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Research Article

Decomposition of Automorphisms of Certain Solvable Subalgebra of Symplectic Lie Algebra over Commutative Rings

Xing Tao Wang and Lei Zhang

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Correspondence should be addressed to Xing Tao Wang, xingtao@hope.hit.edu.cn

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Let $C_{l+1}(R)$ be the $2(l+1)\times 2(l+1)$ matrix symplectic Lie algebra over a commutative ring R with 2 invertible. Then $\mathbf{t}_{l+1}^{(C)}(R)=\{\left(\begin{smallmatrix}\overline{m}_1 & \overline{m}_2\\ 0 & -\overline{m}_1^T\end{smallmatrix}\right) \mid \overline{m}_1$ is an $l+1$ upper triangular matrix, $\overline{m}_2^T=\overline{m}_2$, over $R\}$ is the solvable subalgebra of $C_{l+1}(R)$. In this paper, we give an explicit description of the automorphism group of $\mathbf{t}_{l+1}^{(C)}(R)$.

1. Introduction

Classical Lie algebras occupy an important place in matrix algebras. Let R be a commutative ring R with the identity 1 and R^* the group of invertible elements in R. Let $M_n(R)$ be the R-algebra of n by n matrices over R that has a structure of a Lie algebra over R with bracket operation $[x,y]=xy-yx$ for any $x,y\in M_n(R)$. The symplectic Lie algebra

$$C_{l+1}(R)=\left\{X \mid X\begin{pmatrix}0 & I_l\\ -I_l & 0\end{pmatrix}+\begin{pmatrix}0 & I_l\\ -I_l & 0\end{pmatrix}X^T=0,\ X\in M_{2l+2}(R)\right\} \tag{1.1}$$

is one of classical Lie algebras, where T denotes the matrix transpose. It is easy to show that the following subalgebra of $C_{l+1}(R)$ such that

$$\mathbf{t}_{l+1}^{(C)}(R)=\left\{\begin{pmatrix}\overline{m}_1 & \overline{m}_2\\ 0 & -\overline{m}_1^T\end{pmatrix} \mid \overline{m}_1 \text{ is an } l+1 \text{ upper triangular matrix}, \overline{m}_2^T=\overline{m}_2\right\} \tag{1.2}$$

is solvable.

Let e be the identity matrix in $M_n(R)$ and let $e_{ij}^{(n)}$ denotes the matrix in $M_n(R)$ all of whose entries are 0, except the (i,j)th entry which is 1. Let

$$\begin{aligned} \alpha_{i,i+k} &= e_{i,i+k}^{(n)} - e_{l+i+k+1,l+i+1}^{(n)}, \quad i = 1,\dots,l-k+1,\ k = 0,1,\dots,l, \\ \gamma_{i,i+k} &= e_{i,l+i+k+1}^{(n)} + e_{i+k,l+i+1}^{(n)}, \quad i = 1,\dots,l-k+1,\ k = 1,\dots,l, \\ \gamma_{ii} &= e_{i,l+i+1}^{(n)}, \quad i = 1,\dots,l+1, \end{aligned} \tag{1.3}$$

where $n = 2(l+1)$, $l \geq 1$. For discussion latter, we rewrite $\mathbf{t}_{l+1}^{(C)}(R)$ as

$$\mathbf{t}_{l+1}^{(C)}(R) = \sum_{k=0}^{l} \sum_{i=1}^{l-k+1} R\alpha_{i,i+k} + \sum_{k=0}^{l} \sum_{i=1}^{l-k+1} R\gamma_{i,i+k}. \tag{1.4}$$

Automorphisms of associative algebras have been explored in many articles [1–8]. Encouraged by Doković [9] and Cao's [10] papers which described the automorphism groups of Lie algebra consisting of all upper triangular $n \times n$ matrices of trace 0 over a connected commutative ring and a commutative ring with n invertible, respectively, in this paper we use similar techniques to those in [11] to prove that any automorphism φ of $\mathbf{t}_{l+1}^{(C)}(R)$ can be uniquely expressed as $\varphi = \theta\lambda_D$, where θ and λ_D are inner and diagonal automorphisms, respectively, for $l \geq 1$ and R is a commutative ring with 2 invertible. We also give an explicit description of the remaining case $l = 0$.

Theorem 1.1. *For any automorphism φ of $\mathbf{t}_{l+1}^{(C)}(R)$ $(l \geq 1)$ there are unique inner and diagonal automorphisms, θ and λ_D, respectively, of $\mathbf{t}_{l+1}^{(C)}(R)$ such that $\varphi = \theta\lambda_D$.*

Theorem 1.2. *Let $\mathcal{I}$ and $\mathfrak{D}$ be the inner and diagonal automorphism groups, respectively. Then $\mathrm{Aut}(\mathbf{t}_{l+1}^{(C)}(R)) = \mathcal{I} \ltimes \mathfrak{D}$, where $\mathrm{Aut}(\mathbf{t}_{l+1}^{(C)}(R))$ denotes the automorphism group of $\mathbf{t}_{l+1}^{(C)}(R)$.*

2. Preliminaries

Let

$$\begin{aligned} P_n &= \{\alpha_{i,i+k} \mid i = 1,\dots,l-k+1, k = 0,1,\dots,l\}, \\ W_n &= \{\gamma_{i,i+k} \mid i = 1,\dots,l-k+1, k = 0,1,\dots,l\}. \end{aligned} \tag{2.1}$$

Then the set $P_n \cup W_n$ is a basis of $\mathbf{t}_{l+1}^{(C)}(R)$.

Lemma 2.1. *Let H_n be the set generated by the set $\{\alpha_{jj}, \alpha_{i,i+1}, \gamma_{l+1,l+1} \mid 1 \leq j \leq l+1, 1 \leq i \leq l\}$, where $n = 2(l+1)$. Then $H_n = \mathbf{t}_{l+1}^{(C)}(R)$.*

Proof. We only need to show that $\mathbf{t}_{l+1}^{(C)}(R) \subseteq H_n$. It is obvious that $\alpha_{i,i+k} \in H_n$, when $k = 0, 1$. When $k = 2$, we have

$$\begin{aligned}\alpha_{i,i+2} &= e_{i,i+2}^{(n)} - e_{l+i+3,l+i+1}^{(n)} \\ &= \left[\left(e_{i,i+1}^{(n)} - e_{l+i+2,l+i+1}^{(n)}\right), \left(e_{i+1,i+2}^{(n)} - e_{l+i+3,l+i+2}^{(n)}\right)\right] \\ &= \left[\alpha_{i,i+1}, \alpha_{i+1,i+2}\right] \in H_n.\end{aligned} \tag{2.2}$$

Assume that $\alpha_{i,i+k-1} \in H_n$, then $\alpha_{i,i+k} = [\alpha_{i,i+k-1}, \alpha_{i+k-1,i+k}] \in H_n$, that is, $P_n \subseteq H_n$. Since $\gamma_{l+1,l+1} \in H_n$, for any $\gamma_{l-k+1,l-k+2} \in W_n$, when $k = 1$,

$$\begin{aligned}\gamma_{l,l+1} &= e_{l,2l+2}^{(n)} + e_{l+1,2l+1}^{(n)} \\ &= \left[\left(e_{l,l+1}^{(n)} - e_{2l+2,2l+1}^{(n)}\right), e_{l+1,2l+2}^{(n)}\right] \\ &= \left[\alpha_{l,l+1}, \gamma_{l+1,l+1}\right] \in H_n.\end{aligned} \tag{2.3}$$

