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Research Article

Nilpotent Elements of Residuated Lattices

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Some properties of the nilpotent elements of a residuated lattice are studied. The concept of cyclic residuated lattices is introduced, and some related results are obtained. The relation between finite cyclic residuated lattices and simple MV-algebras is obtained. Finally, the notion of nilpotent elements is used to define the radical of a residuated lattice.

1. Introduction

Ward and Dilworth [1] introduced the concept of residuated lattices as generalization of ideal lattices of rings. The residuated lattice plays the role of semantics for a multiple-valued logic called residuated logic. Residuated logic is a generalization of intuitionistic logic. Therefore it is weaker than classical logic. Important examples of residuated lattices related to logic are Boolean algebras corresponding to basic logic, BL-algebras corresponding to Hajek's basic logic, and MV-algebras corresponding to Lukasiewicz many valued logic. The residuated lattices have been widely studied (see [2–8]).

In this paper, we study the properties of nilpotent elements of residuated lattices. In Section 2, we recall some definitions and theorems which will be needed in this paper. In Section 3, we study the nilpotent elements of a residuated lattice and study its properties. In Section 4, we define the notion of cyclic residuated lattice and we obtain some related results. In particular, we will prove that a finite residuated lattice is cyclic if and only if it is a simple MV-algebra. In Section 5, we investigate the relation between nilpotent elements and the radical of a residuated lattice.

2. Preliminaries

In this section, we review some basic concepts and results which are needed in the later sections.

A residuated lattice is an algebraic structure $(A, \wedge, \vee, \rightarrow, *, 0, 1)$ such that

(1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1,

(2) $(A, *, 1)$ is a commutative monoid where 1 is a unit element,

(3) $x * y \leq z$ if and only if $x \leq y \rightarrow z$, for all $x, y, z \in A$.

We denote the residuated lattice $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ by A. We use the notation $L(A)$ for the bounded lattice $(A, \wedge, \vee, 0, 1)$.

Proposition 2.1 (see [5, 9]). *Let A be a residuated lattice. Then one has the following properties: for all $x, y, z \in A$,*

(1) $x \leq y$ *if and only if* $x \rightarrow y = 1$,

(2) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$,

(3) $x * y \leq x, y$, $x * (x \rightarrow y) \leq x \wedge y$,

(4) $x * (y \vee z) = (x * y) \vee (x * z)$,

(5) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$,

(6) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$,

(7) *if* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$.

An MV-algebra is an algebra $(A, \oplus, \neg, 0)$ with one binary operation $\oplus$, one unary operation $\neg$, and one constant 0 such that $(A, \oplus, 0)$ is a commutative monoid and, for all $x, y \in A$, $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. If A is an MV-algebra, then the binary operations $*, \wedge, \vee, \rightarrow$ and the constant 1 are defined by the following relations: for all $x, y \in A$, $x * y = \neg(\neg x \oplus \neg y)$, $x \wedge y = (x \oplus \neg y) * y$, $x \vee y = (x * \neg y) \oplus y$, $x \rightarrow y = \neg x \oplus y$, $1 = \neg 0$.

Remark 2.2. A residuated lattice A is an MV-algebra if it satisfies the additional condition: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for any $x, y \in A$.

Definition 2.3. A nonempty subset F of A is called a filter of A if and only if it satisfies the following conditions:

(i) for all $x \in F$ and all $y \in A$, if $x \leq y$ then $y \in F$,

(ii) for all $x, y \in F$, $x * y \in F$.

Let F be a filter of A. For all $x, y \in A$, we denote $x \equiv y$ and say that x and y are congruent if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$. $\equiv$ is a congruence relation on A. The quotient residuated lattice with respect to the congruence relation $\equiv$ is denoted by A/F and its elements are denoted by $[x]$, for $x \in A$.

For all elements x of a residuated lattice A, define $x^0 = 1$ and $x^n = x^{n-1} * x$ for all $n \in \aleph$. The order of $x \in A$, in symbols $\text{ord}(x)$, is the smallest positive integer n such that $x^n = 0$. If such n does not exist, we say x has infinite order.

