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Research Article

Trigonometric Approximation of Signals (Functions) Belonging to $W(L^r, \xi(t))$ Class by Matrix $(C^1 \cdot N_p)$ Operator

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Various investigators such as Khan (1974), Chandra (2002), and Liendler (2005) have determined the degree of approximation of 2π-periodic signals (functions) belonging to $\mathrm{Lip}(\alpha, r)$ class of functions through trigonometric Fourier approximation using different summability matrices with monotone rows. Recently, Mittal et al. (2007 and 2011) have obtained the degree of approximation of signals belonging to $\mathrm{Lip}(\alpha, r)$- class by general summability matrix, which generalize some of the results of Chandra (2002) and results of Leindler (2005), respectively. In this paper, we determine the degree of approximation of functions belonging to $\mathrm{Lip}\,\alpha$ and $W(L^r, \xi(t))$ classes by using Cesáro-Nörlund $(C^1 \cdot N_p)$ summability without monotonicity condition on $\{p_n\}$, which in turn generalizes the results of Lal (2009). We also note some errors appearing in the paper of Lal (2009) and rectify them in the light of observations of Rhoades et al. (2011).

1. Introduction

For a given signal (function) $f \in L^r := L^r[0, 2\pi]$, $r \geq 1$, let

$$s_n(f) = s_n(f; x) = \left(\frac{a_0}{2}\right) + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} u_k(f; x) \tag{1.1}$$

denote the partial sum, called trigonometric polynomial of degree (or order) n, of the first $(n+1)$ terms of the Fourier series of f. Let $\{p_n\}$ be a nonnegative sequence of real numbers such that $P_n (= \sum_{k=0}^{n} p_k \neq 0) \to \infty$ as $n \to \infty$ and $P_{-1} = 0 = p_{-1}$.

Define

$$N_n(f) = N_n(f;x) = P_n^{-1}\sum_{k=0}^{n} p_{n-k}s_k(f;x), \quad \forall n \geq 0, \tag{1.2}$$

the Nörlund (N_p) means of the sequence $s_n(f)$ or Fourier series of f. The Fourier series of f is said to be Nörlund (N_p) summable to $s(x)$ if $N_n(f;x) \to s(x)$ as $n \to \infty$. The Fourier series of f is called Cesáro-Nörlund ($C^1 \cdot N_p$) summable to $S(x)$ if

$$t_n^{CN}(f) = (n+1)^{-1}\sum_{k=0}^{n} P_k^{-1}\sum_{i=0}^{k} p_{k-i}s_i(f;x) \longrightarrow S(x) \quad \text{as } n \longrightarrow \infty. \tag{1.3}$$

We note that $N_n(f)$ and $t_n^{CN}(f)$ are also trigonometric polynomials of degree (or order) n.

Some interesting applications of the Cesáro summability can be seen in [1, 2].

The L^r-norm of signal f is defined by

$$\|f\|_r = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r dx\right)^{1/r} \quad (1 \leq r < \infty), \qquad \|f\|_\infty = \sup_{x\in[0,2\pi]} |f(x)|. \tag{1.4}$$

A signal (function) f is approximated by trigonometric polynomials $T_n(f)$ of degree n, and the degree of approximation $E_n(f)$ is given by

$$E_n(f) = \operatorname*{Min}_{n} \|f(x) - T_n(f)\|_r. \tag{1.5}$$

This method of approximation is called trigonometric Fourier approximation.

A signal (function) f is said to belong to the class Lip α if $|f(x+t) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1$, and $f \in \text{Lip}(\alpha, r)$ if $\|f(x+t) - f(x)\|_r = O(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1$.

For a positive increasing function $\xi(t)$ and $r \geq 1$, $f \in \text{Lip}(\xi(t), r)$ if $\|f(x+t) - f(x)\|_r = O(\xi(t))$, and $f \in W(L^r, \xi(t))$ if $\|[f(x+t) - f(x)]\sin^\beta(x/2)\|_r = O(\xi(t)), \beta \geq 0$.

