Research Article

On the Modified Jump Problem for the Laplace Equation in the Exterior of Cracks in a Plane

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The boundary value problem for the Laplace equation outside several cracks in a plane is studied. The jump of the solution of the Laplace equation and the boundary condition containing the jump of its normal derivative are specified on the cracks. The problem has unique classical solution under certain conditions. The new integral representation for the unique solution of this problem is obtained. The problem is reduced to the uniquely solvable Fredholm equation of the second kind and index zero. The integral representation and integral equation are essentially simpler than those derived for this problem earlier. The singularities at the ends of the cracks are investigated.

1. Introduction

The Dirichlet and Neumann problems outside cracks in a plane for Laplace and Helmholtz equations were treated in [1–10]. The jump problem for the Laplace equation outside cracks in a plane has been studied in [11]. Two boundary conditions of jump type, namely, the jump of the unknown function and the jump of its normal derivative were specified at the cracks in [11]. The so-called modified jump problem has been studied in [12]. In this problem the jump of the normal derivative of the unknown function was not given exactly, but was expressed through limit values of this function on the cracks. The problem in [12] was reduced to the uniquely solvable Fredholm integral equation and the integral representation for a solution of the problem was obtained. However, both integral representation and integral equation derived in [12] were rather complicated. In the present paper we reduce the problem to the simpler integral equation and derive new integral representation for the solution of the problem. The explicit formulas for singularities of the solution gradient at the ends of cracks are presented.

2. Formulation of the Problem

By a simple open curve we mean a nonclosed smooth arc of finite length without self-intersections [10]. In the plane $x = (x_1, x_2) \in R^2$ we consider simple open curves $\Gamma_1, \ldots, \Gamma_N \in C^{2,\lambda}$, $\lambda \in (0, 1]$, so that they do not have common points. We put $\Gamma = \bigcup_{n=1}^{N} \Gamma_n$. We assume that each curve Γ_n is parametrized by the are length s:

$$\Gamma_n = \{x : x = x(s) = (x_1(s), x_2(s)), \ s \in [a_n, b_n]\}, \quad n = 1, \ldots, N, \tag{2.1}$$

so that $a_1 < b_1 < \cdots < a_N < b_N$. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence. Below the set of the intervals on the Os axis $\bigcup_{n=1}^{N} [a_n, b_n]$ will be denoted by Γ also.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos \alpha(s), \sin \alpha(s))$, where $\cos \alpha(s) = x_1'(s)$, $\sin \alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin \alpha(s), -\cos \alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$. We consider Γ as a set of cracks. The side of Γ which is on the left when the parameter s increases will be denoted by Γ^+ and the opposite side will be denoted by Γ^-.

We say that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if the following conditions are satisfied:

(1) $u(x) \in C^0(\overline{R^2 \setminus \Gamma}) \cap C^2(R^2 \setminus \Gamma)$ and $u(x)$ is continuous at the ends of Γ;

(2) $\nabla u \in C^0(\overline{R^2 \setminus \Gamma} \setminus X)$, where X is a point set, consisting of the endpoints of Γ : $X = \bigcup_{n=1}^{N} (x(a_n) \cup x(b_n))$;

(3) in the neighbourhood of any point $x(d) \in X$, for some constants $C > 0$ and $\varepsilon > -1$, the inequality

$$|\nabla u| < C|x - x(d)|^\varepsilon \tag{2.2}$$

holds, where $x \to x(d)$ and $d = a_n$ or $d = b_n$ for $n = 1, \ldots, N$.

Remark 2.1. In the definition of the class $\mathbf{K}$ we consider Γ as a set of cracks in a plane. In particular, the notation $C^0(\overline{R^2 \setminus \Gamma})$ denotes a class of functions, which are continuously extended on Γ from the left and right, but their values on Γ from the left and right can be different, so that the functions may have a jump across Γ.

Let us formulate the jump problem for the harmonic functions in $R^2 \setminus \Gamma$.