Definition 2.4. The residuated lattice A is called simple if the only filters of A are $\{1\}$ and A.

Proposition 2.5 (see [10]). *A residuated lattice A is simple if and only if $\text{ord}(a) < \infty$, for every $a \in A$ such that $a \neq 1$.*

A filter M of A is called a maximal filter if and only if it is a maximal element of the set of all proper filters of A. The set of all maximal filters of A is called the maximal spectrum of A and is denoted by $\mathrm{Max}(A)$. For any $X \subseteq A$, we will denote $S_{\mathrm{Max}}(X) = \{M \in \mathrm{Max}(A) \mid X \not\subseteq M\}$. For any $x \in A$, $S_{\mathrm{Max}}(\{x\})$ will be denoted by $S_{\mathrm{Max}}(x)$.

Proposition 2.6 (see [10]). *Let A be a residuated lattice and M a proper filter of A. Then the followings are equivalent:*

(i) *A/M is a simple residuated lattice,*

(ii) *M is a maximal filter,*

(iii) *for any $x \in A$, $x \notin M$ if and only if $x^n \to 0 \in M$, for some $n \geq 1$.*

Definition 2.7. A residuated lattice is said to be local if and only if it has exactly one maximal filter.

Proposition 2.8 (see [10]). *Any simple residuated lattice is local.*

We denote by $B(A)$ the Boolean center of A, that is the set of all complemented elements of the lattice $L(A)$. The complements of the elements in the Boolean center of a residuated lattice are unique.

Theorem 2.9 (see [10]). *If A is a local residuated lattice, then $B(A) = \{0, 1\}$.*

3. Nilpotent Elements of Residuated Lattices

We recall that an element $x \in A$ is called nilpotent if and only if $\mathrm{ord}(x)$ is finite. We denote by $N(A)$ the set of the nilpotent elements of A.

Example 3.1 (see [5]). Consider the residuated lattice A with the universe $\{0, a, b, c, d, e, f, 1\}$. Lattice ordering is such that $0 < d < c < b < a < 1$, $0 < d < e < f < a < 1$, and elements from $\{b, c\}$ and $\{e, f\}$ are pairwise incomparable. The operations of $*$ and $\to$ are given by the following:

$$
\begin{array}{c|cccccccc}
* & 0 & a & b & c & d & e & f & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & c & c & c & 0 & d & d & a \\
b & 0 & c & c & c & 0 & 0 & d & b \\
c & 0 & c & c & c & 0 & 0 & 0 & c \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
e & 0 & d & 0 & 0 & 0 & d & d & e \\
f & 0 & d & d & 0 & 0 & d & d & f \\
1 & 0 & a & b & c & d & e & f & 1
\end{array}
\qquad
\begin{array}{c|cccccccc}
\longrightarrow & 0 & a & b & c & d & e & f & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & d & 1 & a & a & f & f & f & 1 \\
b & e & 1 & 1 & a & f & f & f & 1 \\
c & f & 1 & 1 & 1 & f & f & f & 1 \\
d & a & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
e & b & 1 & a & a & a & 1 & 1 & 1 \\
f & c & 1 & a & a & a & a & 1 & 1 \\
1 & 0 & a & b & c & d & e & f & 1
\end{array}
\tag{3.1}
$$

Then $\mathrm{ord}(0) = 1$, $\mathrm{ord}(d) = 2$, $\mathrm{ord}(e) = \mathrm{ord}(f) = 3$, and $\mathrm{ord}(a) = \mathrm{ord}(b) = \mathrm{ord}(c) = \mathrm{ord}(1) = \infty$. Hence $N(A) = \{0, d, e, f\}$.

Remark 3.2. Let A be a residuated lattice. Then $x \in A$ has order n if and only if $x^n \to 0 = 1$, for some $n \in \aleph$.

