

Research Article

A Note on Some Identities of Frobenius-Euler Numbers and Polynomials

J. Choi,¹ D. S. Kim,² T. Kim,³ and Y. H. Kim¹

¹ Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

² Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

³ Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Correspondence should be addressed to D. S. Kim, dskim@sogang.ac.kr
and T. Kim, tkkim@kw.ac.kr

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The purpose of this paper is to give some identities on the Frobenius-Euler numbers and polynomials by using the fermionic p -adic q -integral equation on \mathbb{Z}_p .

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will denote the ring of p -adic rational integers, the field of p -adic rational numbers, and the completion of algebraic closure of \mathbb{Q}_p , respectively. Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p -adic absolute value on \mathbb{C}_p is normalized so that $|p|_p = 1/p$. Assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

Let f be a continuous function on \mathbb{Z}_p . Then the fermionic p -adic q -integral on \mathbb{Z}_p for f is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \quad (\text{see [1]}). \quad (1.1)$$

From (1.1), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + (1+q) \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \quad (1.2)$$

where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$ (see [1]). The ordinary Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.3)$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1–10]). In the special case, $x = 0$, $E_n(0) = E_n$ is called the n th Euler number.

As the extension of (1.3), the Frobenius-Euler polynomials are defined by

$$\frac{1-q}{e^t - q} e^{xt} = \sum_{n=0}^{\infty} H_n(q, x) \frac{t^n}{n!}, \quad (\text{see [2]}). \quad (1.4)$$

In the special case, $x = 0$, $H_n(q, 0) = H_n(q)$ is called the n th Frobenius-Euler number.

By (1.3) and (1.4), we easily get $H_n(-1, x) = E_n(x)$.

From (1.4), we note that

$$H_n(q, x) = \sum_{l=0}^n \binom{n}{l} H_l(q) x^{n-l} = (H(q) + x)^n, \quad (\text{see [2]}), \quad (1.5)$$

with the usual convention about replacing $H(q)^n$ by $H_n(q)$.

In this paper, we consider the fermionic p -adic q -integral on \mathbb{Z}_p for the Frobenius-Euler numbers and polynomials. From these p -adic q -integrals on \mathbb{Z}_p , we derive some new and interesting identities on the Frobenius-Euler numbers and polynomials.

2. Identities on the Frobenius-Euler Numbers

From (1.2) and (1.4), we can derive the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{1+q^{-1}}{e^t + q^{-1}} e^{xt} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!}. \quad (2.1)$$

Thus, by (2.1), we get Witt's formula for $H_n(-q^{-1}, x)$ as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = H_n(-q^{-1}, x), \quad n \in \mathbb{Z}_+. \quad (2.2)$$

By (1.5) and (2.1), we get

$$(H(-q^{-1}) + 1)^n + q^{-1} H_n(-q^{-1}) = \begin{cases} 1 + q^{-1}, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad (2.3)$$

with the usual convention about replacing $H(-q^{-1})^n$ by $H_n(-q^{-1})$.

From (1.5) and (2.3), we note that

$$H_0(-q^{-1}) = 1, \quad H_n(-q^{-1}, 1) + q^{-1}H_n(-q^{-1}) = 0, \quad \text{if } n > 0. \quad (2.4)$$

By (2.1) and (2.2), we get

$$\int_{\mathbb{Z}_p} (\mathbf{y} + 1 - x)^n d\mu_{-q}(\mathbf{y}) = (-1)^n \int_{\mathbb{Z}_p} (\mathbf{y} + x)^n d\mu_{-q^{-1}}(\mathbf{y}). \quad (2.5)$$

Therefore, by (2.5), we obtain the following lemma.

Lemma 2.1. For $n \in \mathbb{Z}_+$, one has

$$H_n(-q^{-1}, 1 - x) = (-1)^n H_n(-q, x). \quad (2.6)$$

From (2.3), we can derive the following:

$$\begin{aligned} q^2 H_n(-q^{-1}, 2) - q^2 - q &= q^2 \sum_{l=0}^n \binom{n}{l} (H(-q^{-1}) + 1)^l - q^2 - q \\ &= q \sum_{l=1}^n \binom{n}{l} q (H(-q^{-1}) + 1)^l - q \\ &= -q \sum_{l=0}^n \binom{n}{l} H_l(-q^{-1}) \\ &= -(1 + q)\delta_{0,n} + H_n(-q^{-1}), \end{aligned} \quad (2.7)$$

where $\delta_{k,n}$ is the Kronecker symbol.

