Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 784028, 15 pages
doi:10.1155/2012/784028

Research Article

Some Integral Formulas for the $(r + 1)$th Mean Curvature of a Closed Hypersurface

Akram Mohammadpouri and S. M. B. Kashani

Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran

Correspondence should be addressed to S. M. B. Kashani, kashanim@modares.ac.ir

Received 24 April 2012; Accepted 27 June 2012

Academic Editor: Luc Vrancken

By using the operator L_r, we define the notions of rth order and rth type of a Euclidean hypersurface. By the use of these notions, we are able to obtain some sharp estimates of the $(r+1)$th mean curvature for a closed hypersurface of the Euclidean space in terms of rth order.

1. Introduction

The study of submanifolds of finite type began in 1970, with Chen's attempts to find the best possible estimate of the total mean curvature of a compact submanifold of the Euclidean space and to find a notion of "degree" for Euclidean submanifolds [1, 2].

In algebraic geometry varieties are the main objects to study. Since an algebraic variety is defined by using algebraic equations, one can define the degree of an algebraic variety by its algebraic structure, and it is well known that the concept of degree plays a fundamental role in algebraic geometry [3].

On the other hand, in differential geometry, the main objects to study are Riemannian (sub) manifolds. According to Nash's immersion Theorem, every Riemannian manifold can be realized as a submanifold of the Euclidean space via an isometric immersion [4], but there is no notion of degree for submanifolds of the Euclidean space in general.

So inspired by algebraic geometry, in 1970, Chen defined the notions of order and type for submanifolds of the Euclidean space by the use of the Laplace operator. After that, Chen was able to obtain some sharp estimates of the total mean curvature for compact submanifolds of the Euclidean space in terms of their orders. Moreover, he could introduce submanifolds and maps of finite type [1, 2].

On one hand finite-type submanifolds provides a natural way to exploit the spectral theory to study the geometry of submanifolds and smooth maps, in particular the Gauss map. On the other hand the techniques of the submanifold theory can be used in the study of spectral geometry via the study of finite-type submanifolds.

As is well known, the Laplace operator of a hypersurface M immersed into $\mathbb{R}^{n+1}$ is an (intrinsic) second-order linear elliptic differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. From this point of view, the Laplace operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see [5]). These operators are given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the rth Newton transformation associated to the second fundamental form of the hypersurface, and $\nabla^2 f$ is the hessian of f (see the next section for details).

In contrast to the operator Δ, the operators L_r are not elliptic in general, but they still share some nice properties with Laplacian of M; moreover, under appropriate natural geometric hypotheses on the hypersurface, they are elliptic [6]. Therefore, from this point of view, it seems natural and interesting to generalize the definition of finite-type hypersurface by replacing Δ by the operator L_r. Having this idea, for the first time in [7], the second author, inspired by a private communication with Alías, introduced such hypersurfaces and called them "L_r-finite type" hypersurfaces.

In this paper, by using the operator L_r, we define the notions of rth order and rth type of a Euclidean hypersurface. Then we are able to obtain some sharp estimates of the $(r+1)$th mean curvature for closed hypersurfaces of the Euclidean space in terms of their rth orders when L_r is elliptic. The paper generalizes the results of [8, 9].

2. Preliminaries

In this section, we recall some prerequisites about Newton transformations P_r and their associated second-order differential operators L_r from [10].

Consider an orientable isometrically immersed hypersurface $x : M^n \to \mathbb{R}^{n+1}$ in the Euclidean space, with the Gauss map N. We denote by ∇^0 and ∇ the Levi-Civita connections on $\mathbb{R}^{n+1}$ and M, respectively. Then, the basic Gauss and Weingarten formulae of the hypersurface are written as

$$\begin{gathered}\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle N,\\ SX = -\nabla^0_X N\end{gathered} \tag{2.1}$$

for all tangent vector fields $X, Y \in \chi(M)$, where $S : \chi(M) \to \chi(M)$ is the shape operator of M with respect to the Gauss map N. As is well known, S defines a self-adjoint linear operator on tangent space T_pM, and its eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$ are called the principal curvatures of the hypersurface. Associated to the shape operator S, there are n algebraic invariants given by

$$s_r(p) = \sigma_r(\kappa_1(p), \ldots, \kappa_n(p)), \quad 1 \le r \le n, \tag{2.2}$$

where $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_r(x_1, \ldots, x_n) = \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1} \cdots x_{i_r}. \tag{2.3}$$