Example 3.3. Let $A_{n+1} = \{a_0, \dots, a_n\}$, for $n \in \aleph$ and $a_i \leq a_{i+1}$ for all $0 \leq i \leq n$. Also, we have $a_0 = 0$ and $a_n = 1$. Define

$$a_i * a_j = \begin{cases} 0 & \text{if } i + j \leq n, \\ a_{i+j-n} & \text{if } n < i + j, \end{cases}$$
$$a_i \longrightarrow a_j = \begin{cases} a_{n-i+j} & \text{if } j < i, \\ 1 & \text{if } i \leq j. \end{cases} \tag{3.2}$$

Then A_{n+1} becomes a residuated lattice. Let $1 \neq a_i \in A_{n+1}$. Then there exists $k \in \aleph$ such that $ki \leq (k-1)n$. We get that $(a_i)^k = a_i * (a_i)^{k-1} = a_i * a_{(k-1)i-(k-2)n} = 0$, because $i + (k-1)i - (k-2)n = ki - (k-2)n \leq n$. Hence A_{n+1} is a simple residuated lattice.

Theorem 3.4. *$N(A)$ is a lattice ideal of the residuated lattice A.*

Proof. It is clear that $0 \in N(A)$. Suppose that $x \leq y$ and $y \in N(A)$. There exists $m \in \aleph$ such that $y^m = 0$. We have $x^m \leq y^m$. Therefore $x \in N(A)$.

Suppose that $x, y \in N(A)$. Then there exists $m, n \in \aleph$ such that $x^m = y^n = 0$. By Proposition 2.1(4), we have

$$(x \vee y)^{m+n} = x^{m+n} \vee \left(x^{m+n-1} * y\right) \vee \cdots \vee (x^m * y^n) \vee \cdots \vee y^{m+n} = 0. \tag{3.3}$$

Hence $x \vee y \in N(A)$, and then $N(A)$ is a lattice ideal of L. □

Remark 3.5. An element x of a residuated lattice A is nilpotent if and only if there is no proper filter F of A such that $x \in F$.

Theorem 3.6. *Let A be a residuated lattice and $\{A_i : i \in I\}$ a family of residuated lattices. Then*

(1) $B(A) \cap N(A) = \{0\}$,

(2) $N(\Pi_{i \in I} A_i) = \Pi_{i \in I} N(A_i)$.

Proof. (1) Let $x \in B(A) \cap N(A)$. Since $x \in B(A)$, then $x = x * x$. Hence we get that $x^n = x$ for all $n \in \aleph$. Also, we have $x \in N(A)$. So there exists $m \in N(A)$ such that $x^m = 0$. Therefore, we obtain that $x = 0$.

(2) Suppose that $x \in \Pi_{i \in I} N(A_i)$. Then $x = (x_1, \dots, x_n)$ where x_i is nilpotent in A_i. Thus, for each $1 \leq i \leq n$, there exists m_i such that $(x_i)^{m_i} = 0$. Put $m = \text{Max}\{m_i : 1 \leq i \leq n\}$. Then $x^m = 0$, that is $x \in N(\Pi_{i \in I} A_i)$. The proof of reverse inclusion is straightforward. □

For a nonempty subset $X \subseteq A$, the smallest filter of A which contains X is said to be the filter of A generated by X and will be denoted by $\langle X \rangle$. If $X = \{x\}$ with $x \in A$, we denote by $\langle x \rangle$ the filter generated by $\{x\}$. Also, we have $\langle x \rangle = \{y \in A : y \geq x^n \text{ for some } n \in \aleph\}$.

Theorem 3.7. *Let x be an element of a residuated lattice A. Then x is nilpotent if and only if $A = \langle x \rangle$.*

Proof. Let x be a finite order. Then there exists integer $n > 0$ such that $x^n = 0$, that is $0 \in \langle x \rangle$. Therefore $A = \langle x \rangle$.