Assume that when $k = m-1$, $\gamma_{l-m+2,l-m+3} \in H_n$, then when $k = m$, $\gamma_{l-m+1,l-m+2} = [\alpha_{l-m+1,l-m+3}, \gamma_{l-m+2,l-m+3}] \in H_n$, that is, $\gamma_{i,i+1} \in H_n$, $i = 1, \ldots, l$. For any $\gamma_{i,i+k} \in H_n$ $(k \geq 1)$, when $k = 1$, $\gamma_{i,i+1} \in H_n$. When $k \geq 2$, $\gamma_{i,i+k} = [\alpha_{i,i+k-1}, \gamma_{i+k-1,i+k}] \in H_n$. Since $2\gamma_{ii} = [\alpha_{i,i+1}, \gamma_{i,i+1}] \in H_n$, and 2 is invertible, we have $\gamma_{ii} \in H_n$ for $1 \leq i \leq l$. Thus $W_n \subseteq H_n$. Because $P_n \cup W_n$ is a basis of $\mathbf{t}_{l+1}^{(C)}(R)$, we obtain $\mathbf{t}_{l+1}^{(C)}(R) \subseteq H_n$. □

Now, denote $\mathbf{t}_{l+1}^{(C)}(R)$ by $\mathbf{n}_0^{(C)}$. Let $\mathbf{n}_1^{(C)} = [\mathbf{n}_0^{(C)}, \mathbf{n}_0^{(C)}]$, $\mathbf{n}_2^{(C)} = [\mathbf{n}_1^{(C)}, \mathbf{n}_1^{(C)}]$, $\mathbf{n}_j^{(C)} = [\mathbf{n}_1^{(C)}, \mathbf{n}_{j-1}^{(C)}]$, $j = 3, \ldots, 2l+1$. It is not difficult to know

$$\begin{aligned}\mathbf{n}_j^{(C)} = &\sum_{k=j}^{l}\sum_{i=1}^{l-k+1} R\alpha_{i,i+k} + \sum_{k=j}^{l}\sum_{i=l+2-[(k+1)\ /2]}^{l+1} R\gamma_{2l-k-i+3,i} \\ &+ \sum_{k=l+1}^{2l+1}\sum_{i=1}^{l+1-[k/2]} R\gamma_{i,2l-k-i+3}, \quad 1 \leq j \leq l, \\ \mathbf{n}_j^{(C)} = &\sum_{k=j}^{2l+1}\sum_{i=1}^{l+1-[k/2]} R\gamma_{i,2l-k-i+3}, \quad l+1 \leq j \leq 2l+1 \ (l \geq 2), \\ \mathbf{n}_j^{(C)} = &\ 0, \quad 2l+2 \leq j.\end{aligned} \tag{2.4}$$

It is easy to check that $[\mathbf{n}_m^{(C)}, \mathbf{n}_l^{(C)}] \subseteq \mathbf{n}_{m+l}^{(C)}$ for $m + l \leq 2l + 1$ or $[\mathbf{n}_m^{(C)}, \mathbf{n}_l^{(C)}] = 0$ for $m + l \geq 2l + 2$. For any $\varphi \in \operatorname{Aut}(\mathbf{n}_0^{(C)})$, we have $\varphi(\mathbf{n}_1^{(C)}) = [\varphi(\mathbf{n}_0^{(C)}), \varphi(\mathbf{n}_0^{(C)})] = [\mathbf{n}_0^{(C)}, \mathbf{n}_0^{(C)}] = \mathbf{n}_1^{(C)}$ and $\varphi(\mathbf{n}_j^{(C)}) = \mathbf{n}_j^{(C)}$, $j = 2, \ldots, 2l+1$. Therefore, $\varphi(\mathbf{n}_{j-1}^{(C)} \setminus \mathbf{n}_j^{(C)}) = \mathbf{n}_{j-1}^{(C)} \setminus \mathbf{n}_j^{(C)}$, $j = 1, \ldots, 2l+1$. Note that if $\gamma_{ij} \in W_n$, then $\gamma_{ij} \in \mathbf{n}_{2l-i-j+3}^{(C)} \setminus \mathbf{n}_{2l-i-j+4}^{(C)}$.

For any maximal ideal M of R, $\overline{R} = R/M$ is a field. The natural homomorphism $\pi : R \to \overline{R}$ induces a homomorphism $\varphi_M : \mathbf{t}_{l+1}^{(C)}(R) \to \mathbf{t}_{l+1}^{(C)}(\overline{R})$ which is surjective. So every automorphism ψ of $\mathbf{t}_{l+1}^{(C)}(R)$ may induce an automorphism $\overline{\psi}$ of $\mathbf{t}_{l+1}^{(C)}(\overline{R})$. Using this fact and that $\mathbf{n}_{2l+1}^{(C)} = R\gamma_{11}$ (for $l \geq 1$), we have that $\psi(\gamma_{11}) = c_{11}\gamma_{11}$, where $c_{11} \in R^*$. Otherwise, c_{11} should be contained in a maximal ideal M of R, then $\overline{\psi}(\overline{\gamma}_{11}) = 0$ on $\mathbf{t}_{l+1}^{(C)}(\overline{R})$, where $\overline{\gamma}_{11}$ is the image of γ_{11} in $\mathbf{t}_{l+1}^{(C)}(\overline{R})$, which is impossible.

Lemma 2.2. *Let ψ be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If $\psi(\alpha_{jj})$, $\psi(\alpha_{j,j+1})$ and $\psi(\gamma_{l+1,l+1})$ are expressed, respectively, as*

$$\psi(\alpha_{jj}) = \sum_{i=1}^{l+1} a_{ii}^{(j)}\alpha_{ii} \bmod \mathbf{n}_1^{(C)}, \quad j = 1, \ldots, l+1, \tag{2.5}$$

$$\psi(\alpha_{j,j+1}) = \sum_{i=1}^{l} \widetilde{a}_{i,i+1}^{(j)}\alpha_{i,i+1} + \widetilde{c}_{l+1,l+1}^{(j)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)}, \quad j = 1, \ldots, l, \tag{2.6}$$

$$\psi(\gamma_{l+1,l+1}) = \sum_{i=1}^{l} \widehat{a}_{i,i+1}^{(l+1)}\alpha_{i,i+1} + \widehat{c}_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)}, \tag{2.7}$$

then the following matrices are invertible.

(i) $A = (a_{ji})_{(l+1)\times(l+1)}$, *where* $a_{ji} = a_{ii}^{(j)}$, $j = 1, \ldots, l+1$, $i = 1, \ldots, l+1$;

(ii) $B = (b_{ji})_{(l+1)\times(l+1)}$, *where* $b_{ji} = \widetilde{a}_{i,i+1}^{(j)}$, $j = 1, \ldots, l$, $i = 1, \ldots, l$, $b_{j,l+1} = \widetilde{c}_{l+1,l+1}^{(j)}$, $j = 1, \ldots, l$, $b_{l+1,i} = \widehat{a}_{i,i+1}^{(l+1)}$, $i = 1, \ldots, l$ *and* $b_{l+1,l+1} = \widehat{c}_{l+1,l+1}^{(l+1)}$.