Problem **U**

To find a function $u(x)$ of class $\mathbf{K}$, so that $u(x)$ obeys the Laplace equation in $R^2 \setminus \Gamma$

$$\Delta u = 0, \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \tag{2.3}$$

satisfies the jump boundary conditions

$$u(x)\big|_{x(s)\in\Gamma^+} - u(x)\big|_{x(s)\in\Gamma^-} = f_1(s), \tag{2.4a}$$

$$\frac{\partial u}{\partial \mathbf{n}}\bigg|_{x(s)\in\Gamma^+} - \frac{\partial u}{\partial \mathbf{n}}\bigg|_{x(s)\in\Gamma^-} = \beta(s)u(x)\big|_{x(s)\in\Gamma^+} + f_2(s),$$
$$\beta(s) \in C^{0,\lambda}(\Gamma), \quad \lambda \in (0,1]; \quad \beta(s)\big|_{s\in\Gamma} \not\equiv 0, \quad \beta(s)\big|_{s\in\Gamma} \le 0, \tag{2.4b}$$

and satisfies the following conditions at infinity:

$$|u(x)| < C_1, \qquad |\nabla u(x)| < C_2|x|^{-2}, \tag{2.5}$$

where C_1, C_2 are some constants and $|x| = \sqrt{x_1^2 + x_2^2} \to \infty$. Functions $f_1(s)$, $f_2(s)$ are given on Γ.

All conditions of the problem must be fulfiled in a classical sense. The case $\beta(s) \equiv 0$ has been studied in [11]. Gradient inequality in (2.5) is not necessary and follows from estimates for harmonic functions. We write this inequality for convenience only.

The problem **U** has been formulated in [12]. The goal of the present paper is to obtain new integral representation for the solution of the problem **U** and to reduce problem **U** to the new integral equation.

Conditions (2.2) at the ends of Γ in the formulation of the class **K** ensure the absence of point sources at the ends of Γ. If $\beta(s) \equiv f_1(s) \equiv f_2(s) \equiv 0$ on $\gamma \subset \Gamma$, then (2.3) holds on γ and $u(x)$ is analytic on γ.

Remark 2.2. Instead of the boundary condition (2.4b) we may consider another boundary condition:

$$\frac{\partial u}{\partial \mathbf{n}}\bigg|_{x(s)\in\Gamma^+} - \frac{\partial u}{\partial \mathbf{n}}\bigg|_{x(s)\in\Gamma^-} = \beta(s)u(x)\big|_{x(s)\in\Gamma^-} + f_0(s),$$
$$\beta(s) \in C^{0,\lambda}(\Gamma), \quad \lambda \in (0,1]; \quad \beta(s)\big|_{s\in\Gamma} \not\equiv 0, \quad \beta(s)\big|_{s\in\Gamma} \le 0. \tag{2.6}$$

However, this boundary condition can be easily reduced to (2.4b). Indeed, we substitute $u(x)\big|_{x(s)\in\Gamma^-}$ from (2.4a) to (2.6), then we arrive at (2.4b), where

$$f_2(s) = f_0(s) - \beta(s)f_1(s). \tag{2.7}$$

Theorem 2.3. *Problem* **U** *has at most one solution.*

By $\int_\Gamma \cdots d\sigma$ one means

$$\sum_{n=1}^{N} \int_{a_n}^{b_n} \cdots d\sigma. \tag{2.8}$$

Now we prove the theorem. The limit values of functions on Γ^+ and Γ^- will be denoted by the superscripts "+" and "−", respectively.

Let $u_0(x)$ be a solution of the homogeneous problem **U**. We will prove that $u_0(x) \equiv 0$. To prove this with the help of energy equalities for harmonic functions [13], we envelope open curves by closed contours, tend contours to the curves and use the smoothness of the solution of the problem **U**. In this way we obtain

$$
\begin{aligned}
\|\nabla u_0\|^2_{L_2(C_r\backslash\Gamma)} &= \int_\Gamma u_0^+ \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ ds - \int_\Gamma u_0^- \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- ds + \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\, d\varphi \\
&= \int_\Gamma \left\{ (u_0^+ - u_0^-)\left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- + u_0^+ \left[\left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ - \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^-\right]\right\} ds + \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\, d\varphi,
\end{aligned}
\tag{2.9}
$$

where C_r is the circle of the large radius r with the center in the origin, and φ is a polar angle. We suppose that $\Gamma \subset C_r$.