Conversely, if $A = \langle x \rangle$, then $0 \in \langle x \rangle$. So there exists an integer $n > 0$ such that $x^n \leq 0$. Hence x has finite order. □

Theorem 3.8. *Let x be an element of order n of a residuated lattice A. Then the elements $x^0 \to 0 = 0$, $x^n \to 0 = 1$, $x^i \to 0$, $i = 1, \ldots, n-1$ of A are pairwise distinct.*

Proof. Suppose that $x^i \to 0 = x^j \to 0$, for some $0 \leq i < j \leq n$. Then $1 = x^n \to 0 = (x^j * x^{n-j}) \to 0 = x^{n-j} \to (x^j \to 0) = x^{n-j} \to (x^i \to 0) = x^{n-j+i} \to 0$. But $n - j + i < n$ which is a contradiction with the order of a. Hence $x^i \to 0 \neq x^j \to 0$. □

4. Cyclic Residuated Lattices

The order of a residuated lattice A is the cardinality of A and denoted by $O(A)$.

Definition 4.1. Let A be a finite residuated lattice. A is called cyclic, if there exists $x \in A$ such that $\operatorname{ord}(x) = O(A) - 1$. x is called a generator of A.

Theorem 4.2. *Let A be a cyclic residuated lattice of order $n+1$. Then there exists an element $x \in L$ of order n such that $A = \{x^i \to 0 : 0 \leq i \leq n\}$ where $x^0 \to 0 = 0$, $x^n \to 0 = 1$.*

Proof. Since A is cyclic of order $n+1$, then there exists an element $x \in L$ of order n. By Theorem 3.8, the elements $x^0 \to 0 = 0$, $x^n \to 0 = 1$ and $x^i \to 0$, $i = 1, \ldots, n-1$ of A are pairwise distinct. Hence the cardinality of $X = \{x^i \to 0 : i = 0, \ldots, n-1\}$ is $n+1$. Since $O(A) = n+1$ and $X \subseteq A$, we get that $A = \{x^i \to 0 : i = 0, \ldots, n\}$. □

An element a of a residuated lattice A is called a coatom if it is maximal among elements in $L(A) - \{1\}$.

Theorem 4.3. *Let A be a cyclic residuated lattice of order $n+1$. Then A is linearly ordered. Moreover the generator of A is a unique coatom.*

Proof. Since A is a cyclic residuated lattice of order $n+1$, then there exists an element $x \in A$ such that $A = \{x^i \to 0 : i = 0, \ldots, n\}$ by Theorem 4.2. If $0 \leq j \leq i \leq n$, then $x^i \leq x^j$. By Proposition 2.1(7), $x^j \to 0 \leq x^i \to 0$. Hence A is linearly ordered.

It is clear that $x^{n-1} \to 0$ is a coatom of A. We will show that $x = x^{n-1} \to 0$. Since $1 \neq x \in \{x^i \to 0 : i = 0, \ldots, n\}$, then there exists $0 \leq i < n$ such that $x = x^i \to 0$. We have

$$x^{i+1} \longrightarrow 0 = x \longrightarrow \left(x^i \longrightarrow 0\right) = x \longrightarrow x = 1. \tag{4.1}$$

Therefore $x^{i+1} \to 0 = 1$. By definition order of x, we get that $n \leq i+1$. Hence $n - 1 \leq i$. Therefore $x = x^{n-1} \to 0$. □

In the following example, we will show that the converse of the above theorem may not be true in general.

Example 4.4. Consider the residuated lattice A with the universe $\{0, a, b, c, 1\}$. Lattice ordering is such that $0 < a < b < c < 1$. The operations of $*$ and $\to$ are given by the following:

$$\begin{array}{c|ccccc} * & 0 & a & b & c & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & a \\ b & 0 & 0 & 0 & 0 & b \\ c & 0 & 0 & 0 & a & c \\ 1 & 0 & a & b & c & 1 \end{array} \qquad \begin{array}{c|ccccc} \longrightarrow & 0 & a & b & c & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 \\ a & c & 1 & 1 & 1 & 1 \\ b & c & c & 1 & 1 & 1 \\ c & b & c & c & 1 & 1 \\ 1 & 0 & a & b & c & 1 \end{array} \tag{4.2}$$

A is a finite linearly residuated lattice of order 5 but it is not cyclic because we have $\mathrm{ord}(0) = 1$, $\mathrm{ord}(a) = \mathrm{ord}(b) = 2$, $\mathrm{ord}(d) = 3$, and $\mathrm{ord}(1) = \infty$.