Proof. (i) That A is invertible follows from the fact that ψ induces an automorphism of the free R-module $\mathbf{n}_0^{(C)}/\mathbf{n}_1^{(C)}$ of rank $l+1$ on the basis $\{\alpha_{jj} + \mathbf{n}_1^{(C)} \mid j = 1, \ldots, l+1\}$. (ii) Note that ψ induces an automorphism of the free R-module $\mathbf{n}_1^{(C)}/\mathbf{n}_2^{(C)}$ of rank $l+1$ on the basis $\{\alpha_{j,j+1} + \mathbf{n}_2^{(C)}, \gamma_{l+1,l+1} + \mathbf{n}_2^{(C)} \mid j = 1, \ldots, l\}$. □

Lemma 2.3. *Let $\psi \in$* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$ *($l \geq 2$). Write $\psi(\alpha_{jj})$, $\psi(\alpha_{j,j+1})$, and $\psi(\gamma_{l+1,l+1})$ as in* (2.5)–(2.7), *respectively. Then the following conclusions hold.*

(i) *For $1 \leq m, k, h \leq l$, $\widetilde{a}_{h,h+1}^{(m)}\widetilde{a}_{k,k+1}^{(m)} = 0$ $(h \neq k)$, $\widetilde{a}_{h,h+1}^{(m)}\widetilde{c}_{l+1,l+1}^{(m)} = 0$, $\widehat{a}_{h,h+1}^{(l+1)}\widehat{a}_{k,k+1}^{(l+1)} = 0$ $(h \neq k)$ and $\widehat{a}_{h,h+1}^{(l+1)}\widehat{c}_{l+1,l+1}^{(l+1)} = 0$.*

(ii) *For $1 \leq k, h \leq l$, $(a_{hh}^{(i)} - a_{h+1,h+1}^{(i)})(a_{kk}^{(i)} - a_{k+1,k+1}^{(i)}) = 0$ $(h \neq k)$ and $(a_{hh}^{(i)} - a_{h+1,h+1}^{(i)})a_{l+1,l+1}^{(i)} = 0$ $(1 \leq h \leq l$, here $l \geq 1)$, where $i = 1, l+1$.*

(iii) *For $2 \leq m \leq l$ and $1 \leq i, k, h \leq l$, $(a_{ii}^{(m)} - a_{i+1,i+1}^{(m)})(a_{hh}^{(m)} - a_{h+1,h+1}^{(m)})(a_{kk}^{(m)} - a_{k+1,k+1}^{(m)}) = 0$ $(i \neq h \neq k \neq i$, here $l \geq 3)$ and $(a_{ii}^{(m)} - a_{i+1,i+1}^{(m)})(a_{hh}^{(m)} - a_{h+1,h+1}^{(m)})a_{l+1,l+1}^{(m)} = 0$ $(1 \leq i \neq h \leq l)$.*

Proof. (i) When $j \neq m, m+1$, $[\psi(\alpha_{jj}), \psi(\alpha_{m,m+1})] = 0$. So

$$\widetilde{a}_{i,i+1}^{(m)}\left(a_{ii}^{(j)} - a_{i+1,i+1}^{(j)}\right) = 0, \qquad \widetilde{c}_{l+1,l+1}^{(m)}a_{l+1,l+1}^{(j)} = 0. \tag{2.8}$$

From $[\psi(\alpha_{mm}),\psi(\alpha_{m,m+1})]=\psi(\alpha_{m,m+1})$ and $[\psi(\alpha_{m+1,m+1}),\psi(\alpha_{m,m+1})]=-\psi(\alpha_{m,m+1})$, we have

$$\begin{aligned}\widetilde{a}^{(m)}_{i,i+1}\left(a^{(m)}_{ii}-a^{(m)}_{i+1,i+1}\right)&=\widetilde{a}^{(m)}_{i,i+1}, \qquad 2\widetilde{c}^{(m)}_{l+1,l+1}a^{(m)}_{l+1,l+1}=\widetilde{c}^{(m)}_{l+1,l+1},\\ \widetilde{a}^{(m)}_{i,i+1}\left(a^{(m+1)}_{ii}-a^{(m+1)}_{i+1,i+1}\right)&=-\widetilde{a}^{(m)}_{i,i+1}, \qquad 2\widetilde{c}^{(m)}_{l+1,l+1}a^{(m+1)}_{l+1,l+1}=-\widetilde{c}^{(m)}_{l+1,l+1}.\end{aligned} \tag{2.9}$$

When $j\neq l+1$, $[\psi(\alpha_{jj}),\psi(\gamma_{l+1,l+1})]=0$. So

$$\widehat{a}^{(l+1)}_{i,i+1}\left(a^{(j)}_{ii}-a^{(j)}_{i+1,i+1}\right)=0, \qquad \widehat{c}^{(l+1)}_{l+1,l+1}a^{(j)}_{l+1,l+1}=0. \tag{2.10}$$

From $[\psi(\alpha_{l+1,l+1}),\psi(\gamma_{l+1,l+1})]=2\psi(\gamma_{l+1,l+1})$, we have

$$\widehat{a}^{(l+1)}_{i,i+1}\left(a^{(l+1)}_{ii}-a^{(l+1)}_{i+1,i+1}\right)=2\widehat{a}^{(l+1)}_{i,i+1}, \qquad \widehat{c}^{(l+1)}_{l+1,l+1}a^{(l+1)}_{l+1,l+1}=\widehat{c}^{(l+1)}_{l+1,l+1}. \tag{2.11}$$

Let $C=(c_{ji})_{(l+1)\times(l+1)}$, where $c_{ji}=a^{(j)}_{ii}-a^{(j)}_{i+1,i+1}$, $j=1,\dots,l+1$, $i=1,\dots,l$, and $c_{j,l+1}=2a^{(j)}_{l+1,l+1}$, $j=1,\dots,l+1$. By Lemma 2.2, $\det A\in R^*$, so $\det C\in R^*$. Investigating $\widetilde{a}^{(m)}_{h,h+1}\widetilde{a}^{(m)}_{k,k+1}\det C$, we may find that hth column and kth column are linearly dependent (both are the form $(0,\dots,0,\widetilde{a}^{(m)}_{j,j+1},-\widetilde{a}^{(m)}_{j,j+1},0,\dots,0)^t$, $j=h,k$) by (2.6) and (2.7), so $\widetilde{a}^{(m)}_{h,h+1}\widetilde{a}^{(m)}_{k,k+1}\det C=0$. Similarly, $\widetilde{a}^{(m)}_{h,h+1}\widetilde{c}^{(m)}_{l+1,l+1}\det C=0$, $\widehat{a}^{(l+1)}_{h,h+1}\widehat{a}^{(l+1)}_{k,k+1}\det C=0$ $(h\neq k)$ and $\widehat{a}^{(l+1)}_{h,h+1}\widehat{c}^{(l+1)}_{l+1,l+1}\det C=0$. Then $\widetilde{a}^{(m)}_{h,h+1}\widetilde{a}^{(m)}_{k,k+1}=0$ $(h\neq k)$, $\widetilde{a}^{(m)}_{h,h+1}\widetilde{c}^{(m)}_{l+1,l+1}=0$, $\widehat{a}^{(l+1)}_{h,h+1}\widehat{a}^{(l+1)}_{k,k+1}=0$ $(h\neq k)$, and $\widehat{a}^{(l+1)}_{h,h+1}\widehat{c}^{(l+1)}_{l+1,l+1}=0$.

(ii) When $i=1$, from $(a^{(1)}_{hh}-a^{(1)}_{h+1,h+1})(a^{(1)}_{kk}-a^{(1)}_{k+1,k+1})\det B=0$ $(h\neq k)$, we have $(a^{(1)}_{hh}-a^{(1)}_{h+1,h+1})(a^{(1)}_{kk}-a^{(1)}_{k+1,k+1})=0$ $(h\neq k)$. Similarly, we have $(a^{(1)}_{hh}-a^{(1)}_{h+1,h+1})(a^{(1)}_{ll}+a^{(1)}_{l+1,l+1})=0$ $(1\leq h\leq l)$. When $i=l+1$, we get the results similarly.