Since $u_0(x)$ satisfies zero boundary conditions of the homogeneous problem **U**, we get:

$$
\|\nabla u_0\|^2_{L_2(C_r\backslash\Gamma)} = -\int_\Gamma |\beta(s)| |u_0^+(x(s))|^2 ds + \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\, d\varphi.
\tag{2.10}
$$

Here we took into account that $\beta(s) \leq 0$ for any $s \in \Gamma$. Putting $r \to \infty$, we have

$$
\|\nabla u_0\|^2_{L_2(R^2\backslash\Gamma)} = \lim_{r\to\infty} \|\nabla u_0\|^2_{L_2(C_r\backslash\Gamma)} = -\int_\Gamma |\beta(s)| |u_0^+(x(s))|^2 ds
\tag{2.11}
$$

because

$$
\lim_{r\to\infty} \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\, d\varphi = 0,
\tag{2.12}
$$

according to conditions at infinity (2.5). It follows from (2.11) that $\|\nabla u_0\|^2_{L_2(R^2\backslash\Gamma)} = 0$ and, therefore $u_0(x) \equiv$ const. Since $\beta(s)$ is continuous on Γ and $\beta(s) \not\equiv 0$, there exists $s_0 \in \Gamma$, so that $\beta(s_0) \neq 0$. Hence, $\beta(s) \neq 0$ in the neighbourhood of s_0 on Γ. Thanks to (2.11), $u_0^+(x(s_0)) = 0$. Consequently, const $= 0$ and $u_0(x) \equiv 0$ in R^2 (we used the smoothness of the function $u_0(x)$ ensured by the class **K**). Now the statement of the theorem follows from the linearity of the problem **U**.

3. The Solution of the Problem

To construct a solution of the problem **U** we impose additional assumptions on the functions $f_1(s)$, $f_2(s)$ in the boundary conditions (2.4a) and (2.4b):

$$
f_1(s) \in C^{1,\lambda}(\Gamma), \qquad f_2(s) \in C^{0,\lambda}(\Gamma), \qquad \lambda \in (0,1];
\tag{3.1a}
$$

$$
f_1(a_n) = f_1(b_n) = 0, \qquad n = 1,\dots,N.
\tag{3.1b}
$$

The solution of the problem **U** can be constructed in the form of a sum of a single layer potential, an angular potential [4, 14] and a constant. Set

$$f_1'(\sigma) = \frac{d}{d\sigma} f_1(\sigma) \tag{3.2}$$

and consider a function

$$u[\mu](x) = v[f_1'](x) + w[\mu](x) + \int_\Gamma \mu(\sigma) d\sigma, \tag{3.3}$$

where

$$w[\mu](x) = -\frac{1}{2\pi} \int_\Gamma \theta[\mu](\sigma) \ln|x - y(\sigma)| d\sigma \tag{3.4}$$

is a single layer potential for (2.3), and

$$v[f_1'](x) = -\frac{1}{2\pi} \int_\Gamma f_1'(\sigma) V(x, \sigma) d\sigma \tag{3.5}$$

is the angular potential [4, 14] for (2.3),

$$\theta[\mu](\sigma) = \mu(\sigma) - \frac{\int_\Gamma \mu(\xi) d\xi}{\int_\Gamma 1\, d\xi}. \tag{3.6}$$

We will look for the unknown density $\mu(s)$ in the Hölder space $C^{0,\omega}(\Gamma)$ with some $\omega \in (0, 1]$. The kernel $V(x, \sigma)$ is defined (up to indeterminacy $2\pi m$, $m = \pm 1, \pm 2, \ldots$) by the formulae

$$\cos V(x, \sigma) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|}, \qquad \sin V(x, \sigma) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|}, \tag{3.7}$$

where

$$y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma, \qquad |x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}. \tag{3.8}$$