If $\beta = 0$, then $W(L^r, \xi(t))$ reduces to $\text{Lip}(\xi(t), r)$, and if $\xi(t) = t^\alpha$ $(0 < \alpha \leq 1)$, then $\text{Lip}(\xi(t), r)$ class coincides with the class $\text{Lip}(\alpha, r)$. $\text{Lip}(\alpha, r) \to \text{Lip}\,\alpha$ for $r \to \infty$.

We also write

$$\begin{gathered}
\phi(x,t) = \phi(t) = f(x+t) + f(x-t) - 2f(x), \\
K(n,t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{i=0}^{k} p_i \frac{\sin(k-i+1/2)t}{\sin(t/2)},
\end{gathered} \tag{1.6}$$

$\tau = [1/t]$, the greatest integer contained in $1/t$, $P_\tau = P[1/t]$, $\Delta p_k \equiv p_k - p_{k+1}$.

2. Known Results

Chandra [3] and Khan [4] have obtained the error estimates $\|N_n(f;x) - f(x)\|_r = O(n^{-\alpha})$ in $\text{Lip}(\alpha, r)$ class using monotonicity conditions on the means generating sequence $\{p_n\}$, which was generalized by Leindler [5] to almost monotone weights $\{p_n\}$ and by Mittal et al. [6] to general summability matrix. Further, Mittal et al. [7] have extended the results of Leindler [5] to general summability matrix, which in turn generalizes some results of Chandra [3] and Mittal et al. [6]. Recently, Lal [8] has determined the degree of approximation of the functions belonging to Lip α and $W(L^r, \xi(t))$ classes using Cesáro-Nörlund ($C^1 \cdot N_p$) summability with nonincreasing weights $\{p_n\}$. He proved the following theorem.

Theorem 2.1. *Let N_p be a regular Nörlund method defined by a sequence $\{p_n\}$ such that*

$$P_\tau \sum_{k=\tau}^{n} P_k^{-1} = O(n+1). \tag{2.1}$$

Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function belonging to Lip α $(0 < \alpha \leq 1)$, then the degree of approximation of f by $C^1 \cdot N_p$ means of its Fourier series is given by

$$\sup_{x\in[0,2\pi]} \left| t_n^{CN}(x) - f(x) \right| = \left\| t_n^{CN} - f \right\|_\infty = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0 < \alpha < 1, \\ O\left(\dfrac{\log(n+1)\pi e}{(n+1)}\right), & \alpha = 1. \end{cases} \tag{2.2}$$

Theorem 2.2. *If f is a 2π-periodic function and Lebesgue integrable on [0, 2π] and is belonging to $W(L^r, \xi(t))$ class, then its degree of approximation by $C^1 \cdot N_p$ means of its Fourier series is given by*

$$\left\| t_n^{CN} - f \right\|_r = O\left((n+1)^{\beta+1/r} \xi\left((n+1)^{-1} \right) \right), \tag{2.3}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ be a decreasing sequence,} \tag{2.4}$$

$$\left\{ \int_0^{1/(n+1)} \left(\frac{t|\phi(t)| \sin^\beta t}{\xi(t)} \right)^r dt \right\}^{1/r} = O\left((n+1)^{-1} \right), \tag{2.5}$$

$$\left\{ \int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right)^r dt \right\}^{1/r} = O\left((n+1)^{\delta} \right), \tag{2.6}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $r^{-1} + s^{-1} = 1$, $1 \leq r \leq \infty$, conditions (2.5) *and* (2.6) *hold uniformly in x.*

Remark 2.3. In the proof of Theorem 2.1 of Lal [8, page 349], the estimate for $\alpha = 1$ is obtained as

$$O\left(\frac{1}{n+1}\right) + O\left(\frac{\log(n+1)\pi}{n+1}\right) = O\left(\frac{\log e}{n+1}\right) + O\left(\frac{\log(n+1)\pi}{n+1}\right) = O\left(\frac{\log(n+1)\pi e}{n+1}\right). \tag{2.7}$$

Since $1/(n+1) \leq \log(n+1)\pi/(n+1)$, the e is not needed in (2.2) for the case $\alpha = 1$ (cf. [9, page 6870]).