(iii) The proving process is similar to (i) and (ii). □

Lemma 2.4. *Let* $\psi\in\operatorname{Aut}(\mathbf{n}^{(C)}_0)$. *Then*

(i) *when* $l\geq 1$, $\psi(\alpha_{12})=\widetilde{a}^{(1)}_{12}\alpha_{12}\bmod \mathbf{n}^{(C)}_2$, *where* $\widetilde{a}^{(1)}_{12}\in R^*$;

(ii) *if* $\psi(\alpha_{12})=\widetilde{a}^{(1)}_{12}\alpha_{12}\bmod\mathbf{n}^{(C)}_2$, *where* $\widetilde{a}^{(1)}_{12}\in R^*$, *then* $\psi(\alpha_{i,i+1})=\widetilde{a}^{(i)}_{i,i+1}\alpha_{i,i+1}\bmod\mathbf{n}^{(C)}_2$ *and* $\psi(\gamma_{l+1,l+1})=\widehat{c}^{(l+1)}_{l+1,l+1}\gamma_{l+1,l+1}\bmod\mathbf{n}^{(C)}_2$, *where* $\widetilde{a}^{(i)}_{i,i+1},\widehat{c}^{(l+1)}_{l+1,l+1}\in R^*$.

Proof. (i) Noting that $\alpha_{i,i+1},\gamma_{l+1,l+1}\in\mathbf{n}^{(C)}_1\setminus\mathbf{n}^{(C)}_2$ and $\gamma_{12}\in\mathbf{n}^{(C)}_{2l}\setminus\mathbf{n}^{(C)}_{2l+1}$, we have $\psi(\alpha_{12})\in\mathbf{n}^{(C)}_1\setminus\mathbf{n}^{(C)}_2$ and $\psi(\gamma_{12})=\widehat{c}^{(12)}_{12}\gamma_{12}\bmod\mathbf{n}^{(C)}_{2l+1}\in\mathbf{n}^{(C)}_{2l}\setminus\mathbf{n}^{(C)}_{2l+1}$, where $\widehat{c}^{(12)}_{12}\in R^*$. Using (2.7), from $[\psi(\alpha_{11}),\psi(\gamma_{12})]=\psi(\gamma_{12})$, we have $\widehat{c}^{(12)}_{12}(a^{(1)}_{11}+a^{(1)}_{22})=\widehat{c}^{(12)}_{12}$, that is, $a^{(1)}_{11}+a^{(1)}_{22}=1$. Write $\psi(\alpha_{11})$ and $\psi(\gamma_{11})$ as $\psi(\alpha_{11})=\sum_{i=1}^{l+1}a^{(1)}_{ii}\alpha_{ii}\bmod\mathbf{n}^{(C)}_1$ and $\psi(\gamma_{11})=c^*_{11}\gamma_{11}\in\mathbf{n}^{(C)}_{2l+1}$, where $c^*_{11}\in R^*$. From $2\psi(\gamma_{11})=[\psi(\alpha_{11}),\psi(\gamma_{11})]=2a^{(1)}_{11}c^*_{11}\gamma_{11}$, we have $a^{(1)}_{11}=1$. Then $a^{(1)}_{22}=0$. By Lemma 2.3 we have $a^{(1)}_{ii}-a^{(1)}_{i+1,i+1}=0$, $i=2,\dots,l$ (here $l\geq 2$) and $a^{(1)}_{l+1,l+1}=0$. So $a^{(1)}_{ii}=0$, $i=2,\dots,l+1$, that is, $\psi(\alpha_{11})=a^{(1)}_{11}\bmod\mathbf{n}^{(C)}_1$. Then $\psi(\alpha_{12})=[\psi(\alpha_{11}),\psi(\alpha_{12})]=\widetilde{a}^{(1)}_{12}\alpha_{12}\bmod\mathbf{n}^{(C)}_2$ and $\widetilde{a}^{(1)}_{12}\in R^*$. By Lemma 2.3, (i) holds.

(ii) Write $\psi(\alpha_{j,j+1})$ and $\psi(\gamma_{l+1,l+1})$ as (2.6) and (2.7), respectively. From $\psi(\alpha_{13}) = [\psi(\alpha_{12}), \psi(\alpha_{23})]$, we have $\psi(\alpha_{13}) = \tilde{a}_{12}^{(1)}\tilde{a}_{23}^{(2)}\alpha_{13} \bmod \mathbf{n}_3^{(C)}$. Since $\alpha_{13} \in \mathbf{n}_2^{(C)} \setminus \mathbf{n}_3^{(C)}$, $\psi(\alpha_{13}) \in \mathbf{n}_2^{(C)} \setminus \mathbf{n}_3^{(C)}$. So $\tilde{a}_{12}^{(1)}\tilde{a}_{23}^{(2)} \in R^*$, that is, $\tilde{a}_{23}^{(2)} \in R^*$. In general, for $m = 2,\dots,l$, we have

$$\psi(\alpha_{1,m+1}) = \prod_{i=1}^{m}\tilde{a}_{i,i+1}^{(i)}\alpha_{1,m+1} \bmod \mathbf{n}_{m+1}^{(C)} \in \mathbf{n}_m^{(C)} \setminus \mathbf{n}_{m+1}^{(C)}, \quad m = 1,\dots,l,$$
$$\psi(\gamma_{1,l+1}) = [\psi(\alpha_{1,l+1}), \psi(\gamma_{l+1,l+1})] = \prod_{i=1}^{l}\tilde{a}_{i,i+1}^{(i)}\hat{c}_{l+1,l+1}^{(l+1)}\gamma_{1,l+1} \bmod \mathbf{n}_{l+2}^{(C)} \in \mathbf{n}_{l+1}^{(C)} \setminus \mathbf{n}_{l+2}^{(C)}, \tag{2.12}$$

here $\tilde{a}_{i,i+1}^{(i)}$, $\hat{c}_{l+1,l+1}^{(l+1)}$ should be in R^*, $i = 1,\dots,l$. By Lemma 2.3 we have that $\tilde{a}_{i,i+1}^{(j)} = 0 (i \neq j)$, $\tilde{c}_{l+1,l+1}^{(j)} = 0$, $j = 1,\dots,l$, and $\hat{a}_{i,i+1}^{(l+1)}$, $i = 1,\dots,l$. Hence $\psi(\alpha_{l,l+1})$ and $\psi(\gamma_{l+1,l+1})$ have the required forms, respectively. □

3. The Standard Automorphisms of $\mathbf{t}_{l+1}^{(C)}(R)$

Now let us introduce two types of Lie automorphisms of $\mathbf{t}_{l+1}^{(C)}(R)$.