One can see that $V(x, \sigma)$ is the angle between the vector $\overrightarrow{(x - y(\sigma))}$ and the direction of the Ox_1 axis. More precisely, $V(x, \sigma)$ is a many-valued harmonic function of x connected with $\ln|x - y(\sigma)|$ by the Cauchy-Riemann relations. Below by $V(x, \sigma)$ we denote an arbitrary fixed branch of this function, which varies continuously with σ varying along each curve Γ_n ($n = 1, \ldots, N$) for given fixed $x \notin \Gamma$. Under this definition of $V(x, \sigma)$, the potential $v[f_1'](x)$ is a multivalued function. In order that the potential $v[f_1'](x)$ be single-valued, the following additional conditions [10] must hold:

$$\int_{a_n}^{b_n} f_1'(\sigma) d\sigma = f_1(b_n) - f(a_n) = 0, \quad n = 1, \ldots, N. \tag{3.9}$$

Clearly, these conditions are satisfied due to our assumptions (3.1b). Integrating $v[f_1'](x)$ by parts and using (3.1b) we express the angular potential in terms of a double-layer potential:

$$v[f_1'](x) = \frac{1}{2\pi}\int_{\Gamma} f_1(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x - y(\sigma)|d\sigma. \tag{3.10}$$

Consequently, the angular potential $v[f_1'](x)$ satisfies (2.3) outside Γ and conditions at infinity (2.5). The single-layer potential $w[\mu](x)$ satisfies conditions (2.5).

   It follows from properties of single-layer and angular potentials [4, 13] that the function (3.3) belongs to the class **K** and obeys (2.3). Besides, function (3.3) satisfies conditions (2.5) at infinity. Hence, the function (3.3) satisfies all conditions of the problem **U** except for the boundary conditions (2.4a) and (2.4b). To satisfy the boundary conditions we, at first, derive the jump formulas for $u(x)$ and its normal derivative on Γ using limit properties of single layer potential and angular potential. According to [4], normal derivative of the angular potential $v[f_1'](x)$ is continuous across Γ. The single-layer potential $w[\mu](x)$ is continuous across Γ in our assumptions. On the basis of the jump relations on Γ for the angular potential and for the normal derivative of the single-layer potential, we obtain [4]

$$u[\mu](x)\big|_{x(s)\in\Gamma_n^+} - u[\mu](x)\big|_{x(s)\in\Gamma_n^-} = v[f_1'](x)\big|_{x(s)\in\Gamma_n^+} - v[f_1'](x)\big|_{x(s)\in\Gamma_n^-}$$

$$= \int_{a_n}^{s}\left(\frac{d}{d\sigma}f_1(\sigma)\right)d\sigma = f_1(s), \quad n = 1,\ldots,N, \tag{3.11}$$

$$\frac{\partial u[\mu](x)}{\partial \mathbf{n}_x}\bigg|_{x(s)\in\Gamma^+} - \frac{\partial u[\mu](x)}{\partial \mathbf{n}_x}\bigg|_{x(s)\in\Gamma^-} = \frac{\partial}{\partial \mathbf{n}_x}w[\mu](x)\bigg|_{x(s)\in\Gamma^+}$$

$$- \frac{\partial}{\partial \mathbf{n}_x}w[\mu](x)\bigg|_{x(s)\in\Gamma^-} = \theta[\mu](s), \tag{3.12}$$

where conditions (3.1b) for $f_1(s)$ have been applied. It follows from (3.11) that the function (3.3) satisfies the boundary condition (2.4a). Substituting (3.3) in (2.4b) and using (3.12) and limit formulas for potentials [4], we obtain the integral equation for the function $\mu(s)$ on Γ

$$\mu(s) + \beta(s)\frac{1}{2\pi}\int_{\Gamma}\theta[\mu](\sigma)\ln|x(s) - y(\sigma)|d\sigma$$

$$- \left(\beta(s) + \frac{1}{\int_{\Gamma}1\,d\xi}\right)\int_{\Gamma}\mu(\sigma)d\sigma = F(s), \quad s \in \Gamma, \tag{3.13}$$

where

$$F(s) = f_2(s) + \beta(s)\left[\frac{1}{2}f_1(s) + \frac{1}{2\pi}\int_{\Gamma}f_1(\sigma)\frac{\partial\ln|x(s) - y(\sigma)|}{\partial\mathbf{n}_y}d\sigma\right], \tag{3.14}$$

and conditions (3.1b) have been used. The term in the square brackets is the limit value of a double layer potential (3.10) on Γ^+. The last term in square brackets is the direct value of the double layer potential (3.10) on Γ. This term belongs to $C^{0,\lambda}(\Gamma)$ in s [see Lemma 1 in [5] and