In the following theorems, we characterize cyclic residuated lattices.

Theorem 4.5. *Let A be a cyclic residuated lattice of order $n + 1$. Then A is isomorphic to the simple residuated lattice A_{n+1} of Example 3.3.*

Proof. By Theorems 4.2 and 4.3, there exists an element $x \in A$ such that $A = \{x^i \to 0 : i = 0, \ldots, n\}$, where $x^0 \to 0 = 0 < x^1 \to 0 < \cdots < x^{n-1} \to 0 < x^n \to 0 = 1$. We denote $a_0 = 0$, $a_n = 1$, and $a_i = x^i \to 0$ for $1 \leq i \leq n - 1$. By Theorem 4.3, $x = a_{n-1}$ is the generator of A and it is a coatom. We will show that

$$a_i \longrightarrow a_j = \begin{cases} a_{n-i+j} & \text{if } j < i, \\ 1 & \text{if } i \leq j. \end{cases} \tag{4.3}$$

It is clear that $a_i \to a_j = 1$, if $i \leq j$. We will prove that $a_i \to a_j = a_{n-i+j}$, if $j < i$. Suppose that $i = j + 1$. Then

$$\begin{aligned} a_{n-1} \longrightarrow (a_i \longrightarrow a_j) &= a_i \longrightarrow (a_{n-1} \longrightarrow a_j) = a_i \longrightarrow \left(x \longrightarrow \left(x^j \longrightarrow 0\right)\right) \\ &= a_i \longrightarrow \left(x^{j+1} \longrightarrow 0\right) = a_{j+1} \longrightarrow a_{j+1} = 1. \end{aligned} \tag{4.4}$$

Hence $x = a_{n-1} \leq a_i \to a_j \leq 1$. Since x is a coatom, then $a_i \to a_j = 1$ or $x = a_i \to a_j$. If $a_i \to a_j = 1$, then $a_i \leq a_j$. Since $j < i$, then $a_j < a_i$ which is a contradiction by Theorem 3.8. Hence $a_{n-1} = x = a_i \to a_j$.

Now, suppose that $i = j + k$ where $k > 1$. Since $a_i \to a_j \in A = \{x^i \to 0 : i = 0, \ldots, n\}$, then there exists $0 \leq m \leq n$ such that $a_i \to a_j = x^m \to 0$. We have

$$\begin{aligned} x^{k-1} \longrightarrow (a_i \longrightarrow a_j) &= a_i \longrightarrow \left(x^{k-1} \longrightarrow a_j\right) = a_i \longrightarrow \left(x^{k-1} \longrightarrow \left(x^j \longrightarrow 0\right)\right) \\ &= a_i \longrightarrow \left(\left(x^{k-1} * x^j\right) \longrightarrow 0\right) = a_i \longrightarrow \left(x^{j+k-1} \longrightarrow 0\right) = a_i \longrightarrow a_{i-1} = x. \end{aligned} \tag{4.5}$$

Therefore $x = x^{k-1} \to (a_i \to a_j) = x^{k-1} \to (x^m \to 0) = x^{m+k-1} \to 0$. We get that $1 = x \to x = x^{m+k} \to 0$. Thus, $n \leq m+k$. On the other hand, since $1 \neq x = x^{n-1} \to 0$ and $x^t \to 0 = 1$ for all $t \geq n$, then $m+k-1 \leq n-1$. Therefore $m = n-k$. We obtain that $a_i \to a_j = x^{n-k} \to 0 = a_{n-k} = a_{n-i+j}$.

