2\left(\frac{1}{m}, t\right) \frac{d}{dt} \left( \frac{B_{n,[1/t]}}{t} \right) dt \\
&\quad - \left\{ \int_{1/m}^\nu \int_{1/n}^\eta \Psi(s, t) \frac{d}{ds} \left( \frac{A_{m,[1/s]}}{s} \right) \frac{d}{dt} \left( \frac{B_{n,[1/t]}}{t} \right) dt ds \right\} \\
&= I_{1.4.1} + I_{1.4.2} + I_{1.4.3} + I_{1.4.4} + I_{1.4.5} + I_{1.4.6} + I_{1.4.7} + I_{1.4.8} + I_{1.4.9} \quad (\text{say}). \tag{4.11}
\end{aligned}$$

Then,

$$\begin{aligned}
I_{1.4.1} + I_{1.4.2} &= O\left[\frac{B_{n,[1/\eta]} A_{m,[1/\nu]} \Psi(\nu, \eta)}{\nu \eta} + \frac{m B_{n,[1/\eta]} A_{m,m} \Psi(1/m, \eta)}{\eta}\right] \\
&= o\left[\frac{B_{n,[1/\eta]} A_{m,[1/\nu]}}{\alpha(1/\nu)\beta(1/\eta)}\right] + o\left[m B_{n,[1/\eta]} \left\{ \frac{1}{m\xi(m)} \cdot \frac{\eta}{\chi(1/\eta)} \right\}\right] \\
&= o\left[\frac{B_{n,[1/\eta]} A_{m,[1/\nu]}}{\xi(1/\nu)\chi(1/\eta)}\right] + o\left[B_{n,[1/\eta]} \frac{1}{\xi(m)} \cdot \frac{\eta}{\chi(1/\eta)}\right]
\end{aligned}$$

$$\begin{aligned}
&= o(1) + o(1) \left[ \frac{1}{\xi(m)} \right] \\
&= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty \text{ by the regularity of } (T, S), \\
I_{1.4.3} &= -\frac{B_{n,[1/\eta]}}{\eta} \int_{1/m}^v \Psi(s, \eta) \frac{d}{ds} \left( \frac{A_{m,[1/s]}}{s} \right) ds \\
&= o\left(\frac{B_{n,[1/\eta]}}{\eta}\right) \int_{1/m}^v \left( \frac{s}{\xi(1/s)} \cdot \frac{\eta}{\chi(1/\eta)} \right) \left( \frac{A_{m,[1/s]}}{s^2} \right) ds \\
&\quad + o\left(\frac{B_{n,[1/\eta]}}{\eta}\right) \int_{1/m}^v o\left(\frac{s}{\xi(1/s)} \cdot \frac{\eta}{\chi(1/\eta)}\right) \frac{1}{s} \frac{d}{ds} (A_{m,[1/s]}) ds \\
&= o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right) \int_{1/m}^v \frac{A_{m,[1/s]}}{s\xi(1/s)} ds + o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right) \int_{1/m}^v \frac{1}{\xi(1/s)} \frac{d}{ds} (A_{m,[1/s]}) ds \\
&= o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right) \int_{1/v}^m \frac{A_{m,s}}{s\xi(s)} ds + o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right) \int_{1/v}^m \frac{1}{\xi(s)} d(A_{m,s}) \\
&= o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right) O(1) + o\left(\frac{B_{n,[1/\eta]}}{\chi(1/\eta)}\right) O(A_{m,m}) \\
&= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.12}$$

Thus, we get

$$I_{1.4.3} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.13}$$

Similarly

$$I_{1.4.4} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.14}$$

Now,

$$\begin{aligned}
I_{1.4.5} &= mnB_{n,n}A_{m,m}\Psi\left(\frac{1}{m}, \frac{1}{n}\right) \\
&= o\left(mnB_{n,n}A_{m,m}\frac{1}{mn\xi(m)\chi(n)}\right) \\
&= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.15}$$