(i) Inner Automorphisms

Let $r = I_n + a\alpha_{ij} (i \neq j)$ or $r = I_n + a\gamma_{ij}$. It is easy to check that $ryr^{-1} \in \mathbf{n}_0^{(C)}$. The map $\theta_r\colon \mathbf{t}_{l+1}^{(C)}(R) \to \mathbf{t}_{l+1}^{(C)}(R)$ such that $x \mapsto rxr^{-1}$, $x \in \mathbf{n}_0^{(C)}$, defines an automorphism of $\mathbf{n}_0^{(C)}$, which is called an *inner automorphism* (note that r is a symplectic matrix defined by $\begin{pmatrix} 0 & I_{l+1} \\ -I_{l+1} & 0 \end{pmatrix}$). We denote θ_r by $\theta_{a\alpha_{ij}}$, $\theta_{a\gamma_{ij}}$, respectively. In these cases, we have $\theta_{a\alpha_{ij}}^{-1} = \theta_{-a\alpha_{ij}}$, $\theta_{a\gamma_{ij}}^{-1} = \theta_{-a\gamma_{ij}}$, respectively, and that $\theta_{a\alpha_{ij}}(\alpha_{ii}) = \alpha_{ii} - a\alpha_{ij}$, $\theta_{a\alpha_{ij}}(\alpha_{jj}) = \alpha_{jj} + a\alpha_{ij}$, $\theta_{a\alpha_{ij}}(\alpha_{kk}) = \alpha_{kk}\ (k \neq i, j)$, $\theta_{a\alpha_{ij}}(\alpha_{k,k+1}) = \alpha_{k,k+1}\ (k \neq j, i-1)$, $\theta_{a\alpha_{ij}}(\alpha_{j,j+1}) = \alpha_{j,j+1} + a\alpha_{i,j+1}$, $\theta_{a\alpha_{ij}}(\alpha_{i-1,i}) = \alpha_{i-1,i} - a\alpha_{i-1,j}$, $\theta_{a\gamma_{ii}}(\alpha_{ii}) = \alpha_{ii} - 2a\gamma_{ii}$, $\theta_{a\gamma_{i,i+1}}(\alpha_{i,i+1}) = \alpha_{i,i+1} - 2a\gamma_{ii}$ and $\theta_{a\alpha_{i,i+1}}(\gamma_{i,i+1}) = \gamma_{i,i+1} + 2a\gamma_{ii}$. All inner automorphisms of $\mathbf{t}_{l+1}^{(C)}(R)$ generate a subgroup of $\mathrm{Aut}(\mathbf{n}_0^{(C)})$, which is denoted by $\mathcal{I}$.

(ii) Diagonal Automorphisms

Let $d_i \in R^*$, $i = 0, 1, \dots, l+1$, $d = \mathrm{diag}(d_1,\dots,d_{l+1})$ and $D = \mathrm{diag}(d, d^{-1}d_0)$. The map λ_D: $\mathbf{t}_{l+1}^{(C)}(R) \to \mathbf{t}_{l+1}^{(C)}(R)$ such that $x \mapsto DxD^{-1}$, $x \in \mathbf{n}_0^{(C)}$, defines an automorphism of $\mathbf{t}_{l+1}^{(C)}(R)$, which is called a *diagonal automorphism*. It is clear that $\lambda_D\lambda_{\overline{D}} = \lambda_{D\overline{D}}$. So the set of diagonal automorphisms of $\mathbf{t}_{l+1}^{(C)}(R)$ is a subgroup of $\mathrm{Aut}(\mathbf{n}_0^{(C)})$, which is denoted by $\mathfrak{D}$.

4. Lemmas for Main Results

Lemma 4.1. *Let $\psi \in \mathrm{Aut}(\mathbf{n}_0^{(C)})$. The following two statements are equivalent:*

(i) $\psi(\alpha_{j,j+1}) = \tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} \bmod \mathbf{n}_2^{(C)}$ and $\psi(\gamma_{l+1,l+1}) = \hat{c}_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)}$, where $\tilde{a}_{j,j+1}^{(j)}$, $\hat{c}_{l+1,l+1}^{(l+1)} \in R^*$, $j = 1,\dots,l$;

(ii) $\psi(\alpha_{jj}) = \alpha_{jj} \bmod \mathbf{n}_1^{(C)}$, $j = 1,\dots,l+1$.

Proof. (i)⇒(ii). Write $\psi(\alpha_{jj})$ as in (2.5). By the process of proving Lemma 2.3, we have $\tilde{a}_{12}^{(1)}(a_{11}^{(1)}-a_{22}^{(1)})=\tilde{a}_{12}^{(1)}$, $\tilde{a}_{i,i+1}^{(i)}(a_{ii}^{(i+1)}-a_{i+1,i+1}^{(i+1)})=-\tilde{a}_{i,i+1}^{(i)}$, $\tilde{a}_{i+1,i+2}^{(i+1)}(a_{i+1,i+1}^{(i+1)}-a_{i+2,i+2}^{(i+1)})=\tilde{a}_{i+1,i+2}^{(i+1)}$, $i=1,\ldots,l-1$ and $\tilde{a}_{l,l+1}^{(l)}(a_{ll}^{(l+1)}-a_{l+1,l+1}^{(l+1)})=-\tilde{a}_{l,l+1}^{(l)}$, $\hat{c}_{l+1,l+1}^{(l+1)}a_{l+1,l+1}^{(l+1)}=\hat{c}_{l+1,l+1}^{(l+1)}$. Then we obtain that $a_{11}^{(1)}-a_{22}^{(1)}=1$, $a_{ii}^{(i+1)}-a_{i+1,i+1}^{(i+1)}=-1$, $a_{i+1,i+1}^{(i+1)}-a_{i+2,i+2}^{(i+1)}=1$, $i=1,\ldots,l-1$ and $a_{l+1,l+1}^{(l+1)}=1$. By Lemma 2.3, we have $a_{jj}^{(j)}=1$ $(1\le j\le l+1)$ and $a_{ii}^{(j)}=0$ $(i\neq j)$.

(ii)⇒(i). Write $\psi(\alpha_{j,j+1})$ and $\psi(\gamma_{l+1,l+1})$, respectively, as in (2.6) and (2.7). Then

$$\begin{aligned}\psi(\alpha_{j,j+1})&=[\psi(\alpha_{jj}),\psi(\alpha_{j,j+1})]=\tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} \bmod \mathbf{n}_2^{(C)},\quad j=1,\ldots,l,\\ 2\psi(\gamma_{l+1,l+1})&=[\psi(\alpha_{l+1,l+1}),\psi(\gamma_{l+1,l+1})]=2\hat{c}_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)},\end{aligned}\tag{4.1}$$

that is, $\psi(\gamma_{l+1,l+1})=\hat{c}_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)}$. By the method of modularizing a maximal ideal of R to a residue field, we know that $\tilde{a}_{j,j+1}^{(j)}, \hat{c}_{l+1,l+1}^{(l+1)}\in R^*$, $j=1,\ldots,l$. □

Lemma 4.2. *Let ψ be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If $\psi(\alpha_{jj})=\alpha_{jj} \bmod \mathbf{n}_1^{(C)}$, then*

$$\begin{aligned}\psi(\alpha_{11})&=\alpha_{11}+a_{12}^{(1)}\alpha_{12} \bmod \mathbf{n}_2^{(C)},\\ \psi(\alpha_{jj})&=\alpha_{jj}-a_{j-1,j}^{(j-1)}\alpha_{j-1,j}+a_{j,j+1}^{(j)}\alpha_{j,j+1} \bmod \mathbf{n}_2^{(C)},\quad j=2,\ldots,l\ (l\ge 2),\\ \psi(\alpha_{l+1,l+1})&=\alpha_{l+1,l+1}-a_{l,l+1}^{(l)}\alpha_{l,l+1}+c_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)}.\end{aligned}\tag{4.2}$$

Proof. We express $\psi(\alpha_{jj})$ as

$$\psi(\alpha_{jj})=\alpha_{jj}+\sum_{i=1}^{l}a_{i,i+1}^{(j)}\alpha_{i,i+1}+c_{l+1,l+1}^{(j)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)},\quad j=1,\ldots,l+1.\tag{4.3}$$