Lemma 2 in [4]] if conditions (3.1a) hold. Taking into account conditions (3.1a) we observe that $F(s) \in C^{0,\lambda}(\Gamma)$. The kernel of the integral term in (3.13) has logarithmic singularity if $s = \sigma$. It follows from [Lemma 2 in [15]] that the integral operator in (3.13):

$$\mathbf{A}\mu = \frac{1}{2\pi} \int_\Gamma \theta[\mu](\sigma)\beta(s) \ln|x(s) - y(\sigma)| d\sigma \tag{3.15}$$

is a compact operator mapping $C^0(\Gamma)$ into itself. Moreover, this integral operator maps $C^0(\Gamma)$ into $C^{0,\omega}(\Gamma)$ with $\omega = \min\{\lambda, 1/4\}$. Hence if $\mu(s)$ is a solution of (3.13) in $C^0(\Gamma)$, then proceeding from identity (3.13) for $\mu(s)$ we observe that $\mu(s)$ automatically belongs to $C^{0,\omega}(\Gamma)$, where $\omega = \min\{\lambda, 1/4\}$. Consequently, if $\mu(s) \in C^0(\Gamma)$ is a solution of (3.13), then according to properties of potentials [4, 11, 13] the function (3.3) belongs to the class **K** and satisfies all conditions of the problem **U**.

We arrive at the following lemma.

Lemma 3.1. *(1) If $F(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$, then any solution of (3.13) in $C^0(\Gamma)$ automatically belongs to $C^{0,\omega}(\Gamma)$ with $\omega = \min\{\lambda, 1/4\}$.*

*(2) Let conditions (3.1a) and (3.1b) hold. If $\mu(s)$ is a solution of (3.13) in $C^0(\Gamma)$, where $F(s)$ is given by (3.14), then the function (3.3) is a solution of the problem **U**.*

Remark 3.2. Notice that in [12] the integral representation for the solution to the problem **U** has been obtained in the form

$$u[\mu](x) = v[f_1'](x)$$

$$- \frac{1}{2\pi} \int_\Gamma \mu(\sigma) \ln|x - y(\sigma)| d\sigma + \frac{(1/2\pi) \int_\Gamma \mu(\sigma) \int_\Gamma \beta(s) \ln|x(s) - y(\sigma)| ds\, d\sigma}{\int_\Gamma \beta(s) ds}$$

$$- \frac{\int_\Gamma (\beta(s)[(1/2)f_1(s) + (1/2\pi) \int_\Gamma f_1(\sigma)((\partial/\partial\mathbf{n}_y) \ln|x(s) - y(\sigma)| d\sigma)] + f_2(s)) ds}{\int_\Gamma \beta(s) ds}.$$

$$\tag{3.16}$$

This function *a priori* does not satisfy conditions at infinity (2.5) for an arbitrary μ unlike the function (3.3). However being substituted into the boundary condition (2.4b) this integral representation for a solution leads to such an integral equation for the unknown function μ (see [12]) that this integral equation satisfies two important properties. First, any solution μ of the integral equation in [12] falls in the class of functions that ensure satisfaction of conditions at infinity for the solution of the boundary value problem. Second, the integral equation in [12] is uniquely solvable. In fact, the integral representation for a solution of the problem **U** has been taken in [12] in the above form in order to obtaining integral equation satisfying two aforementioned properties (see details and proofs in [12]). However, the integral representation for a solution of the problem **U** given by (3.3) is much simpler than the integral representation for a solution of the problem **U** suggested in [12] and written out above. Finding simple integral representation for a solution is very important for further numerical treatment of the problem by the boundary integral equation method.