Consider

$$\begin{aligned}
I_{1.4.6} &= nB_{n,n} \int_{1/m}^v \Psi\left(s, \frac{1}{n}\right) \frac{d}{ds} \left( \frac{A_{m,[1/s]}}{s} \right) ds \\
&= nB_{n,n} \int_{1/m}^v \Psi\left(s, \frac{1}{n}\right) \frac{A_{m,[1/s]}}{s^2} ds + nB_{n,n} \int_{1/m}^v \Psi\left(s, \frac{1}{n}\right) \frac{1}{s} \frac{d}{ds} (A_{m,[1/s]}) ds \\
&= nB_{n,n} \int_{1/m}^v o\left(\frac{s}{\xi(1/s)} \cdot \frac{1}{n\chi(\eta)}\right) \frac{A_{m,[1/s]}}{s^2} ds \\
&\quad + nB_{n,n} \int_{1/m}^v \left( \frac{s}{\xi(1/s)n\chi(\eta)} \right) \frac{1}{s} \frac{d}{ds} (A_{m,[1/s]}) ds \\
&= o\left(\frac{nB_{n,n}}{n\chi(n)}\right) \int_{1/m}^v \frac{A_{m,[1/s]}}{s\xi(1/s)} ds + o\left(\frac{1}{\chi(n)}\right) \int_{1/m}^v \frac{1}{\xi(1/s)} \frac{d}{ds} (A_{m,[1/s]}) ds \\
&= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.16}$$

As similar to  $I_{1.4.3}$ ,

$$I_{1.4.7} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.17}$$

As similar to  $I_{1.4.6}$ ,

$$I_{1.4.8} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.18}$$

Now,

$$\begin{aligned}
I_{1.4.9} &= O\left[\left\{\int_{1/m}^v \int_{1/n}^\eta \Psi(s, t) \left( \frac{A_{m,[1/s]}}{s^2} \right) + \frac{1}{s} \frac{d}{ds} (A_{m,[1/s]})\right\} \cdot \left\{ \frac{B_{n,[1/t]}}{t^2} + \frac{1}{t} \frac{d}{dt} (B_{n,[1/t]}) \right\} dt ds\right] \\
&= I_{1.4.9.1} + I_{1.4.9.2} + I_{1.4.9.3} + I_{1.4.9.4} \quad (\text{say}).
\end{aligned} \tag{4.19}$$

Then,

$$\begin{aligned}
I_{1.4.9.1} + I_{1.4.9.2} &= O\left[\int_{1/m}^v \int_{1/n}^\eta \Psi(s, t) \left\{ \frac{A_{m,[1/s]}}{s^2} \cdot \frac{B_{n,[1/t]}}{t^2} \right\} dt ds\right] \\
&\quad + O\left[\int_{1/m}^v \int_{1/n}^\eta \Psi(s, t) \left\{ \frac{A_{m,[1/s]}}{s^2} \frac{1}{t} \cdot \frac{d}{dt} (B_{n,[1/t]}) \right\} dt ds\right] \\
&= O\left[\int_{1/m}^v \int_{1/n}^\eta o\left\{ \frac{s}{\xi(1/s)} \cdot \frac{t}{\chi(1/t)} \right\} \frac{A_{m,[1/s]}}{s^2} \cdot \frac{B_{n,[1/t]}}{t^2} dt ds\right]
\end{aligned}$$

$$\begin{aligned}
& + O \left[ \int_{1/m}^{\nu} \int_{1/n}^{\eta} o \left\{ \frac{s}{\xi(1/s)} \cdot \frac{t}{\chi(1/t)} \right\} \frac{A_{m,[1/s]}}{s^2} \frac{1}{t} \cdot \frac{d}{dt} (B_{n,[1/t]}) dt ds \right] \\
& = o \left[ \int_{1/m}^{\nu} \int_{1/n}^{\eta} \frac{A_{m,[1/s]} B_{n,[1/t]}}{s \xi(1/s) t \chi(1/t)} dt ds \right] \\
& \quad + o \left[ \int_{1/m}^{\nu} \int_{1/n}^{\eta} \frac{A_{m,[1/s]}}{s \xi(1/s) \chi(1/t)} \frac{d}{dt} (B_{n,[1/t]}) dt ds \right] \\
& = o \left[ \int_{1/\nu}^m \frac{A_{m,s}}{s \xi(s)} ds \int_{1/\eta}^n \frac{B_{n,t}}{t \chi(t)} dt \right] \\
& \quad + o \left[ \int_{1/\nu}^m \frac{A_{m,s}}{s \xi(s)} ds \int_{1/\eta}^n \frac{1}{\chi(t)} \frac{d}{dt} (B_{n,[1/t]}) dt \right] \\
& = o(1) + o(1) \int_{1/\eta}^n O(1) \frac{d}{dt} (B_{n,[1/t]}) dt \\
& = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.20}$$