Observe that the characteristic polynomial of S can be written in terms of the s_r, s as

$$Q_S(t) = \det(tI - S) = \sum_{r=0}^{n} (-1)^r s_r t^{n-r}, \tag{2.4}$$

where $s_0 = 1$ by definition. The rth mean curvature H_r of the hypersurface is then defined by

$$\binom{n}{k} H_r = s_r, \quad 0 \le r \le n. \tag{2.5}$$

In particular, when $r = 1$,

$$H_1 = \frac{1}{n}\sum_{i=1}^{n} \kappa_i = \frac{1}{n} tr(S) = H \tag{2.6}$$

is nothing but the mean curvature of M, which is the main extrinsic curvature of the hypersurface. On the other hand, $H_n = \kappa_1 \cdots \kappa_n$ is called the Gauss-Kronecker curvature of M. A hypersurface with zero $(r+1)$th mean curvature in $\mathbb{R}^{n+1}$ is called r-minimal.

The classical Newton transformations $P_r : \chi(M) \to \chi(M)$ are defined inductively by

$$P_0 = I, \qquad P_r = s_r I - S \circ P_{r-1} = \binom{n}{r} H_r I - S \circ P_{r-1} \tag{2.7}$$

for $r = 1, \ldots, n$ where I denotes the identity of $\chi(M)$. Equivalently, we have

$$P_r = \sum_{j=0}^{r} (-1)^j s_{r-j} S^j = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j. \tag{2.8}$$

Note that, by the Cayley-Hamilton Theorem stating that any operator T is annihilated by its characteristic polynomial, we have $P_n = 0$ from (2.4).

Each $P_r(p)$ is also a self-adjoint linear operator on the tangent space T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized if $\{e_1, \ldots, e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$, respectively. Then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r}(p) = \sum_{1 \le i_1 < \cdots < i_r \le n,\, i_j \ne i} \kappa_{i_1}(p) \cdots \kappa_{i_r}(p), \tag{2.9}$$

for every r, $1 \leq i \leq n$. We have the following formulas for the Newton transformation P_r [5]:

$$tr(P_r) = c_r H_r, \tag{2.10}$$

$$tr(S \circ P_r) = c_r H_{r+1}, \tag{2.11}$$

$$tr\left(S^2 \circ P_r\right) = \binom{n}{r+1}(nH_1 H_{r+1} - (n-r-1)H_{r+2}), \tag{2.12}$$

where

$$c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}. \tag{2.13}$$

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ given by

$$L_r(f) = tr\left(P_r \circ \nabla^2 f\right). \tag{2.14}$$

$\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$\left\langle \nabla^2 f(X),\ Y \right\rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad \forall X, Y \in \chi(M). \tag{2.15}$$

Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M and observe that

$$\begin{aligned} \operatorname{div}(P_r(\nabla f)) &= \sum_{i=1}^{n} \langle (\nabla_{e_i} P_r)(\nabla f), e_i \rangle + \sum_{i=1}^{n} \langle P_r(\nabla_{e_i} \nabla f), e_i \rangle \\ &= \langle \operatorname{div} P_r, \nabla f \rangle + L_r(f), \end{aligned} \tag{2.16}$$

where div denotes the divergence operator on M.

Since $\operatorname{div} P_r = 0$ (see [10]), as a consequence, from (2.16), one gets that

$$L_r(f) = \operatorname{div}(P_r(\nabla f)). \tag{2.17}$$

3. The rth Order and the rth Type of a Hypersurface

As mentioned in the introduction, there is no notion of degree for submanifolds of the Euclidean space in general. However, Chen could use the induced Riemannian structure on a submanifold to introduce a pair of well-defined numbers p and q associated with a submanifold (see [1] for the precise definition). p is a natural number and $q \geq p$ or $+\infty$. The pair $[p, q]$ is called the order of the submanifold M; more precisely, p is the lower order, and q is the upper order of the submanifold. The submanifold is said to be of finite type if its upper order is finite, and it is of infinite type if its upper order is $+\infty$ (see [1, 2] for details).

Consider an isometrically immersed closed orientable hypersurface $x : M^n \longrightarrow \mathbb{R}^{n+1}$ in the Euclidean space, with the Gauss map N, and assume that, for a fixed r, $1 \le r \le n-1$, L_r is an elliptic differential operator on $\mathcal{C}^\infty(M)$, the ring of all smooth real functions on M.