Below we look for a solution of (3.13) in $C^0(\Gamma)$ owing to **Lemma 3.1**. The operator

$$\mathbf{B}\mu = \left(\beta(s) + \frac{1}{\int_\Gamma 1\, d\xi}\right) \int_\Gamma \mu(\sigma)d\sigma \tag{3.17}$$

is finite dimensional operator mapping $C^0(\Gamma)$ into itself, and so it is compact [16]. Consequently, the integral operator $(\mathbf{A} - \mathbf{B})\mu$ in (3.13) is a compact operator mapping $C^0(\Gamma)$ into itself. Therefore (3.13) is a Fredholm integral equation of the second kind and index zero in $C^0(\Gamma)$, that is, (3.13) is subject to the Fredholm alternative in $C^0(\Gamma)$ (see [16, 17]).

Let us show that the homogeneous equation (3.13) has only a trivial solution in $C^0(\Gamma)$. Let $\mu^0(s)$ be a solution of the homogeneous equation (3.13). According to lemma, $\mu^0(s)$ automatically belongs to $C^{0,\omega}(\Gamma)$. It follows from point 2 of **Lemma 3.1** that

$$
\begin{aligned}
u\left[\mu^0\right](x) &= w\left[\mu^0\right](x) + \int_\Gamma \mu^0(\sigma)d\sigma \\
&= -\frac{1}{2\pi} \int_\Gamma \theta\left[\mu^0\right](s) \ln|x(s) - y(\sigma)| d\sigma + \int_\Gamma \mu^0(\sigma)d\sigma
\end{aligned}
\tag{3.18}
$$

is a solution of the homogeneous problem **U**. Taking into account **Theorem 2.3**, we observe that

$$u\left[\mu^0\right](x) \equiv 0, \qquad x \in R^2 \setminus \Gamma, \tag{3.19}$$

since the homogeneous problem **U** has only a trivial solution. Using the jump relations for the normal derivative of the single layer potential on Γ, we obtain

$$
\begin{aligned}
\left.\frac{\partial u\left[\mu^0\right](x)}{\partial \mathbf{n}_x}\right|_{x\in\Gamma^+} - \left.\frac{\partial u\left[\mu^0\right](x)}{\partial \mathbf{n}_x}\right|_{x\in\Gamma^-} &= \theta\left[\mu^0\right](s) \\
&= \mu^0(s) - \frac{\int_\Gamma \mu^0(\xi)d\xi}{\int_\Gamma 1\, d\xi} \equiv 0, \quad s \in \Gamma.
\end{aligned}
\tag{3.20}
$$

Consequenly,

$$\mu^0(s) = \frac{\int_\Gamma \mu^0(\xi)d\xi}{\int_\Gamma 1\, d\xi}, \qquad s \in \Gamma. \tag{3.21}$$

Substituting (3.18) into (3.19) we have

$$\int_\Gamma \mu^0(\xi)d\xi = 0. \tag{3.22}$$

It follows from (3.21) that

$$\mu^0(s) \equiv 0, \quad s \in \Gamma. \tag{3.23}$$

Thus, we have proved that the homogeneous equation (3.13) has only a trivial solution. According to Fredholm alternative, the inhomogeneous equation (3.13) is uniquely solvable in $C^0(\Gamma)$ for any $F(s) \in C^0(\Gamma)$.

Theorem 3.3. *Equation* (3.13) *has unique solution* $\mu(s) \in C^0(\Gamma)$ *for any* $F(s) \in C^0(\Gamma)$. *In addition, if* $F(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$, *then the unique solution of* (3.13) *in* $C^0(\Gamma)$ *belongs to* $C^{0,\omega}(\Gamma)$ *with* $\omega = \min\{\lambda, 1/4\}$.

The latter statement of Theorem 3.3 follows from Lemma 3.1. As aforementioned, if conditions (3.1a) and (3.1b) hold, then $F(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$. Using Lemma 3.1 we obtain solvability of the problem **U**.