Remark 2.4. (i) The author has used monotonicity condition on sequence $\{p_n\}$ in the proof of Theorem 2.1 and Theorem 2.2, but not mentioned it in the statements. Further in condition (2.4), $\{\xi(t)/t\}$ is a function of t not a sequence.

(ii) The condition (2.5) of Theorem 2.2 leads to the divergent integral $\int_{\varepsilon}^{1/(n+1)} t^{-(\beta+1)s}dt$ as $\varepsilon \to 0$ and $\beta \geq 0$ [8, page 349]. Also in [8, pages 349-350], the author while writing the proof of Theorem 2.2 has used $\sin t \geq 2t/\pi$ in the interval $[1/(n+1), \pi]$, which is not valid for $t = \pi$.

3. Main Results

The observations of Remarks 2.3 and 2.4 motivated us to determine a proper set of conditions to prove Theorems 2.1 and 2.2 without monotonocity on $\{p_n\}$. More precisely, we prove the following theorem.

Theorem 3.1. *Let N_p be the Nörlund summability matrix generated by the nonnegative sequence $\{p_n\}$, which satisfies*

$$(n+1)p_n = O(P_n), \quad \forall n \geq 0. \tag{3.1}$$

Then the degree of approximation of a 2π-periodic signal (function) $f \in \mathrm{Lip}\,\alpha$ by $C^1 \cdot N_p$ means of its Fourier series is given by

$$\left\| t_n^{CN}(f) - f(x) \right\|_{\infty} = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1. \end{cases} \tag{3.2}$$

Theorem 3.2. *Let the condition* (3.1) *be satisfied. Then the degree of approximation of a 2π-periodic signal (function) $f \in W(L^r, \xi(t))$ with $0 \leq \beta \leq 1 - 1/r$ by $C^1 \cdot N_p$ means of its Fourier series is given by*

$$\left\| t_n^{CN}(f) - f(x) \right\|_r = O\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right), \tag{3.3}$$

provided positive increasing function $\xi(t)$ satisfies the condition (2.4) *and*

$$\left\{ \int_0^{\pi/n} \left(\frac{|\phi(t)| \sin^{\beta}(t/2)}{\xi(t)} \right)^r dt \right\}^{1/r} = O(1), \tag{3.4}$$

$$\left\{ \int_{\pi/n}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^r dt \right\}^{1/r} = O\left(n^{\delta}\right), \tag{3.5}$$

where δ is an arbitrary number such that $s(\beta-\delta)-1>0$, $r^{-1}+s^{-1}=1$, $r\geq 1$, conditions (3.4) *and* (3.5) *hold uniformly in x.*

Remark 3.3. For nonincreasing sequence $\{p_n\}$, we have

$$P_n=\sum_{k=0}^{n}p_k\geq p_n\sum_{k=0}^{n}1=(n+1)p_n, \quad \text{that is, } (n+1)p_n=O(P_n). \tag{3.6}$$

Thus condition (3.1) holds for nonincreasing sequence $\{p_n\}$; hence our Theorems 3.1 and 3.2 generalize Theorems 2.1 and 2.2, respectively.

Note 1. Using condition (2.4), we get$(n/\pi)\xi(\pi/n)\leq n\xi(1/n)$.

4. Lemmas

For the proof of our Theorems, we need the following lemmas.