It is well known that the eigenvalues of $-L_r$ when it is elliptic (see [1], chapter 3, for properties of an elliptic operator) form a discrete infinite sequence

$$0 = \lambda_0^{L_r} < \lambda_1^{L_r} < \lambda_2^{L_r} < \cdots \nearrow \infty. \tag{3.1}$$

Let $V_k^{L_r} = \{f \in \mathcal{C}^\infty(M) : L_r f + \lambda_k^{L_r} f = 0\}$ be the eigenspace of $-L_r$ with eigenvalue $\lambda_k^{L_r}$. Then $V_k^{L_r}$ is finite dimensional. Define an inner product $(,)$ on $\mathcal{C}^\infty(M)$ by

$$(f, h) = \int_M f h \, dM, \tag{3.2}$$

where dM is the volume element of M. Then $\sum_{k=0}^{\infty} V_k^{L_r}$ is dense in $\mathcal{C}^\infty(M)$ (in L^2- sense). If we denote by $\widehat{\oplus} V_k^{L_r}$ the completion of $\sum V_k^{L_r}$, we have

$$\mathcal{C}^\infty(M) = \widehat{\oplus}_k V_k^{L_r}. \tag{3.3}$$

For each function $f \in \mathcal{C}^\infty(M)$, let f_t denote the projection of f onto the subspace $V_t^{L_r}$ $(t = 0, 1, 2, \ldots)$. Then we have the following spectral decomposition:

$$f = \sum_{t=0}^{\infty} f_t \left(\text{in } L^2 - \text{sence}\right). \tag{3.4}$$

Because $V_0^{L_r}$ is 1-dimensional, for any nonconstant function $f \in \mathcal{C}^\infty(M)$, there is a positive integer $p \ge 1$ such that $f_p \ne 0$ and

$$f - f_0 = \sum_{t \ge p} f_t, \tag{3.5}$$

where $f_0 \in V_0^{L_r}$ is a constant. If there are infinitely many f_t's which are nonzero, we put $q = +\infty$; otherwise, there is an integer $q, q \ge p$, such that $f_q \ne 0$ and

$$f - f_0 = \sum_{t=p}^{q} f_t. \tag{3.6}$$

Consider the set

$$T_f^{L_r} = \{t \in N : f_t \not\equiv 0\}. \tag{3.7}$$

The smallest element of $T_f^{L_r}$ is called the lower rth order of f and is denoted by $l.o.^{L_r}(f)$, and the supremum of $T_f^{L_r}$ is called the upper rth order of f and is denoted by $u.o.^{L_r}(f)$. A function f in $C^\infty(M)$ is said to be of L_r-finite type if $T_f^{L_r}$ is a finite set, that is, if its spectral decomposition contains only finitely many nonzero terms. Otherwise, f is said to be of L_r-infinite type. f is said to be of L_r-k type if $T_f^{L_r}$ contains exactly k elements.

For an isometrically immersed closed hypersurface $x : M^n \to \mathbb{R}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$, we put

$$x = (x_1, \ldots, x_{n+1}), \tag{3.8}$$

where x_A is the A-th component of x. For each x_A, we have

$$x_A - (x_A)_0 = \sum_{t=p_A}^{q_A} (x_A)_t, \quad A = 1, \ldots, n+1. \tag{3.9}$$

For the isometric immersion $x : M \to \mathbb{R}^{n+1}$, we put

$$p = \inf_A \{p_A\}, \qquad q = \sup_A \{q_A\}. \tag{3.10}$$

It is easy to see that p and q are independent of the choice of the Euclidean coordinate system on $\mathbb{R}^{n+1}$, and p is a positive integer and q is either $+\infty$ or $q \geq p$. Thus, p and q are well defined. Consequently, for each closed hypersurface M in $\mathbb{R}^{n+1}$ (or, more precisely, for each isometric immersion $x : M^n \to \mathbb{R}^{n+1}$), we have a pair $[p, q]$ associated with M. We call the pair $[p, q]$ the order of the hypersurface M.

By using the above notation, we have the following spectral decomposition of x in vector form:

$$x = x_0 + \sum_{t=p}^{q} x_t. \tag{3.11}$$

We define $T^{L_r}(x)$ by

$$T_x^{L_r} = \{t \in N : x_t \not\equiv 0\}. \tag{3.12}$$

The immersion x or the hypersurface M is said to be of L_r-k type if $T_x^{L_r}$ contains exactly k elements. Similarly, we can define the lower rth order and the upper rth order of the immersion.

The immersion x is said to be of L_r finite type if its upper rth order q is finite, and it is said to be of L_r infinite type if its upper order is $+\infty$.

The following Lemma states that for an isometrically immersed closed orientable hypersurface $x : M^n \to \mathbb{R}^{n+1}$, the constant vector x_0 in (3.11) is exactly the "center of mass" of M in $\mathbb{R}^{n+1}$ (i.e., $\int_M x\,dM/\mathrm{vol}(M)$, where dM is a chosen volume form of M).