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$, one has

$$H_n(-q^{-1}, 2) = 1 + q^{-1} - q^{-2}(1 + q)\delta_{0,n} + q^{-2}H_n(-q^{-1}). \quad (2.8)$$

First we consider the fermionic p -adic q -integral on \mathbb{Z}_p for the n th Frobenius-Euler polynomials as follows:

$$\begin{aligned}
 I_1 &= \int_{\mathbb{Z}_p} H_n(-q^{-1}, x) d\mu_{-q}(x) \\
 &= \sum_{l=0}^n \binom{n}{l} H_l(-q^{-1}) \int_{\mathbb{Z}_p} x^{n-l} d\mu_{-q}(x) \\
 &= \sum_{l=0}^n \binom{n}{l} H_l(-q^{-1}) H_{n-l}(-q^{-1}), \quad \text{where } n \in \mathbb{Z}_+.
 \end{aligned} \tag{2.9}$$

On the other hand, by (2.5) and Lemma 2.1, we get

$$\begin{aligned}
 I_1 &= \int_{\mathbb{Z}_p} H_n(-q^{-1}, x) d\mu_{-q}(x) = (-1)^n \int_{\mathbb{Z}_p} H_n(-q, 1-x) d\mu_{-q}(x) \\
 &= (-1)^n \sum_{l=0}^n \binom{n}{l} H_{n-l}(-q) \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-q}(x) \\
 &= \sum_{l=0}^n \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) \int_{\mathbb{Z}_p} (x-1)^l d\mu_{-q}(x) \\
 &= \sum_{l=0}^n \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) H_l(-q^{-1}, -1).
 \end{aligned} \tag{2.10}$$

From Lemma 2.1, Theorem 2.2, and (2.10), we note that

$$\begin{aligned}
 I_1 &= \sum_{l=0}^n \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) H_l(-q^{-1}, -1) \\
 &= \sum_{l=0}^n \binom{n}{l} (-1)^n H_{n-l}(-q) H_l(-q, 2) \\
 &= \sum_{l=0}^n \binom{n}{l} (-1)^n H_{n-l}(-q) \{1 + q - q^2(1 + q^{-1})\delta_{0,l} + q^2 H_l(-q)\} \\
 &= (-1)^n (1+q) ((1+q)\delta_{0,n} - qH_n(-q)) - H_n(-q)(q+q^2)(-1)^n \\
 &\quad + (-1)^n q^2 \sum_{l=0}^n \binom{n}{l} H_{n-l}(-q) H_l(-q).
 \end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. For $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} \sum_{l=0}^n \binom{n}{l} H_l(-q^{-1}) H_{n-l}(-q^{-1}) &= (-1)^n (1+q) ((1+q)\delta_{0,n} - 2qH_n(-q)) \\ &+ (-1)^n q^2 \sum_{l=0}^n \binom{n}{l} H_{n-l}(-q) H_l(-q). \end{aligned} \tag{2.12}$$

In particular, for $n \in \mathbb{N}$, one has

$$\begin{aligned} \sum_{l=0}^n \binom{n}{l} H_l(-q^{-1}) H_{n-l}(-q^{-1}) &= 2(-1)^{n+1} q(1+q) H_n(-q) \\ &+ (-1)^n q^2 \sum_{l=0}^n \binom{n}{l} H_{n-l}(-q) H_l(-q). \end{aligned} \tag{2.13}$$

Let us consider the following fermionic p -adic q -integral on \mathbb{Z}_p for the product of Bernoulli and Frobenius-Euler polynomials as follows:

$$\begin{aligned} I_2 &= \int_{\mathbb{Z}_p} B_m(x) H_n(-q^{-1}, x) d\mu_{-q}(x) \\ &= \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q^{-1}) \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\ &= \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q^{-1}) H_{k+l}(-q^{-1}). \end{aligned} \tag{2.14}$$

It is known that $B_n(x) = (-1)^n B_n(1-x)$.

On the other hand, by Lemma 2.1, we get

$$\begin{aligned} I_2 &= (-1)^{m+n} \int_{\mathbb{Z}_p} B_m(1-x) H_n(-q, 1-x) d\mu_{-q}(x) \\ &= (-1)^{m+n} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) \int_{\mathbb{Z}_p} (1-x)^{k+l} d\mu_{-q}(x) \\ &= (-1)^{m+n} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) \{ (1+q) - q^2(1+q^{-1})\delta_{0,k+l} + q^2 H_{k+l}(-q) \} \\ &= (-1)^{m+n} (1+q) B_m(1) H_n(-q, 1) - (q^2+q) (-1)^{m+n} B_m H_n(-q) \\ &\quad + q^2 (-1)^{m+n} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) H_{k+l}(-q). \end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. For $m, n \in \mathbb{Z}_+$, one has

$$\begin{aligned} & \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q^{-1}) H_{k+l}(-q^{-1}) \\ &= (-1)^{m+n} (1+q) B_m(1) H_n(-q, 1) - (q^2 + q) (-1)^{m+n} B_m H_n(-q) \\ & \quad + q^2 (-1)^{m+n} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) H_{k+l}(-q). \end{aligned} \quad (2.16)$$