From $[\psi(\alpha_{jj}),\psi(\alpha_{kk})]=0$ $(j\neq k)$ we have

$$\begin{aligned}\psi(\alpha_{11})&=\alpha_{11}+a_{12}^{(1)}\alpha_{12} \bmod \mathbf{n}_2^{(C)},\\ \psi(\alpha_{jj})&=\alpha_{jj}+a_{j-1,j}^{(j)}\alpha_{j-1,j}+a_{j,j+1}^{(j)}\alpha_{j,j+1} \bmod \mathbf{n}_2^{(C)},\quad j=2,\ldots,l\ (l\ge 2),\\ \psi(\alpha_{l+1,l+1})&=\alpha_{l+1,l+1}+a_{l,l+1}^{(l+1)}\alpha_{l,l+1}+c_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)},\end{aligned}\tag{4.4}$$

where $a_{j,j+1}^{(j)}+a_{j,j+1}^{(j+1)}=0$, $j=1,\ldots,l$. Lemma 4.2 is proved. □

Lemma 4.3. *Let ψ be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If every $\psi(\alpha_{jj})$ is expressed as the form in Lemma 4.2, one may find an inner automorphism*

$$\theta=\prod_{j=1}^{l}\theta_{a_{j,j+1}^{(j)}\alpha_{j,j+1}}\theta_{2^{-1}c_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1}}\tag{4.5}$$

such that

$$\theta\psi(\alpha_{jj}) = \alpha_{jj} \bmod \mathbf{n}_2^{(C)}, \quad j = 1, \ldots, l+1. \tag{4.6}$$

Proof. Note that $\theta_{2^{-1}c_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1}}(\alpha_{l+1,l+1}) = \alpha_{l+1,l+1} - c_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1}$. Then, by Lemma 4.2, it is not difficult to prove Lemma 4.3. □

Lemma 4.4. *Let* ψ *be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If* $\psi(\alpha_{jj}) = \alpha_{jj} \bmod \mathbf{n}_k^{(C)}$, $j = 1, \ldots, l+1$ $(1 \le k \le l+1)$, *then*

$$\begin{aligned}
&\psi(\alpha_{jj}) = \alpha_{jj} + a_{j,j+k}^{(j)}\alpha_{j,j+k} \bmod \mathbf{n}_{k+1}^{(C)}, \quad j = 1, \ldots, \min\{k, l-k+1\}(k \le l, l \ge 1),\\
&\psi(\alpha_{jj}) = \alpha_{jj} - a_{j-k,j}^{(j-k)}\alpha_{j-k,j} + a_{j,j+k}^{(j)}\alpha_{j,j+k} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = k+1, \ldots, l-k+1 \left(k \le \left[\frac{l+1}{2}\right], l \ge 2\right),\\
&\psi(\alpha_{jj}) = \alpha_{jj} - a_{j-k,j}^{(j-k)}\alpha_{j-k,j} + c_{j,2l-k-j+3}^{(j)}\gamma_{j,2l-k-j+3} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = l-k+2, \ldots, l - \left[\frac{k}{2}\right] + 1 \left(k \le \left[\frac{l+1}{2}\right], l \ge 1\right),\\
&\psi(\alpha_{jj}) = \alpha_{jj} + c_{j,2l-k-j+3}^{(j)}\gamma_{j,2l-k-j+3} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = l-k+2, \ldots, l - \left[\frac{k}{2}\right] + 1 \left(1 + \left[\frac{l+1}{2}\right] \le k, l \ge 2\right),\\
&\psi(\alpha_{jj}) = \alpha_{jj} + c_{j,2l-k-j+3}^{(j)}\gamma_{j,2l-k-j+3} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = \frac{l}{2} + 1 \left(k = \frac{l}{2} + 1, l \ge 4, \text{ here } p \text{ even}\right),\\
&\psi(\alpha_{jj}) = \alpha_{jj} - a_{j-k,j}^{(j-k)}\alpha_{j-k,j} + c_{j,2l-k-j+3}^{(j)}\gamma_{j,2l-k-j+3} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = \frac{l}{2} + 2, \ldots, l - \left[\frac{k}{2}\right] + 1\left(k = \frac{l}{2} + 1, l \ge 4, \text{ here } p \text{ even}\right),\\
&\psi(\alpha_{jj}) = \alpha_{jj} + c_{2l-k-j+3,j}^{(2l-k-j+3)}\gamma_{2l-k-j+3,j} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = l - \left[\frac{k}{2}\right] + 2, \ldots, k\left(l+2 \le k + \left[\frac{k}{2}\right], l \ge 3\right),\\
&\psi(\alpha_{jj}) = \alpha_{jj} - a_{j-k,j}^{(j-k)}\alpha_{j-k,j} + c_{2l-k-j+3,j}^{(2l-k-j+3)}\gamma_{2l-k-j+3,j} \bmod \mathbf{n}_{k+1}^{(C)},\\
&\quad j = \max\left\{k+1, l - \left[\frac{k}{2}\right] + 2\right\}, \ldots, l+1\left(k \le p, \left[\frac{k}{2}\right] \ge 1, l \ge 2\right).
\end{aligned} \tag{4.7}$$

Proof. We express $\psi(\alpha_{jj})$, $j = 1, \ldots, l+1$, as

$$\psi(\alpha_{jj}) = \alpha_{jj} + \sum_{i=1}^{l-k+1} a_{i,i+k}^{(j)}\alpha_{i,i+k} + \sum_{i=l+2-[(k+1)/2]}^{l+1} c_{2l-k-i+3,i}^{(j)}\gamma_{2l-k-i+3,i} \bmod \mathbf{n}_{k+1}^{(C)}. \tag{4.8}$$

When $k = 1$ that is the case in Lemma 4.2. The conclusion follows from repeating the process of proving Lemma 4.4. □

Lemma 4.5. *Let* ψ *be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If every* $\psi(\alpha_{jj})$ *be expressed as the form in Lemma 4.4, one may find an inner automorphism*

$$\theta = \prod_{j=1}^{l-k+1} \theta_{a_{j,j+k}^{(j)} \alpha_{j,j+k}} \prod_{j=l-k+2}^{l-[k/2]+1} \theta_{(1+\delta_{jl})^{-1} c_{j,2l-k-j+3}^{(j)} \gamma_{j,2l-k-j+3}}, \tag{4.9}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad h = l + 1 - (k-1)/2(k \text{ an odd}). \tag{4.10}$$

Then

$$\theta\psi(\alpha_{jj}) = \alpha_{jj} \bmod \mathbf{n}_{k+1}^{(C)}, \quad i = 1, \dots, l+1. \tag{4.11}$$

Proof. Apply θ to $\psi(\alpha_{jj})$ and use Lemma 4.4 to obtain Lemma 4.5. □

Lemma 4.6. *Let* ψ *be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If* $\psi(\alpha_{jj}) = \alpha_{jj} \bmod \mathbf{n}_k^{(C)}$, $j = 1, \dots, l+1$, $l+1 \le k \le 2l+1 (l \ge 1)$, *then*

$$\begin{aligned} \psi(\alpha_{jj}) &= \alpha_{jj} + c_{j,2l-k-j+3}^{(j)} \gamma_{j,2l-k-j+3} \bmod \mathbf{n}_{k+1}^{(C)}, \quad j = 1, \dots, l - \left[\frac{k}{2}\right] + 1, \\ \psi(\alpha_{jj}) &= \alpha_{jj} + c_{2l-k-j+3,j}^{(2l-k-j+3)} \gamma_{2l-k-j+3,j} \bmod \mathbf{n}_{k+1}^{(C)}, \quad j = l - \left[\frac{k}{2}\right] + 2, \dots, 2l - k + 2, \\ \psi(\alpha_{jj}) &= \alpha_{jj} \bmod \mathbf{n}_{k+1}^{(C)}, \quad j = 2l - k + 3, \dots, l+1. \end{aligned} \tag{4.12}$$