Lemma 3.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a closed orientable hypersurface M into $\mathbb{R}^{n+1}$. Assume that L_r is elliptic, for some $1 \leq r \leq n-1$. Then x_0 in* (3.11) *is the center of mass of M in $\mathbb{R}^{n+1}$.*

Proof. Consider the decomposition

$$x = \sum_{t=0}^{\infty} x_t. \tag{3.13}$$

We have $L_r x_t + \lambda_t^{L_r} x_t = 0$. If $t \neq 0$, then (2.17) and the Divergence Theorem imply that

$$\int_M x_t dM = -\frac{1}{\lambda_t^{L_r}} L_r x_t dM = 0. \tag{3.14}$$

Since x_0 is a constant vector in $\mathbb{R}^{n+1}$, we obtain from (3.13) and (3.14) that

$$x_0 = \int_M \frac{x dM}{\mathrm{vol}(M)}. \tag{3.15}$$

This shows that x_0 is the center of mass of M. □

On the set of all $\mathbb{R}^{n+1}$-valued functions on M which is a real vector space, we define an inner product on such space by

$$(v, w) = \int_M \langle v, w \rangle dM \tag{3.16}$$

for any two $\mathbb{R}^{n+1}$-valued functions v, w on M, where $\langle v, w \rangle(x)$ denotes the Euclidean inner product of $v(x), w(x)$ for any $x \in M$. Then we have the following lemma.

Lemma 3.2. *For an isometric immersion $x : M \to \mathbb{R}^{n+1}$ of a closed orientable hypersurface M into $\mathbb{R}^{n+1}$ the components of the spectral decomposition* (3.11) *are mutually orthogonal, for example,*

$$(x_t, x_s) = 0, \quad t \neq s. \tag{3.17}$$

Proof. Since L_r is self-adjoint with respect to the inner product (3.16), we have

$$\lambda_t^{L_r}(x_t, x_s) = -(L_r x_t, x_s) = -(x_t, L_r x_s) = \lambda_s^{L_r}(x_t, x_s). \tag{3.18}$$

Since $\lambda_t^{L_r} \neq \lambda_s^{L_r}$, we obtain (3.17). □

Before we give our main result and to facilitate the reader, we quote Theorem 3.3 from [11] and present Theorem 3.4 about L_r-finite-type Euclidean hypersurfaces.

Theorem 3.3 (see [11]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable connected hypersurface immersed into the Euclidean space, and let L_r be the linearized operator of the* $(r+1)$-th *mean curvature of M, for some $r = 0, \ldots, n-1$. Then, one has*

$$L_r x + \lambda x = 0 \tag{3.19}$$

for a real constant λ if and only if either $\lambda = 0$ and M is r-minimal in $\mathbb{R}^{n+1}$ (i.e., $H_{r+1} = 0$ on M), or $\lambda \neq 0$ and M is an open piece of a round sphere $S^n(\varrho) \subset \mathbb{R}^{n+1}$ of radius $\varrho = (c_r/\|\lambda|)^{(1/(r+2))}$ centered at the origin of $\mathbb{R}^{n+1}$, where $c_r = (n-r)\binom{n}{r}$.

Theorem 3.4. *There is no compact Euclidean hypersurface of L_r-2 type with constant H_{r+1}, when L_r is elliptic.*

Proof. If M is of L_r-2 type, by using (3.11), the position vector field x of M in $\mathbb{R}^{n+1}$ has the following spectral decomposition:

$$x - x_0 = x_p + x_q, \quad L_r x_p + \lambda_p x_p = 0, \quad L_r x_q + \lambda_q x_q = 0, \quad \text{for some } x_0 \in \mathbb{R}^{n+1},\ p, q \in N, \tag{3.20}$$

so

$$L_r^2 x = -(\lambda_p + \lambda_q) L_r x - \lambda_p \lambda_q (x - x_0). \tag{3.21}$$

From [10], we also have

$$L_r x = c_r H_{r+1} N, \tag{3.22}$$

$$L_r^2 x = -c_r \binom{n}{r+1} H_{r+1} (n H_1 H_{r+1} - (n-r-1) H_{r+2}) N. \tag{3.23}$$

The formula (3.23) holds since H_{r+1} is a nonzero constant, see [10]. Therefore, by using (3.21), (3.22), and (3.23), we obtain that

$$-c_r \binom{n}{r+1} H_{r+1} (n H_1 H_{r+1} - (n-r-1) H_{r+2}) N = c_r (\lambda_p + \lambda_q) H_{r+1} N - \lambda_p \lambda_q (x - x_0). \tag{3.24}$$

Since $\lambda_p \lambda_q \neq 0$, from (3.24), we have $x - x_0$ that is normal to M at every point of M. So $\langle x - x_0, x - x_0 \rangle$ is a positive constant. In this case, M is an open piece of S^n centered at x_0, by Theorem 3.3, M is of L_r-1 type, which is not. □

4. The rth Order and the $(r+1)$th Mean Curvature

In this section, we will relate the notion of the rth order of a Euclidean hypersurface with the $(r+1)$th mean curvature. In particular, we will obtain some sharp estimates of the $(r+1)$th mean curvature for a closed hypersurface of the Euclidean space in terms of rth order of the hypersurface when L_r is elliptic. In the following we will state several results from [11, 12] which guarantee the ellipticity of L_r.