Now, we will show that

$$a_i * a_j = \begin{cases} 0 & \text{if } i+j \leq n, \\ a_{i+j-n} & \text{if } n < i+j. \end{cases} \tag{4.6}$$

Suppose that $i+j \leq n$. Then $a_i \leq a_{n-j}$. Since $a_{n-j} = a_j \to a_0$, we get that $a_i * a_j \leq a_0 = 0$. Thus $a_i * a_j = 0$.

Suppose that $n < i+j$. Since $i \leq n$, we have $i+j-n \leq j$. Therefore $a_j \to a_{i+j-n} = a_i$. We get that $a_i * a_j \leq a_{i+j-n}$. Let $a_i * a_j = a_t$. Then $t \leq i+j-n$. Consider the following cases.

(1) If $j \leq t < i+j-n$, then $0 \leq t-j < i-n \leq 0$ which is a contradiction.

(2) If $i \leq t < i+j-n$, then $0 \leq t-i < j-n \leq 0$ which is a contradiction.

(3) $t \leq i, j$, then $a_j \leq a_i \to a_t = a_{n-i+t}$. We get that $j \leq n-i+t$. Therefore $j+i-n \leq t$ which is a contradiction.

We obtain that $t = i+j-n$.

Hence $a_i \mapsto x_i$ is an isomorphism between A and A_{n+1}. Since A_{n+1} is a simple residuated lattice, then A is a simple residuated lattice. □

Remark 4.6. Consider the residuated lattice A_{n+1} in Example 3.3. Since $(a_i \to a_j) \to a_j = (a_j \to a_i) \to a_i$ for all $0 \leq i, j \leq n$, then A_{n+1} is an MV-algebra.

Corollary 4.7. *Every cyclic residuated lattice is a finite simple MV-algebra.*

Proof. It follows from Theorem 4.5 and Remark 4.6. □

Corollary 4.8. *Every cyclic residuated lattice A is local and $B(A) = \{0, 1\}$.*

Proof. It follows from Theorem 4.5, Proposition 2.8, and Theorem 2.9. □

Theorem 4.9. *Every finite simple MV-algebra is a cyclic residuated lattice.*

Proof. Any simple MV-algebra is isomorphic to a subalgebra of $[0,1]$, and also $L_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$, $(n \geq 2)$ is the only subalgebra of $[0,1]$ with n elements [11]. Since $((n-2)/(n-1))^{n-1} = 0$, then $\text{ord}((n-1)/(n-2)) = O(L_n) - 1$. Therefore it is cyclic. □

Corollary 4.10. *A finite residuated lattice is cyclic if and only if it is a simple MV-algebra.*

Proof. It follows from Corollary 4.7 and Theorem 4.9. □

Theorem 4.11. *Every nonzero subalgebra of a cyclic residuated lattice is cyclic.*

Proof. Suppose that S is a nonzero subalgebra of a cyclic residuated lattice A. Then A is a simple MV-algebra and S is isomorphic to a subalgebra of $[0,1]$. Moreover $L_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$, $(n \geq 2)$ is the only subalgebra of $[0,1]$ with n elements. Hence S is a simple MV-algebra. By Theorem 4.9, S is a cyclic MV-algebra. □

Theorem 4.12. *Every finite MV-algebra is a direct product of cyclic residuated lattices.*

Proof. Every finite MV-algebra is isomorphic to a finite direct product of finite subalgebras of the standard MV-algebra [11]. Theorem 4.11 yields the theorem. □

5. Semisimple Residuated Lattices

The intersection of the maximal filters of residuated lattice A is called the radical of A and will be denoted by $\mathrm{Rad}(A)$.

Theorem 5.1. *Let A be a residuated lattice. Then* $\mathrm{Rad}(A) = \{x \in A : x^n \to 0 \in N(A)$ *for all* $n \in \aleph\}$.