Proof. We express $\psi(\alpha_{jj})$, $j = 1, \dots, l+1$, as

$$\psi(\alpha_{jj}) = \alpha_{jj} + \sum_{i=1}^{l-[k/2]+1} c_{i,2l-k-i+3}^{(j)} \gamma_{i,2l-k-i+3} \bmod \mathbf{n}_{k+1}^{(C)}. \tag{4.13}$$

The process of proving Lemma 4.6 is similar to Lemma 4.2. □

Lemma 4.7. *Let* ψ *be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If every* $\psi(\alpha_{jj})$ *is expressed as the form in Lemma 4.6, one may find an inner automorphism*

$$\theta = \prod_{j=1}^{l-[k/2]+1} \theta_{(1+\delta_{jh})^{-1} c_{j,2l-k-j+3}^{(j)} \gamma_{j,2l-k-j+3}}, \tag{4.14}$$

where $h = l + 1 - (k-1)/2$ *(k an odd). Then*

$$\theta\psi(\alpha_{jj}) = \alpha_{jj} \bmod \mathbf{n}_{k+1}^{(C)}. \tag{4.15}$$

When $k = 2l+1$, $\theta\psi(\alpha_{jj}) = \alpha_{jj}$, $j = 1, \dots, l+1$.

Proof. It is similar to proving Lemma 4.5. □

Lemma 4.8. *When* $l \geq 1$, *let* ψ *be in* $\mathrm{Aut}(\mathbf{n}_0^{(C)})$. *If* $\psi(\alpha_{jj}) = \alpha_{jj}$, $j = 1, \dots, l+1$, *there exists a diagonal automorphism* λ_D *such that* $\lambda_D\psi(\alpha_{j,j+1}) = \alpha_{j,j+1}$, $i = 1, \dots, l$, *and* $\lambda_D\,\psi(\gamma_{l+1,l+1}) = \gamma_{l+1,l+1}$.

Proof. By Lemma 4.1 we know that $\psi(\alpha_{j,j+1}) = \tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} \bmod \mathbf{n}_2^{(C)}$ and $\psi(\gamma_{l+1,l+1}) = \hat{c}_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} \bmod \mathbf{n}_2^{(C)}$, where $\tilde{a}_{j,j+1}^{(j)}, \hat{c}_{l+1,l+1}^{(l+1)} \in R^*$, $j = 1, \dots, l$. We express $\psi(\alpha_{j,j+1})$ and $\psi(\gamma_{l+1,l+1})$, respectively, as

$$\begin{aligned}
\psi(\alpha_{j,j+1}) &= \tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} + \sum_{k=2}^{l}\sum_{i=1}^{l-k+1}\tilde{a}_{i,i+k}^{(j)}\alpha_{i,i+k} \\
&\quad + \sum_{k=2}^{l}\sum_{i=l+2-[(k+1)/2]}^{l+1}\tilde{c}_{2l-k-i+3,i}^{(j)}\gamma_{2l-k-i+3,i} \\
&\quad + \sum_{k=l+1}^{2l+1}\sum_{i=1}^{l+1-[k/2]}\tilde{c}_{i,2l-k-i+3}^{(j)}\gamma_{i,2l-k-i+3}, \quad 1 \leq j \leq l, \\
\psi(\gamma_{l+1,l+1}) &= \hat{c}_{l+1,l+1}^{(l+1)}\gamma_{l+1,l+1} + \sum_{k=2}^{l}\sum_{i=1}^{l-k+1}\hat{a}_{i,i+k}^{(l+1)}\alpha_{i,i+k} \\
&\quad + \sum_{k=2}^{l}\sum_{i=l+2-[(k+1)/2]}^{l+1}\hat{c}_{2l-k-i+3,i}^{(l+1)}\gamma_{2l-k-i+3,i} + \sum_{k=l+1}^{2l+1}\sum_{i=1}^{l+1-[k/2]}\hat{c}_{i,2l-k-i+3}^{(l+1)}\gamma_{i,2l-k-i+3}.
\end{aligned} \tag{4.16}$$

Then

$$\begin{aligned}
\varphi(\alpha_{j,j+1}) &= [\varphi(\alpha_{jj}), [\varphi(\alpha_{j,j+1}), \varphi(\alpha_{j+1,j+1})]] \\
&= \tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} - \tilde{c}_{j,j+1}^{(j)}\gamma_{j,j+1}, \quad j = 1, \dots, l.
\end{aligned} \tag{4.17}$$

In addition,

$$\begin{aligned}
\varphi(\alpha_{j,j+1}) &= \left[\varphi(\alpha_{jj}), \left[\left(\tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} - \tilde{b}_{j,j+1}^{(j)}\beta_{j,j+1}\right), \varphi(\alpha_{j+1,j+1})\right]\right] \\
&= \tilde{a}_{j,j+1}^{(j)}\alpha_{j,j+1} + \tilde{c}_{j,j+1}^{(j)}\gamma_{j,j+1}, \quad j = 1, \dots, l.
\end{aligned} \tag{4.18}$$

Thus $\widetilde{c}_{j,j+1}^{(j)} = 0$, $j = 1, \ldots, l$. So

$$\psi(\alpha_{j,j+1}) = \widetilde{a}_{j,j+1}^{(j)} \alpha_{j,j+1}, \quad j = 1, \ldots, l. \tag{4.19}$$

Furthermore,

$$\begin{aligned} 2\psi(\gamma_{l+1,l+1}) &= [\psi(\alpha_{l+1,l+1}), \psi(\gamma_{l+1,l+1})] \\ &= 2\widehat{c}_{l+1,l+1}^{(l+1)} \gamma_{l+1,l+1} + \sum_{i=1}^{l-1} \widehat{a}_{i,l+1}^{(l+1)} \alpha_{i,l+1} + \sum_{i=1}^{l-1} \widehat{c}_{i,l+1}^{(l+1)} \gamma_{i,l+1}. \end{aligned} \tag{4.20}$$

From $[\psi(\alpha_{ii}), \psi(\gamma_{l+1,l+1})] = 0 (i \neq l-1)$, we have $\widehat{a}_{i,l+1}^{(l+1)} = 0$ and $\widehat{c}_{i,l+1}^{(l+1)} = 0$, $i = 1, \cdots, l-1$, that is,

$$\psi(\gamma_{l+1,l+1}) = \widehat{c}_{l+1,l+1}^{(l+1)} \gamma_{l+1,l+1}. \tag{4.21}$$

Let $d = \operatorname{diag}(d_1, \ldots, d_{l+1})$ and $d_0 = \widehat{c}_{l+1,l+1}^{(l+1)} d_{l+1}^2$, where $d_1 = 1$, $d_j = \prod_{i=2}^{j} \widetilde{a}_{j-i+1,j-i+2}^{(j-i+1)}$, $j = 2, \ldots, l+1$. Applying λ_D to $\psi(\alpha_{j,j+1})$, $j = 1, \ldots, l$, and $\psi(\gamma_{l+1,l+1})$, we get the result. □