A classical theorem of Hadamard [12] gives three equivalent conditions on a closed connected hypersurface M^n immersed into the Euclidean space $\mathbb{R}^{n+1}$ which imply that M is a convex hypersurface (i.e., M is embedded in $\mathbb{R}^{n+1}$ and is the boundary of a convex body).

Theorem 4.1 (Hadamard Theorem, see [12]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface immersed into the Euclidean space. The following assertions are equivalent.*

(i) *The second fundamental form is definite at every point of M.*

(ii) *M is orientable, and its Gauss map is a diffeomorphism onto S^n.*

(iii) *The Gauss-Kronecker curvature never vanishes on M.*

Moreover, any of the above conditions implies that M is a convex hypersurface.

Here we observe that the convexity of a hypersurface in $\mathbb{R}^{n+1}$ is closely related to its Ricci curvature.

Theorem 4.2 (see [11]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface immersed into the Euclidean space. The following assertion is equivalent to any of the assertions ((i)–(iii)) in Hadamard theorem, and therefore it also implies that M is a convex hypersurface*

(iv) *The Ricci curvature of M is positive everywhere on M.*

Corollary 4.3 (see [11]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface isometrically immersed into the Euclidean space. If M has positive Ricci curvature, then each operator L_r on $C^\infty(M)$ is elliptic, and each r-th mean curvatures of M is positive.*

In [8] by using the concept of order, Chen obtained the following best possible lower bound of total mean curvature for a closed Euclidean hypersurface.

Theorem 4.4 (see [8]). *Let $x : M \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface isometrically immersed into the Euclidean space. Then, one has*

$$\int_M H^2 \, dM \geq \frac{\lambda_p}{n} \operatorname{vol}(M), \tag{4.1}$$

where p is the lower order of M, and $\operatorname{vol}(M)$ denotes the n-dimensional volume of M. Equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

Now we establish the corresponding result for the operator L_r (since $L_0 = \Delta$, taking $r = 0$, we recover Theorem 4.4).

Theorem 4.5. *Let $x : M \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface isometrically immersed into the Euclidean space. Assume that L_r is elliptic, for some $1 \leq r \leq n-1$. Then, one has*

$$\int_M H_{r+1}^2 \, dM \geq \frac{\lambda_p^{L_r}}{c_r} \int_M H_r \, dM, \tag{4.2}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$. In particular, if M is embedded in $\mathbb{R}^{n+1}$, then

$$\int_M H_{r+1}^2\, dM \geq (n+1)^2 \frac{\left(\lambda_1^{L_r}\right)^2}{(c_r)^2} \frac{\mathrm{vol}\,(\Omega)^2}{\mathrm{vol}\,(M)}, \tag{4.3}$$

where equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$. Here, $\mathrm{vol}\,(M)$ *denotes the n-dimensional volume of M, and Ω is the compact domain in $\mathbb{R}^{n+1}$ bounded by M, and* $\mathrm{vol}\,(\Omega)$ *denotes its $(n+1)$-dimensional volume of Ω.*

Proof. We will follow the techniques introduced by Chen (Theorem 4.4) in our context, we generalize some properties of Δ and H, respectively, to L_r and H_{r+1}. Since $L_r x = c_r H_{r+1} N$ (see [10]), and

$$x = x_0 + \sum_{t=p}^{q} x_t, \qquad L_r x_t + \lambda_t^{L_r} x_t = 0, \tag{4.4}$$

by using the inner product on the set of all $\mathbb{R}^{n+1}$-valued functions on M, defined by (3.16), we have

$$(c_r)^2 \int_M H_{r+1}^2\, dM = (c_r)^2 (H_{r+1}N, H_{r+1}N) = (L_r x, L_r x) = \sum_{t=p}^{q} \left(\lambda_t^{L_r}\right)^2 \|x_t\|^2. \tag{4.5}$$

Since, by Lemma 3.1, x_0 is the center of mass of M, we have the well-known Minkowski formula [13] as follows:

$$\int (H_r + H_{r+1}\langle x - x_0, N\rangle) dM = 0, \quad r = 0, \dots, n-1, \tag{4.6}$$

so we get that

$$(c_r) \int_M H_r\, dM = -(c_r)(x - x_0, H_{r+1}N) = -(x - x_0, L_r x) = \sum_{t=p}^{q} \lambda_t^{L_r} \|x_t\|^2. \tag{4.7}$$