Lemma 4.1 (see [10, 5.11]). *If $\{p_n\}$ is nonnegative and nonincreasing sequence, then for $0\leq a<b\leq\infty, 0\leq t\leq\pi$ and for any n*

$$\left|\sum_{k=a}^{b}p_k e^{i(n-k)t}\right|=\begin{cases}O(P(t^{-1})), & \text{for any } a,\\ O(t^{-1}p_a), & \text{for } a\geq t^{-1}.\end{cases} \tag{4.1}$$

Lemma 4.2 (see [8, page 348]). *For $0<t\leq\pi/n$, $K(n,t)=O(n)$.*

Lemma 4.3. *If $\{p_n\}$ is nonnegative sequence satisfying* (3.1), *then for $\pi n^{-1}<t\leq\pi$,*

$$K(n,t)=O\left(\frac{t^{-2}}{(n+1)}\right)+O\left(t^{-1}\right). \tag{4.2}$$

Proof. We have

$$\begin{aligned}K(n,t)&=\frac{1}{2\pi(n+1)\sin(t/2)}\sum_{k=0}^{n}P_k^{-1}\sum_{r=0}^{k}p_r\sin\left(k-r+\frac{1}{2}\right)t\\&=\frac{1}{2\pi(n+1)\sin(t/2)}\left(\sum_{k=0}^{\tau}+\sum_{k=\tau+1}^{n}\right)\left(P_k^{-1}\sum_{r=0}^{k}p_r\sin\left(k-r+\frac{1}{2}\right)t\right)\\&=K_1(n,t)+K_2(n,t), \text{say.}\end{aligned} \tag{4.3}$$

Now, using $(\sin t/2)^{-1}\leq\pi/t$, for $0<t\leq\pi$, we get

$$|K_1(n,t)|=O\left((n+1)^{-1}t^{-1}\right)\sum_{k=0}^{\tau}\left(P_k^{-1}\sum_{r=0}^{k}p_r\right)=O\left(\frac{\tau t^{-1}}{(n+1)}\right)=O\left(\frac{t^{-2}}{(n+1)}\right). \tag{4.4}$$

Using $(\sin t/2)^{-1} \le \pi/t$, for $0 < t \le \pi$ and changing the order of summation, we have

$$\begin{aligned}
|K_2(n,t)| &= O\left(t^{-1}(n+1)^{-1}\right)\left|\sum_{k=\tau+1}^{n} P_k^{-1}\sum_{r=0}^{k} p_r \sin\left(k-r+\frac{1}{2}\right)t\right| \\
&= O\left(t^{-1}(n+1)^{-1}\right)\left|\sum_{r=0}^{\tau+1} p_r \sum_{k=\tau+1}^{n} P_k^{-1}\sin\left(k-r+\frac{1}{2}\right)t\right. \\
&\qquad \left. + \sum_{r=\tau+1}^{n} p_r \sum_{k=r}^{n} P_k^{-1}\sin\left(k-r+\frac{1}{2}\right)t\right|.
\end{aligned} \tag{4.5}$$

Again using $(\sin t/2)^{-1} \le \pi/t$, for $0 < t \le \pi$, Lemma 4.1, (in view of P_n being positive and ${P_{n+1}}^{-1} \le {P_n}^{-1}$ for all $n \ge 0$) and $t^{-1} < \tau + 1$, we get

$$\left|\sum_{r=0}^{\tau+1} p_r \sum_{k=\tau+1}^{n} P_k^{-1}\sin\left(k-r+\frac{1}{2}\right)t\right| \le \left(\sum_{r=0}^{\tau+1} p_r \left|\sum_{k=\tau+1}^{n} P_k^{-1} e^{i(k-r)t}\right|\right) = O\left(t^{-1}{P_{\tau+1}}^{-1}\right)\sum_{r=0}^{\tau+1} p_r = O\left(\frac{1}{t}\right). \tag{4.6}$$

Using Abel's transformation, we get

$$\begin{aligned}
\sum_{k=r}^{n} P_k^{-1}\sin\left(k-r+\frac{1}{2}\right)t &= \sum_{k=r}^{n-1}\left(\Delta P_k^{-1}\right)\sum_{j=0}^{k}\sin\left(k-j+\frac{1}{2}\right)t \\
&\quad + P_n^{-1}\sum_{j=0}^{n}\sin\left(k-j+\frac{1}{2}\right)t - P_r^{-1}\sum_{j=0}^{r-1}\sin\left(k-j+\frac{1}{2}\right)t \\
&= O\left(\frac{1}{t}\right)\left(\sum_{k=r}^{n-1}\left|\Delta P_k^{-1}\right| + P_n^{-1} + P_r^{-1}\right) = O\left(\frac{1}{t}\right)\left(P_n^{-1} + P_r^{-1}\right),
\end{aligned} \tag{4.7}$$

in view of $(\sin t/2)^{-1} \le \pi/t$, for $0 < t \le \pi$ and $P_n \ge P_{n-1}$ for all $n \ge 0$.