Theorem 3.4. *If conditions* (3.1a) *and* (3.1b) *hold, then the solution of the problem* **U** *exists and is given by* (3.3), *where* $\mu(s) \in C^{0,\omega}(\Gamma)$ $(\omega = \min\{\lambda, 1/4\})$ *is a solution of* (3.13) *in* $C^0(\Gamma)$ *ensured by Theorem 3.3.*

It follows from [Theorem 5 in [4]] that the condition (2.2) for $|\nabla u|$ is fulfied for any $\varepsilon \in (0,1)$, that is, for any small positive ε. In other words, $\nabla u(x)$ does not have power singularity at the ends of Γ. It will be shown in next section that ∇u has logarithmic singularity or, in certain cases, does not have singularity at all. Explicit formulas for singularities of ∇u at the ends of Γ will be presented and discussed in the next section.

4. Singularities of a Gradient of a Solution at the Ends of Γ

In this section by $u(x)$ we denote the solution of the problem **U** ensured by the Theorem 3.4. According to (2.2), ∇u may be unbounded at the ends of Γ. The explicit expressions for singularities of ∇u can be obtained from the formulas for singularities of derivatives of single layer and angular potentials near edges [4, 5]. Let $x(d)$ be one of the end-points of Γ. In the neighbourhood of $x(d)$ we introduce the system of polar coordinates

$$x_1 = x_1(d) + |x - x(d)| \cos\varphi, \qquad x_2 = x_2(d) + |x - x(d)| \sin\varphi. \tag{4.1}$$

We will assume that $\varphi \in (\alpha(d), \alpha(d) + 2\pi)$ if $d = a_n$ and $\varphi \in (\alpha(d) - \pi, \alpha(d) + \pi)$ if $d = b_n$ for $n = 1, \ldots, N$. Recall that $\alpha(s)$ is the angle between the tangent vector τ_x to Γ at the point $x(s)$ and the direction of the Ox_1 axis. Hence, $\alpha(d) = \alpha(a_n + 0)$ if $d = a_n$ and $\alpha(d) = \alpha(b_n - 0)$ if $d = b_n$. Consequently the angle φ varies continuously in the neighbourhood of the point $x(d)$, cut along the contour Γ.

Recall that X is a set of end-points of Γ. Computing singularities of ∇u in the same way as in [4, 5] we arrive to the following assertion.

Theorem 4.1. *Let $x \to x(d) \in X$. Then in the neigbourhood of the point $x(d)$ the derivatives of the solution of the problem **U** have the following behaviour:*

$$\frac{\partial}{\partial x_1} u(x) = -(-1)^m \frac{f_1'(d)}{2\pi} \left[-\sin\alpha(d)\ln|x - x(d)| + \varphi\cos\alpha(d) \right]$$

$$- (-1)^m \frac{\theta[\mu](d)}{2\pi} \left[\cos\alpha(d)\ln|x - x(d)| + \varphi\sin\alpha(d) \right] + O(1),$$

$$\frac{\partial}{\partial x_2} u(x) = -(-1)^m \frac{f_1'(d)}{2\pi} \left[\cos\alpha(d)\ln|x - x(d)| + \varphi\sin\alpha(d) \right]$$

$$+ (-1)^m \frac{\theta[\mu](d)}{2\pi} \left[-\sin\alpha(d)\ln|x - x(d)| + \varphi\cos\alpha(d) \right] + O(1),$$

$$(4.2)$$

where $m = 0$ if $d = a_n$ and $m = 1$ if $d = b_n$ for $n = 1, \dots, N$.

Remark 4.2. By $O(1)$ we denote functions which are continuous at the point $x(d)$. Furthermore, the functions denoted by $O(1)$ are continuous in the neighbourhood of the point $x(d)$, cut along the contour Γ.

According to Theorem 4.1, ∇u has logarithmic singularities at the ends of cracks Γ in general. However, if $f_1'(d) = \theta[\mu](d) = 0$ at the end $x(d) \in X$, then there is no any singularity of ∇u at the end $x(d)$. Moreover, ∇u is continuous at this end. If $f_1'(d) \neq 0$ or $\theta[\mu](d) \neq 0$, then ∇u has a logarithmic singularity at $x(d) \in X$.