Thus, by (4.5) and (4.7), we find that

$$(c_r)^2 \int_M H_{r+1}^2\, dM - (c_r)\lambda_p^{L_r} \int_M H_r\, dM = \sum_{t=p+1}^{q} \lambda_t^{L_r} \left(\lambda_t^{L_r} - \lambda_p^{L_r}\right) \|x_t\|^2 \geq 0. \tag{4.8}$$

Therefore, we obtain (4.2). Moreover, equality holds if and only if

$$L_r x + \lambda_p^{L_r} x = 0, \tag{4.9}$$

Which, by Theorem 3.3, means that M is a round sphere. If M is embedded in $\mathbb{R}^{n+1}$, by formula (15) of [11], we have

$$\lambda_1^{L_r} \leq \frac{c_r}{(n+1)^2} \frac{\mathrm{vol}(M)}{\mathrm{vol}(\Omega)^2} \int_M H_r \, dM, \tag{4.10}$$

where $\mathrm{vol}(M)$ denotes the n-dimensional volume of M, and Ω is the compact domain in $\mathbb{R}^{n+1}$ bounded by M, and $\mathrm{vol}(\Omega)$ denotes its $(n+1)$-dimensional volume of Ω. So (4.3) is obtained by (4.2) and (4.10) easily. □

By applying Theorem 4.2, Corollary 4.3, and Theorem 4.5 to positively Ricci curved hypersurfaces in $\mathbb{R}^{n+1}$, we have the following Corollary.

Corollary 4.6. *Let $x : M \to \mathbb{R}^{n+1}$ be a closed connected hypersurface of the Euclidean space with positive Ricci curvature, and let Ω be the convex body in $\mathbb{R}^{n+1}$ bounded by M. Then for every $r = 1, \ldots, n-1$, it follows that*

$$\begin{gathered} \int_M H_{r+1}^2 \, dM \geq \frac{\lambda_p^{L_r}}{c_r} \int_M H_r \, dM, \\ \int_M H_{r+1}^2 \, dM \geq \frac{\left(\lambda_1^{L_r}\right)^2}{(c_r)^2} \frac{(n+1)^2 \, \mathrm{vol}\,(\Omega)^2}{\mathrm{vol}\,(M)}, \end{gathered} \tag{4.11}$$

where equalities hold if and only if M is a round sphere in $\mathbb{R}^{n+1}$. Here, $\mathrm{vol}\,(M)$ *denotes the n-dimensional volume of M, and* $\mathrm{vol}\,(\Omega)$ *denotes its $(n+1)$-dimensional volume of Ω.*

By using the concept of order Chen in [9], we obtained the following best possible upper bound of total mean curvature for closed Euclidean hypersurface.

Theorem 4.7 (see [9]). *Let $x : M \to \mathbb{R}^{n+1}$ be an orientable closed connected isometrically immersed hypersurface into the Euclidean space. Then, one has*

$$\int_M H^2 \, dM \leq \frac{\lambda_q}{n} \mathrm{vol}\,(M), \tag{4.12}$$

where q is the upper order of M, and $\mathrm{vol}\,(M)$ *denotes the n-dimensional volume of M. Equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.*

Now we establish the corresponding result for the operator L_r (since $L_0 = \Delta$, taking $r = 0$, we recover Theorem 4.7).

Theorem 4.8. *Let $x : M \to \mathbb{R}^{n+1}$ be an orientable closed connected isometrically immersed hypersurface of the Euclidean space. Assume that L_r is elliptic on M, for some $1 \leq r \leq n-1$. Then, one has*

$$\int_M H_{r+1}^2 \, dM \leq \frac{\lambda_q^{L_r}}{c_r} \int_M H_r \, dM, \tag{4.13}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

Proof. We will follow the techniques introduced by Chen (Theorem 4.7) in our context, we generalize some properties of Δ and H, respectively, to L_r and H_{r+1}. Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface in the Euclidean space with Gauss map N. From [10], we have

$$L_r(H_{r+1}N) = -\binom{n}{r+1} H_{r+1}\nabla H_{r+1} - 2(S\circ P_r)(\nabla H_{r+1}) + \Big((L_rH_{r+1})N - H_{r+1}tr\Big(S^2\circ P_r\Big)N\Big). \tag{4.14}$$

Formula (4.14) implies that

$$\langle L_r(H_{r+1}N), H_{r+1}N\rangle = H_{r+1}L_rH_{r+1} - H_{r+1}^2 tr\Big(S^2\circ P_r\Big). \tag{4.15}$$