Proof. See [12]. □

Theorem 5.2. *Let A be a residuated lattice. Then*

(1) $\mathrm{Rad}(L(A)/N(A)) = \mathrm{Rad}(L(A))/N(A)$,

(2) $\mathrm{Max}(L(A))$ *and* $\mathrm{Rad}(L(A)/N(A))$ *are homomorphic topological spaces.*

Proof. (1) It is easily seen that $\mathrm{Max}(L(A)/N(A)) = \{M/N(A) : M \in \mathrm{Max}(L(A))\}$, thus

$$\mathrm{Rad}\left(\frac{L(A)}{N(A)}\right) = \bigcap_{M \in \mathrm{Max}(L(A))} \frac{M}{N}(A) = \frac{\bigcap_{M \in \mathrm{Max}(L(A))} M}{N(A)} = \mathrm{Rad}\frac{(L(A))}{N(A)}. \tag{5.1}$$

(2) Define $\varphi : \mathrm{Max}(L(A)) \to \mathrm{Max}(L(A)/N(A))$, for all $M \in \mathrm{Max}(L(A))$, $\varphi(M) = M/N(A)$. This function is well defined and surjective. For any $M \in \mathrm{Max}(L(A))$ and any $x \in A$, we have $[x] \in M/N(A)$ if and only if $x \in M$. We get that φ is injective.

Now, we will prove that φ is continuous and open. Let $x \in A$. By using the above, we get

$$\begin{aligned} S_{\mathrm{Max}}([x]) &= \left\{\frac{M}{N}(A) : M \in \mathrm{Max}(L(A)), [x] \notin \frac{M}{N(A)}\right\} \\ &= \left\{\frac{M}{N}(A) : M \in \mathrm{Max}(A), x \notin M\right\} \\ &= \left\{\frac{M}{N}(A) : M \in S_{\mathrm{Max}}(x)\right\} \\ &= \{\varphi(M) : M \in S_{\mathrm{Max}}(x)\} \\ &= \varphi(S_{\mathrm{Max}}(x)). \end{aligned} \tag{5.2}$$

Thus φ is open. Since φ is injective and open, then $\varphi^{-1}(S_{\mathrm{Max}}([x])) = S_{\mathrm{Max}}(x)$. So φ is continuous. □

Definition 5.3. A residuated lattice A is called semisimple if the intersection of all congruences of A is the congruence $\triangle_A$ (where, for all $x, y \in A$, $x \triangle_A y$ if and only if $x = y$).

Remark 5.4 (see [5]). A residuated lattice A is semisimple if and only if $\mathrm{Rad}(A) = \{1\}$.

Lemma 5.5. *Let A be a residuated lattice, $x \in A$, and $F = \{y \in A : y \to x = x\}$. Then F is a filter of A.*

Proof. (1) Suppose that $y_1, y_2 \in F$. Then $y_1 \to x = x = y_2 \to x$. We have

$$y_1 * y_2 \longrightarrow x = y_1 \longrightarrow (y_2 \longrightarrow x) = y_1 \longrightarrow x = x. \tag{5.3}$$

Therefore $y_1 * y_2 \in F$.

(2) Let $y \le z$, where $y \in F$ and $z \in A$. We have $y \to x = x$. Since $y \le z$, then $x \le z \to x \le y \to x = x$. Therefore $z \to x = x$ and $z \in F$.

Hence F is a filter of A. □

Theorem 5.6. *Let A be a residuated lattice such that for all $x \in A$ there exists $n \in \aleph$ such that $(x^n \to 0) \to x = x$. Then A is semisimple.*

Proof. We will show that $\mathrm{Rad}(A) = \{1\}$. Suppose that $x \in \mathrm{Rad}(A)$. By Theorem 5.1, we have that $x^n \to 0$ is nilpotent. By assumption, there exists $n \in \aleph$ such that $(x^n \to 0) \to x = x$. We have $x^n \to 0 \in F = \{y \in A : y \to x = x\}$. Since F is a filter of A by Lemma 5.5 and $x^n \to 0$ is nilpotent, we get that $0 \in F$. Thus $1 = 0 \to x = x$. We obtain that $\mathrm{Rad}(A) = \{1\}$ and then A is semisimple by Remark 5.4. □

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