Similarly,

$$I_{1.4.9.3} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.21}$$

Now,

$$\begin{aligned}
I_{1.4.9.4} & = O \left[ \int_{1/m}^{\nu} \int_{1/n}^{\eta} \Psi(s,t) \frac{1}{s} \frac{d}{ds} (A_{m,[1/s]}) \frac{1}{t} \frac{d}{dt} (B_{n,[1/t]}) dt ds \right] \\
& = o \left[ \int_{1/m}^{\nu} \int_{1/n}^{\eta} \left( \frac{s}{\xi(1/s)} \cdot \frac{t}{\chi(1/t)} \right) \frac{1}{s} \frac{d}{ds} (A_{m,[1/s]}) \frac{1}{t} \cdot \frac{d}{dt} (B_{n,[1/t]}) dt ds \right] \\
& = o \left[ \int_{1/m}^{\nu} \int_{1/n}^{\eta} \left( \frac{1}{\xi(1/s) \cdot \chi(1/t)} \right) \frac{d}{ds} (A_{m,[1/s]}) \frac{d}{dt} (B_{n,[1/t]}) dt ds \right] \\
& = o \left[ \int_{1/m}^{\nu} \frac{1}{\xi(1/s)} \frac{d}{ds} (A_{m,[1/s]}) ds \int_{1/n}^{\eta} \frac{1}{\chi(1/t)} \frac{d}{dt} (B_{n,[1/t]}) dt \right] \\
& = o \left[ \int_{1/m}^{\nu} O(1) \frac{d}{ds} (A_{m,[1/s]}) ds \int_{1/n}^{\eta} O(1) \frac{d}{dt} (B_{n,[1/t]}) dt \right] \\
& = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.22}$$

Hence,

$$I_{1.4} = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.23}$$

Therefore,

$$I_1 = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \quad (4.24)$$

Now,  $1/m < \nu < \pi$ ,  $1/n < \eta < \pi$ , thus we obtain

$$\begin{aligned} |I_3| &\leq \int_{\nu}^{\pi} |K_m(s)| ds \int_0^{1/n} |\psi(s, t)| |K_n(t)| dt + \int_{\nu}^{\pi} |K_m(s)| ds \int_{1/n}^{\nu} |\psi(s, t)| |K_n(t)| dt \\ &= I_{3.1} + I_{3.2} \quad (\text{say}). \end{aligned} \quad (4.25)$$

Using Lemma 3.2 and Lemma 3.5, we have

$$\begin{aligned} I_{3.1} &= \int_{\nu}^{\pi} \frac{A_{m,\sigma}}{s} ds \int_0^{1/n} |\psi(s, t)| \frac{1}{t} dt \\ &= O(n) \int_{\nu}^{\pi} \frac{A_{m,\sigma}}{s} ds \int_0^{1/n} |\psi(s, t)| dt \\ &= O(n) \int_0^{\pi} \Psi_2\left(s, \frac{1}{n}\right) ds \\ &= O\left(\frac{1}{\chi(n)}\right) \\ &= o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \end{aligned} \quad (4.26)$$

Using Lemma 3.5 and Lemma 3.6,

$$\begin{aligned} I_{3.2} &= O\left[\int_{\nu}^{\pi} \frac{A_{m,\sigma}}{s} ds \int_{1/n}^{\eta} |\psi(s, t)| \frac{B_{n,[\tau]}}{t} dt\right] \\ &= O\left[\int_{\nu}^{\pi} ds \left\{ \left(\Psi_2(s, t) \frac{B_{n,[\tau]}}{t}\right)_{1/n}^{\eta} - \int_{1/n}^{\eta} \Psi_2(s, t) \frac{d}{dt} \left(\frac{B_{n,[\tau]}}{t}\right) dt \right\} \right] \\ &= O\left[\int_{\nu}^{\pi} ds \left\{ \Psi_2(s, \eta) \frac{B_{n,[1/n]}}{\eta} - \Psi_2\left(s, \frac{1}{n}\right) n B_{n,n} \right\} \right] \end{aligned}$$

$$\begin{aligned}
& + \left[ \int_v^\pi ds \int_{1/n}^n \Psi_2(s, t) \frac{d}{dt} \left( \frac{B_{n,[\tau]}}{t} \right) dt \right] \\
& = O \left[ \int_v^\pi \Psi_2(s, \eta) \frac{B_{n,[1/n]}}{\eta} ds \right] + O(n) \left[ \int_0^\pi \Psi_2 \left( s, \frac{1}{n} \right) ds \right] \\
& \quad + O \left[ \int_v^\pi ds \int_{1/n}^n \Psi_2(s, t) \frac{d}{dt} \left( \frac{B_{n,[\tau]}}{t} \right) dt \right] \\
& = o(1) + o(1) + o(1) (\text{similar to } I_{1.4.9}) \\
& = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty.
\end{aligned} \tag{4.27}$$

Hence,

$$I_3 = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.28}$$

Similarly,

$$I_2 = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.29}$$

By the regularity conditions of matrix method and Riemann-Lebesgue theorem,

$$I_4 = o(1), \quad \text{as } m \rightarrow \infty, n \rightarrow \infty. \tag{4.30}$$

Therefore,

$$\bar{t}_{m,n}(x, y) - \bar{f}(x, y) = o(1) \tag{4.31}$$

This completes the proof of the theorem.