Furthermore, from (4.4), (4.5), and (4.7), we have

$$\begin{gathered} c_r^2\int_M H_{r+1}^2\, dM = \sum_{t=p}^{q}\Big(\lambda_t^{L_r}\Big)^2\|x_t\|^2, \\ c_r^2\int_M \langle L_r(H_{r+1}N), H_{r+1}N\rangle dM = -\sum_{t=p}^{q}\Big(\lambda_t^{L_r}\Big)^3\|x_t\|^2, \\ \mathrm{c}_r\int_M H_r\, dM = \sum_{t=p}^{q}\lambda_t^{L_r}\|x_t\|^2. \end{gathered} \tag{4.16}$$

Assume that $q \le \infty$. We put

$$\begin{aligned} \Lambda = -c_r^2\int_M \langle L_r(H_{r+1}N), H_{r+1}N\rangle dM - c_r^2\Big(\lambda_p^{L_r} + \lambda_q^{L_r}\Big)\int_M H_{r+1}^2\, dM \\ + c_r\lambda_p^{L_r}\lambda_q^{L_r}\int_M H_r\, dM. \end{aligned} \tag{4.17}$$

Then we have

$$\Lambda = \sum_{t=p+1}^{q-1}\Big(\lambda_t^{L_r} - \lambda_p^{L_r}\Big)\Big(\lambda_t^{L_r} - \lambda_q^{L_r}\Big)\|x_t\|^2 \le 0, \tag{4.18}$$

where equality holds if and only if M is either of L_r-1 type or of L_r-2 type.

Combining (4.15), (4.17), and (4.18), we find that

$$\begin{aligned} -c_r^2\int_M H_{r+1}L_rH_{r+1}\, dM + c_r^2\int_M H_{r+1}^2 tr\Big(S^2\circ P_r\Big)dM - c_r^2\Big(\lambda_p^{L_r} + \lambda_q^{L_r}\Big)\int_M H_{r+1}^2\, dM \\ + c_r\lambda_p^{L_r}\lambda_q^{L_r}\int_M H_r\, dM \le 0. \end{aligned} \tag{4.19}$$

By Proposition 3.1 of [14], we have the following equation:

$$-\int_M H_{r+1} L_r H_{r+1}\, dM = \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1}\rangle dM. \tag{4.20}$$

Since L_r is elliptic, it follows from (2.16) that the Newton transformation P_r is positive definite, so (4.20) implies that $-\int_M H_{r+1} L_r H_{r+1}\, dM \geq 0$. On the other hand, by using (2.12), we have

$$H_{r+1}^2 tr\left(S^2 \circ P_r\right) = \frac{H_{r+1}^4}{H_r} + \frac{\left\{n\binom{n}{r+1} H_1 H_{r+1}^3 H_r - (n-r-1)\binom{n}{r+1} H_{r+2} H_{r+1}^2 H_r - H_{r+1}^4\right\}}{H_r}. \tag{4.21}$$

We suppose that $B = \{n\binom{n}{r+1} H_1 H_{r+1}^3 H_r - (n-r-1)n\binom{n}{r+1} H_{r+2} H_{r+1}^2 H_r - H_{r+1}^4\}/H_r$, and we show that B is positive.

For every $1 \leq j \leq n$, one has the following inequalities (see, for instance, [15, Theorems 51 and 144]):

$$H_{j-1} H_{j+1} \leq H_j^2. \tag{4.22}$$

Since each $H_j > 0$ for $j = 1, \ldots, r$, this is equivalent to

$$\frac{H_{r+1}}{H_r} \leq \frac{H_r}{H_{r-1}} \leq \cdots \frac{H_2}{H_1} \leq H_1. \tag{4.23}$$

And these inequalities imply that

$$H_1 H_r \geq H_{r+1}. \tag{4.24}$$

So by using (4.22) and (4.24), we get that B is positive.