In particular, for $m \in \mathbb{N} - \{1\}$, $n \in \mathbb{N}$, one has

$$\begin{aligned} & \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q^{-1}) H_{k+l}(-q^{-1}) \\ &= 2(-1)^{m+n+1} (q^2 + q) B_m H_n(-q) \\ & \quad + q^2 (-1)^{m+n} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) H_{k+l}(-q). \end{aligned} \quad (2.17)$$

It is known that $x^n = (1/(n+1)) \sum_{l=0}^n \binom{n+1}{l} B_l(x)$. Let us consider the following fermionic p -adic q -integral on \mathbb{Z}_p :

$$\begin{aligned} \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) &= \frac{1}{n+1} \sum_{l=0}^n \binom{n+1}{l} \int_{\mathbb{Z}_p} B_l(x) d\mu_{-q}(x). \\ &= \frac{1}{n+1} \sum_{l=0}^n \binom{n+1}{l} \sum_{k=0}^l \binom{l}{k} B_{l-k} \int_{\mathbb{Z}_p} x^k d\mu_{-q}(x) \\ &= \frac{1}{n+1} \sum_{l=0}^n \binom{n+1}{l} \sum_{k=0}^l \binom{l}{k} B_{l-k} H_k(-q^{-1}). \end{aligned} \quad (2.18)$$

Therefore by (2.18), we obtain the following theorem.

Theorem 2.5. For $n \in \mathbb{Z}_+$, one has

$$H_n(-q^{-1}) = \frac{1}{n+1} \sum_{l=0}^n \binom{n+1}{l} \sum_{k=0}^l \binom{l}{k} B_{l-k} H_k(-q^{-1}). \tag{2.19}$$

From (1.3), we can derive the following:

$$x^n = E_n(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x). \tag{2.20}$$

Let us take the fermionic p -adic q -integral on \mathbb{Z}_p in (2.20) as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} E_n(x) d\mu_{-q}(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \int_{\mathbb{Z}_p} E_l(x) d\mu_{-q}(x) \\ &= \sum_{l=0}^n \binom{n}{l} E_{n-l} H_l(-q^{-1}) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^l \binom{l}{k} E_{l-k} H_k(-q^{-1}). \end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.6. For $n \in \mathbb{N}$, one has

$$H_n(-q^{-1}) = \sum_{l=0}^n \binom{n}{l} E_{n-l} H_l(-q^{-1}) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^l \binom{l}{k} E_{l-k} H_k(-q^{-1}). \tag{2.22}$$

By Theorems 2.5 and 2.6, we obtain the following corollary.

Corollary 2.7. For $n \in \mathbb{N}$, one has

$$\begin{aligned} &\frac{1}{n+1} \sum_{l=0}^n \binom{n+1}{l} \sum_{k=0}^l \binom{l}{k} B_{l-k} H_k(-q^{-1}) \\ &= \sum_{l=0}^n \binom{n}{l} E_{n-l} H_l(-q^{-1}) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^l \binom{l}{k} E_{l-k} H_k(-q^{-1}). \end{aligned} \tag{2.23}$$

By (1.3), we easily get $E_n(x) = (-1)^n E_n(1-x)$.

Thus, we have

$$\begin{aligned}
& \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \\
&= (-1)^n \int_{\mathbb{Z}_p} E_n(1-x) d\mu_{-q}(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} E_l(1-x) d\mu_{-q}(x) \\
&= (-1)^n \sum_{l=0}^n \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-q}(x) \\
&\quad + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^l \binom{l}{k} E_{l-k} \int_{\mathbb{Z}_p} (1-x)^k d\mu_{-q}(x) \\
&= \sum_{l=0}^n \binom{n}{l} E_{n-l} (-1)^{n-l} H_l(-q^{-1}, -1) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^l \binom{l}{k} E_{l-k} (-1)^{l-k} H_k(-q^{-1}, -1) \\
&= \sum_{l=0}^n \binom{n}{l} E_{n-l} (-1)^n H_l(-q, 2) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^l \binom{l}{k} E_{l-k} (-1)^l H_k(-q, 2).
\end{aligned} \tag{2.24}$$

Therefore, by (2.24), we obtain the following theorem.

Theorem 2.8. For $n \in \mathbb{N}$, one has

$$\begin{aligned}
H_n(-q^{-1}) &= \sum_{l=0}^n \binom{n}{l} E_{n-l} (-1)^n H_l(-q, 2) \\
&\quad + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^l \binom{l}{k} E_{l-k} (-1)^l H_k(-q, 2).
\end{aligned} \tag{2.25}$$

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