5. Proofs of Main Results

Proof of Theorem 1.1. By Lemmas 4.3, 4.5, 4.7, and 4.8 we have $\lambda_D \theta \psi(\alpha_{jj}) = \alpha_{jj}$, $j = 1, \ldots, l+1$, $\lambda_D \theta \psi(\alpha_{j,j+1}) = \alpha_{j,j+1}$, $j = 1, \ldots, l$, and $\lambda_D \theta \psi(\gamma_{l+1,l+1}) = \gamma_{l+1,l+1}$. Since the set $\{\gamma_{l+1,l+1}, \alpha_{l+1,l+1}, \alpha_{jj}, \alpha_{j,j+1} \mid j = 1, \ldots, l\}$ generates $\mathbf{t}_{l+1}^{(C)}(R)$, we know that $\lambda_D \theta \psi$ is the identity automorphism of $\mathbf{t}_{l+1}^{(C)}(R)$. Hence $\psi = \theta' \lambda_{D^{-1}}$. The uniqueness of the decomposition follows from Theorem 1.2. □

Proof of Theorem 1.2. By the first part of Theorem 1.1 we have $\operatorname{Aut}(\mathbf{t}_{l+1}^{(C)}(R)) = \mathcal{I}\mathfrak{D}$. For any $x \in \mathbf{t}_{l+1}^{(C)}(R)$ and $\alpha_{ij} \in \mathbf{n}_1^{(C)}$ we have

$$\begin{aligned} \lambda_D \theta_{a\alpha_{ij}} \lambda_D^{-1}(x) &= D(I_n + a\alpha_{ij}) D^{-1} x D (I_n + a\alpha_{ij})^{-1} D^{-1} \\ &= (I_n + a\lambda_D(\alpha_{ij})) x (I_n + a\lambda_D(\alpha_{ij}))^{-1} \\ &= \theta_{a\lambda_D(\alpha_{ij})}(x). \end{aligned} \tag{5.1}$$

So $\lambda_D \theta_{a\alpha_{ij}} = \theta_{a\lambda_D(\alpha_{ij})} \lambda_D$. For $\gamma_{ij} \in \mathbf{n}_1^{(C)}$, we have $\lambda_D \theta_{c\gamma_{ij}} = \theta_{c\lambda_D(\gamma_{ij})} \lambda_D$. Therefore, $\mathcal{I} \lhd \mathcal{I}\mathfrak{D}$. Obviously $\mathcal{I} \cap \mathfrak{D} = 1$. Then, $\mathcal{I}\mathfrak{D} = \mathcal{I} \ltimes \mathfrak{D}$. □

6. Discussion for $l = 0$

In this case, $\mathbf{t}_1^{(C)}(R)$ is generated by α_{11} and γ_{11}. For any automorphism ψ of $\mathbf{t}_1^{(C)}(R)$, write $\psi(\alpha_{11})$ and $\psi(\gamma_{11})$, respectively, as $\psi(\alpha_{11}) = a_{11}\alpha_{11} + c_{11}\gamma_{11}$ and $\psi(\gamma_{11}) = c\gamma_{11}$, where $c \in R^*$. From $2\psi(\gamma_{11}) = [\psi(\alpha_{11}), \psi(\gamma_{11})]$, we have $a_{11} = 1$. Then $\theta_{2^{-1}c_{11}\gamma_{11}} \psi(\alpha_{11}) = \alpha_{11}$ and $\theta_{2^{-1}c_{11}\gamma_{11}} \psi(\gamma_{11}) = c\gamma_{11}$. Also $\eta_c \theta_{2^{-1}c_{11}\gamma_{11}} \psi(\alpha_{11}) = \alpha_{11}$ and $\eta_c \theta_{2^{-1}c_{11}\gamma_{11}} \psi(\gamma_{11}) = \gamma_{11}$. So $\psi = \theta_{-2^{-1}c_{11}\gamma_{11}} \eta_{c^{-1}}$.

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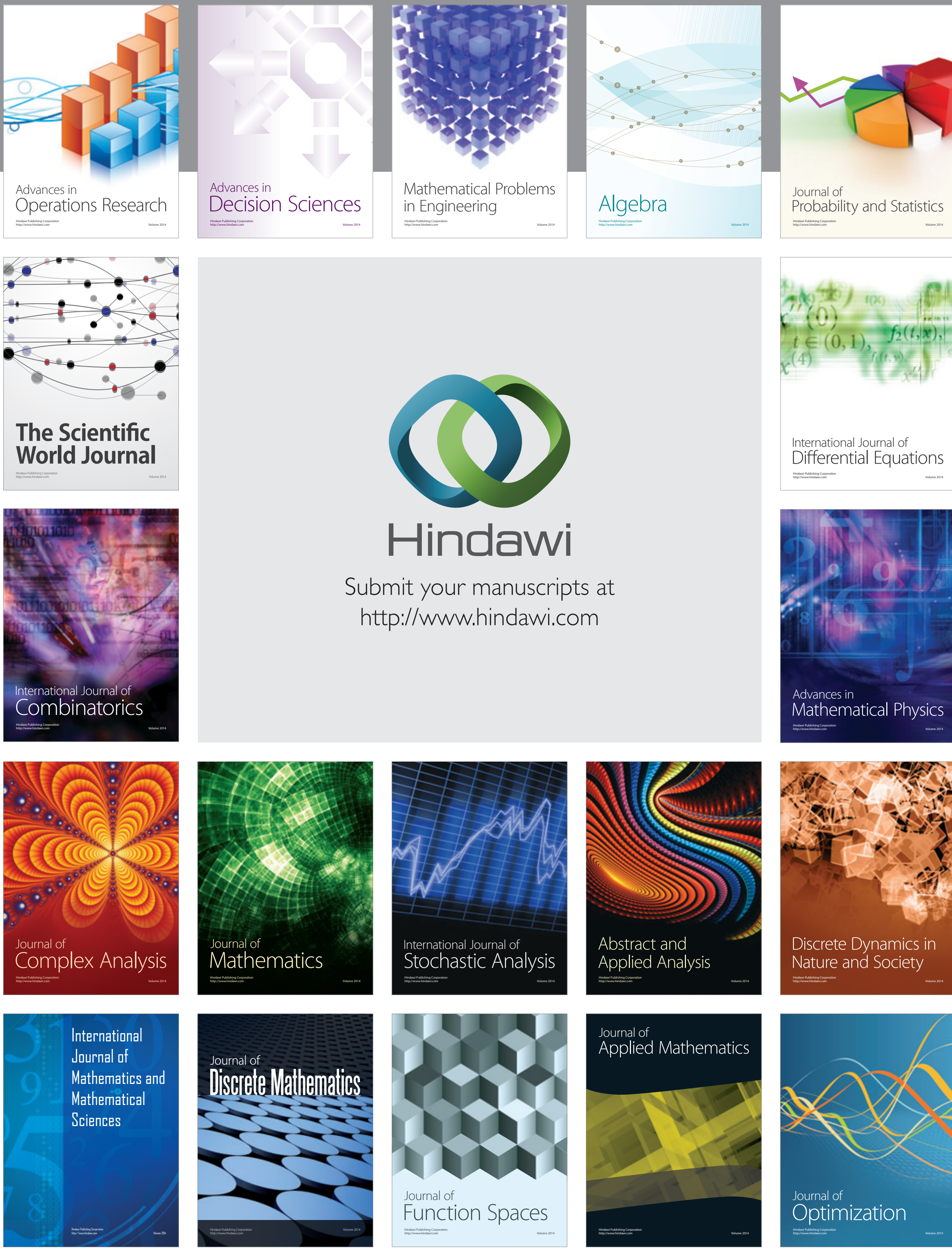
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