Combining (4.19), (4.20), and (4.21) and Schwartz's inequality, we get that

$$\begin{aligned}
0 &\geq c_r^2 \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1}\rangle dM + c_r^3 \int_M \frac{H_{r+1}^4}{H_r\, dM} + c_r^3 \int_M B\, dM \\
&\quad - c_r^2\left(\lambda_p^{L_r} + \lambda_q^{L_r}\right) \int_M H_{r+1}^2\, dM + c_r \lambda_p^{L_r} \lambda_q^{L_r} \int_M H_r\, dM \\
&\geq c_r^2 \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1}\rangle dM + \frac{c_r^3\left(\int_M H_{r+1}^2\, dM\right)^2}{\int_M H_r\, dM} \\
&\quad + c_r^3 \int_M B\, dM - c_r^2\left(\lambda_p^{L_r} + \lambda_q^{L_r}\right) \int_M H_{r+1}^2\, dM + c_r \lambda_p^{L_r} \lambda_q^{L_r} \int_M H_r\, dM.
\end{aligned} \tag{4.25}$$

Hence, we obtain that

$$0 \geq c_r \int_M H_r\, dM \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1} \rangle dM + \int_M H_r\, dM \int_M B dM + \left(c_r \int_M H_{r+1}^2\, d\mathrm{M} - \lambda_p^{L_r} \int_M H_r\, dM \right) \left(c_r \int_M H_{r+1}^2\, dM - \lambda_q^{L_r} \int_M H_r\, dM \right) \tag{4.26}$$

By inequalities (4.2) and (4.26), we obtain (4.13). If in (4.13) the equality holds, then all the inequalities in (4.17) through (4.26) have to be equalities. Thus, we find that M is either of L_r-1 type or of L_r-2 type, and H_{r+1} is constant. So, by Theorems 3.3 and 3.4, M is a round sphere. □

By applying Theorem 4.2, Corollary 4.3, and Theorem 4.8 to positively Ricci curved hypersurfaces in $\mathbb{R}^{n+1}$, we have the following Corollary.

Corollary 4.9. *Let $x : M \to \mathbb{R}^{n+1}$ be a closed connected hypersurface of the Euclidean space with positive Ricci curvature. Then for every $r = 1, \ldots, n-1$ the following inequality holds:*

$$\int_M H_{r+1}^2\, dM \leq \frac{\lambda_q^{L_r}}{c_r} \int_M H_r\, dM, \tag{4.27}$$

where equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

An immediate consequence of Corollaries 4.6 and 4.9 is the following.

Corollary 4.10. *Let $x : M \to \mathbb{R}^{n+1}$ be a closed connected hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with positive Ricci curvature. If H_{r+1} is constant, then*

$$\lambda_p^{L_r} \leq \frac{c_r H_{r+1}^2}{H_r} \leq \lambda_q^{L_r}, \tag{4.28}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

References

[1] B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, vol. 1 of *Series in Pure Mathematics*, World Scientific, Singapore, 1984.
[2] B.-Y. Chen, "A report on submanifolds of finite type," *Soochow Journal of Mathematics*, vol. 22, no. 2, pp. 117–337, 1996.
[3] R. Hartshorne, *Algebraic Geometry*, Springer, New York, NY, USA, 1977.
[4] J. Nash, "C^1 isometric imbeddings," *Annals of Mathematics*, vol. 60, pp. 383–396, 1954.
[5] R. C. Reilly, "Variational properties of functions of the mean curvatures for hypersurfaces in space forms," *Journal of Differential Geometry*, vol. 8, pp. 465–477, 1973.
[6] J. Hounie and M. L. Leite, "Two-ended hypersurfaces with zero scalar curvature," *Indiana University Mathematics Journal*, vol. 48, no. 3, pp. 867–882, 1999.
[7] S. M. B. Kashani, "On some L_1-finite type (hyper)surfaces in R^{n+1}," *Bulletin of the Korean Mathematical Society*, vol. 46, no. 1, pp. 35–43, 2009.
[8] B.-y. Chen, "On total curvature of immersed manifolds. IV. Spectrum and total mean curvature," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 7, no. 3, pp. 301–311, 1979.

[9] B.-Y. Chen, "On the total curvature of immersed manifolds. VI. Submanifolds of finite type and their applications," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 11, no. 3, pp. 309–328, 1983.

[10] L. J. Alías and N. Gürbüz, "An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures," *Geometriae Dedicata*, vol. 121, pp. 113–127, 2006.

[11] L. J. Alías and J. M. Malacarne, "On the first eigenvalue of the linearized operator of the higher order mean curvature for closed hypersurfaces in space forms," *Illinois Journal of Mathematics*, vol. 48, no. 1, pp. 219–240, 2004.

[12] J. Hadamard, "Les surfaces à courbures opposées et leurs lignes géodésiques," *Journal de Mathématiques Pures et Appliquées*, vol. 4, pp. 27–73, 1898.

[13] C.-C. Hsiung, "Some integral formulas for closed hypersurfaces," *Mathematica Scandinavica*, vol. 2, pp. 286–294, 1954.

[14] J. L. M. Barbosa and A. G. Colares, "Stability of hypersurfaces with constant r-mean curvature," *Annals of Global Analysis and Geometry*, vol. 15, no. 3, pp. 277–297, 1997.

[15] G. Harday, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1978.