Combining (4.5)–(4.7), we get

$$\begin{aligned}
|K_2(n,t)| &= O\left(t^{-2}(n+1)^{-1}\right)\left(1 + \sum_{r=\tau+1}^{n} p_r\left(P_n^{-1} + P_r^{-1}\right)\right) \\
&= O\left(t^{-2}(n+1)^{-1}\right)\left(1 + P_n^{-1}\sum_{r=0}^{n} p_r + \sum_{r=\tau+1}^{n}\left(\frac{p_r}{P_r}\right)\right) \\
&= O\left(t^{-2}(n+1)^{-1}\right)\left(1 + \sum_{r=\tau+1}^{n}(r+1)^{-1}\right) \\
&= O\left(\frac{t^{-2}}{n+1}\right)\left(1 + O\left(\frac{n-\tau}{\tau+1}\right)\right) = O\left(t^{-2}(n+1)^{-1}\right) + O\left(t^{-1}\right),
\end{aligned} \tag{4.8}$$

in view of (3.1) and $\tau \le 1/t < (\tau + 1)$.

Finally collecting (4.3), (4.4) and (4.8), we get Lemma 4.3. □

Proof of Theorem 3.1. We have

$$s_n(f) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\left(\frac{\sin(n+1/2)t}{\sin(t/2)}\right)\phi(t)dt. \tag{4.9}$$

Denoting $C^1 \cdot N_p$ means of $\{s_n(f)\}$ by $t_n^{CN}(f)$, we write

$$\begin{aligned}\left|t_n^{CN}(f) - f(x)\right| &= \left(\frac{1}{2\pi(n+1)}\right)\left|\int_0^{\pi}\phi(t)\sum_{k=0}^{n}P_k^{-1}\sum_{i=0}^{k}p_i\left(\frac{\sin(k-i+1/2)t}{\sin(t/2)}\right)dt\right| \\ &\le \int_0^{\pi/n}\left|\phi(t)K(n,t)\right|dt + \int_{\pi/n}^{\pi}\left|\phi(t)K(n,t)\right|dt = I_1 + I_2, \text{say}.\end{aligned} \tag{4.10}$$

Now, using Lemma 4.2 and the fact that $f \in \operatorname{Lip}\alpha \Rightarrow \phi(t) \in \operatorname{Lip}\alpha$ [10], we have

$$I_1 = O(n)\int_0^{\pi/n} t^{\alpha}dt = O(n^{-\alpha}). \tag{4.11}$$

Using Lemma 4.3, we get

$$I_2 = O\left\{\int_{\pi/n}^{\pi} t^{\alpha}\left(\frac{t^{-2}}{(n+1)} + t^{-1}\right)dt\right\} = O(I_{21}) + O(I_{22}), \text{say}, \tag{4.12}$$

where

$$\begin{aligned}I_{21} &= (n+1)^{-1}\int_{\pi/n}^{\pi} t^{\alpha-2}dt = \begin{cases} O(n^{-\alpha}), & 0<\alpha<1, \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1,\end{cases} \\ I_{22} &= \int_{\pi/n}^{\pi} t^{\alpha-1}dt = O\left(\int_{\pi/n}^{\pi}\frac{\pi}{nt}t^{\alpha-1}dt\right) = \begin{cases} O(n^{-\alpha}), & 0<\alpha<1, \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1,\end{cases}.\end{aligned} \tag{4.13}$$

Collecting (4.10)–(4.13) and writing $1/n \le (\log n)/n$, for large values of n, we get

$$\left|t_n^{CN}(f) - f(x)\right| = \begin{cases} O(n^{-\alpha}), & 0<\alpha<1, \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1.\end{cases} \tag{4.14}$$