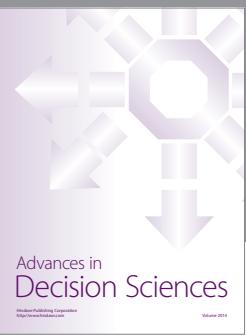
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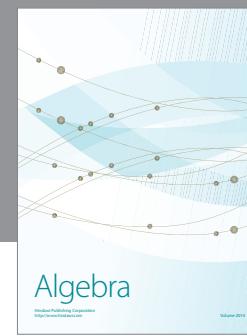
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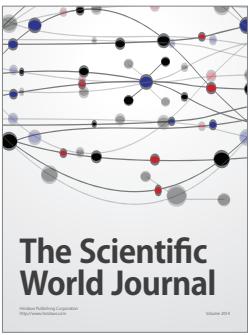
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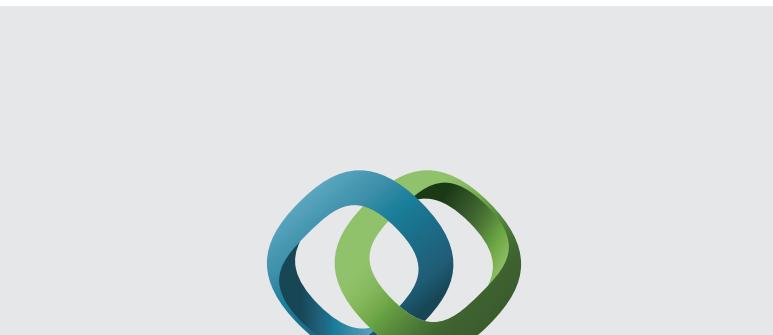
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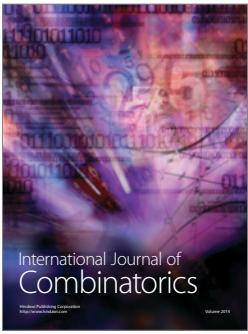


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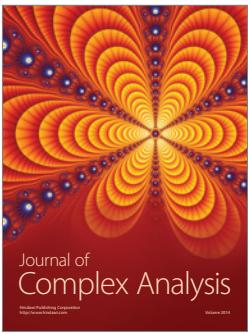
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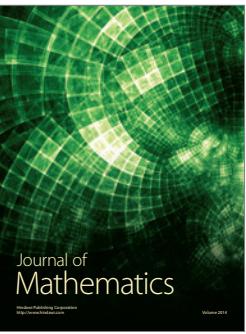
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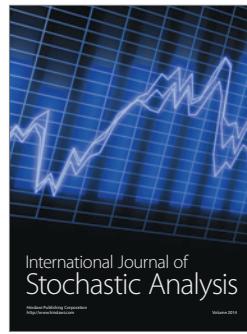
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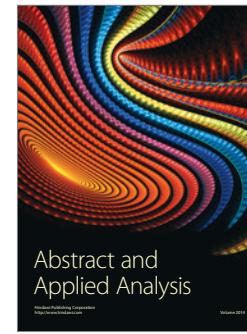
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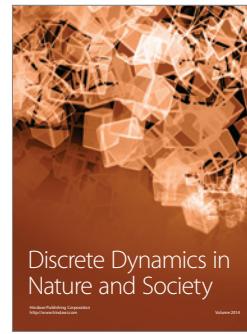
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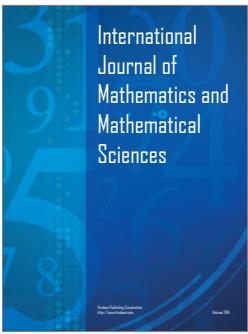
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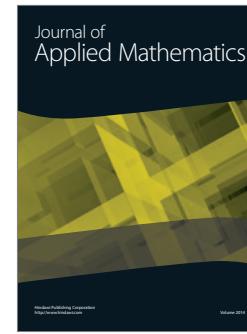
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