Let us compare our results with singularities of a solution gradient in the Dirichlet and Neumann problems at the exterior of cracks in a plane. In these problems either Dirichlet or Neumann boundary condition has been specified on the cracks instead of (2.4a) and (2.4b). It was shown in [4, 5] that the solution gradient in the Dirichlet and Neumann problems in general tends at infinity as $O(|x - x(d)|^{-1/2})$ when $x \to x(d) \in X$. According to Theorem 4.1, the edge singularities of ∇u in the jump problem are generally logarithmic. Thus, the jump problem and Dirichlet (or Neumann) problem have as a rule different orders of singularities at the ends of cracks, so that the singularities in the jump problem are weaker. We can conclude that the behaviour of the solution in the jump problem is essentially different from behaviour of the solution in the Dirichlet (or Neumann) problem. The discussed properties of singularities may be effectively used to select adequate model describing cracked media or media with membranes.

5. Comparison of Results and Conclusions

Let us discuss obtained results and compare them with previous treatments of the problem **U**.

Notice that the case $\beta \equiv 0$ in (2.4b) is excluded from consideration. This case has been studied in [11]. It is curious that the problem **U** is not uniquely solvable in this case under usual conditions at infinity. There exists solvability condition, and if it holds, then the problem **U** has infinitely many solutions that differ by additive constant as in Neumann problem. From the other hand this case admits explicit solution, so that solving of integral equation is not needed.

For the first time the problem **U** with $\beta \not\equiv 0$ has been treated in [12]. To reduce the problem to the uniquely solvable integral equation, very special and unusual technique has

been applied. The solution was looked in the form of single-layer and angular potentials plus some functionals depending on density μ in a single-layer potential and on boundary data. The solution was looked for in the form that *a priori* does not satisfy conditions at infinity. However, being substituted into the boundary condition, the solution in the form of potentials and additional functionals produces such an integral equation, that *any solution μ of the integral equation automatically falls in the class of functions that ensure satisfaction of conditions at infinity for the solution of the boundary value problem*. More precisely, additional functionals of μ and of boundary data in the form of a solution have been chosen *a priori* in such a way to obtain this special property of solutions of the integral equation. Moreover, these additional functionals of μ and of boundary data have been chosen in such a way to obtain uniquely solvable integral equation at the same time. The only problem with approach in [12] was concluded in the fact that the final expression for the solution of the boundary value problem appeared to be rather complicated. In the present paper we derived *another integral representation* for a solution of problem **U** and we have reduced problem **U** to *another uniquely solvable integral equation*. The idea of new approach is the following. We *a priori* choose such a representation for a solution of the problem **U** in the form of potentials that this chosen form satisfy two properties. First, it automatically satisfies conditions at infinity for any density μ in the single-layer potential. Second, it reduces the problem to the uniquely solvable integral equation. It should be stressed that both integral equation and integral representation for a solution obtained in the present paper are completely different from those obtained in [12] and are essentially simpler. Finding simple integral representation for a solution and finding simple uniquely solvable integral equation are very important for further numerical treatment of the problem by the boundary integral equation method.
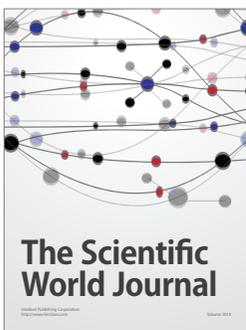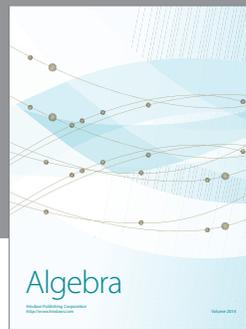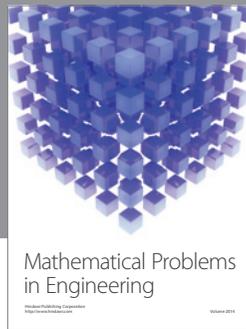
Acknowledgment

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