Hence,

$$\left\|t_n^{CN}(f)-f(x)\right\|_\infty=\sup_{x\in[0,2\pi]}\left|t_n^{CN}(f)-f(x)\right|=\begin{cases}O(n^{-\alpha}), & 0<\alpha<1,\\ O\left(\dfrac{\log n}{n}\right), & \alpha=1.\end{cases}\tag{4.15}$$

This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Following the proof of Theorem 3.1, we have

$$t_n^{CN}(f)-f(x)=\int_0^{\pi/n}\phi(t)K(n,t)dt+\int_{\pi/n}^{\pi}\phi(t)K(n,t)dt=I_3+I_4,\text{say.}\tag{4.16}$$

Using Hölder's inequality, $\phi(t)\in W(L^r,\xi(t))$, condition (3.4), Lemma 4.2, and $(\sin t/2)^{-1}\le\pi/t$, for $0<t\le\pi$, we have

$$\begin{aligned}|I_3|&=\left|\int_0^{\pi/n}\frac{\phi(t)\sin^\beta(t/2)}{\xi(t)}\cdot\frac{\xi(t)K(n,t)}{\sin^\beta(t/2)}dt\right|\\
&\le\left(\int_0^{\pi/n}\left|\frac{\phi(t)\sin^\beta(t/2)}{\xi(t)}\right|^r dt\right)^{1/r}\left(\lim_{\varepsilon\to0}\int_\varepsilon^{\pi/n}\left|\frac{\xi(t)K(n,t)}{\sin^\beta(t/2)}\right|^s dt\right)^{1/s}\\
&=O(1)\left(\lim_{\varepsilon\to0}\int_\varepsilon^{\pi/n}\left(\frac{\xi(t)n}{\sin^\beta(t/2)}\right)^s dt\right)^{1/s}=O\left(n\xi\left(\frac{\pi}{n}\right)\right)\left(\lim_{\varepsilon\to0}\int_\varepsilon^{\pi/n}t^{-\beta s}dt\right)^{1/s}\\
&=O\left(\xi\left(\frac{1}{n}\right)n^{\beta+1-1/s}\right)=O\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right),\end{aligned}\tag{4.17}$$

in view of the mean value theorem for integrals, $r^{-1}+s^{-1}=1$ and Note 1.

Similarly, using Hölder's inequality, Lemma 4.3, $|\sin(t/2)|\le1$, $(\sin(t/2))^{-1}\le\pi/t$, condition (3.5), and the mean value theorem for integrals, we have

$$|I_4|=O\left(\int_{\pi/n}^{\pi}\phi(t)\left(t^{-2}(n+1)^{-1}+t^{-1}\right)dt\right)=O(I_{41})+O(I_{42}),\ \text{say,}\tag{4.18}$$

where

$$
\begin{aligned}
I_{41} &= \left(\int_{\pi/n}^{\pi} \phi(t)t^{-2}(n+1)^{-1}dt\right) = O\left((n+1)^{-1}\right)\left(\int_{\pi/n}^{\pi} \frac{t^{-\delta}\phi(t)\sin^{\beta}(t/2)}{\xi(t)} \frac{\xi(t)}{t^{2-\delta}\sin^{\beta}(t/2)}dt\right) \\
&= O\left(n^{-1}\right)\left(\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^{r}dt\right)^{1/r}\left(\int_{\pi/n}^{\pi}\left(\frac{\xi(t)}{t^{2-\delta+\beta}}\right)^{s}dt\right)^{1/s} \\
&= O\left(n^{\delta-1}\right)\left(\int_{1/\pi}^{n/\pi}\left(\frac{\xi(1/y)}{y^{\delta-2-\beta}}\right)^{s}y^{-2}dy\right)^{1/s} \\
&= O\left(n^{\delta-1}\left(\frac{n}{\pi}\right)\xi\left(\frac{\pi}{n}\right)\right)\left(\int_{\varepsilon_1}^{n/\pi} y^{(1-\delta+\beta)s-2}dy\right)^{1/s} \\
&= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)n^{1-\delta+\beta-1/s}\right) = O\left(\xi\left(\frac{1}{n}\right)n^{\beta+1/r}\right), \\
I_{42} &= \int_{\pi/n}^{\pi}\phi(t)t^{-1}dt = \int_{\pi/n}^{\pi}\frac{t^{-\delta}\phi(t)\sin^{\beta}(t/2)}{\xi(t)}\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}(t/2)}dt \\
&= O\left(\int_{\pi/n}^{\pi}\left|\frac{t^{-\delta}\phi(t)\sin^{\beta}(t/2)}{\xi(t)}\right|^{r}dt\right)^{1/r}\left(\int_{\pi/n}^{\pi}\left|\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}(t/2)}\right|^{s}dt\right)^{1/s} \\
&= O\left(\int_{\pi/n}^{\pi}\left|\frac{t^{-\delta}\phi(t)}{\xi(t)}\right|^{r}dt\right)^{1/r}\left(\int_{\pi/n}^{\pi}\left|\frac{\xi(t)}{t^{1-\delta+\beta}}\right|^{s}dt\right)^{1/s} \\
&= O\left(n^{\delta}\right)\left(\int_{1/\pi}^{n/\pi}\left(\frac{\xi(1/y)}{y^{\delta-1-\beta}}\right)^{s}y^{-2}dy\right)^{1/s} \\
&= O\left(\left(\frac{n^{\delta+1}}{\pi}\right)\xi\left(\frac{\pi}{n}\right)\right)\left(\int_{\varepsilon_2}^{n/\pi} y^{(\beta-\delta)s-2}dy\right)^{1/s} \\
&= O\left(n^{\delta+1}\xi\left(\frac{1}{n}\right)n^{-\delta+\beta-1/s}\right) = O\left(\xi\left(\frac{1}{n}\right)n^{\beta+1/r}\right),
\end{aligned}
\tag{4.19}
$$

in view of increasing nature of $y\xi(1/y)$, $r^{-1}+s^{-1}=1$, where $\varepsilon_1, \varepsilon_2$ lie in $[\pi^{-1}, n\pi^{-1}]$, and Note 1.

Collecting (4.16)–(4.19), we get

$$
\left|t_n^{CN}(f) - f(x)\right| = O\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right). \tag{4.20}
$$

Hence,

$$\left\|t_n^{CN}(f)-f(x)\right\|_r=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|t_n^{CN}(f)-f(x)\right|^r dx\right)^{1/r}=O\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right). \tag{4.21}$$

This completes the proof of Theorem 3.2. □

5. Corollaries

The following corollaries can be derived from Theorem 3.2.

Corollary 5.1. *If* $\beta=0$, *then for* $f\in \mathrm{Lip}(\xi(t),r)$, $\|t_n^{CN}(f)-f(x)\|_r=O(n^{1/r}\xi(1/n))$.

Corollary 5.2. *If* $\beta=0,\xi(t)=t^\alpha\ (0<\alpha\leq 1)$, *then for* $f\in \mathrm{Lip}(\alpha,r)\ (\alpha>1/r)$,

$$\left\|t_n^{CN}(f)-f(x)\right\|_r=O\left(n^{1/r-\alpha}\right). \tag{5.1}$$

Corollary 5.3. *If* $r\ \rightarrow\ \infty$ *in Corollary 5.2, then for* $f\in \mathrm{Lip}\alpha(0<\alpha<1)$, (5.1) *gives*

$$\left\|t_n^{CN}(f)-f(x)\right\|_\infty=O(n^{-\alpha}). \tag{5.2}$$

6. Conclusion

Various results pertaining to the degree of approximation of periodic functions (signals) belonging to the Lipchitz classes have been reviewed and the condition of monotonocity on the means generating sequence $\{p_n\}$ has been relaxed. Further, a proper set of conditions have been discussed to rectify the errors pointed out in Remarks 2.3 and 2.4.

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