

FINITE DIFFERENCE APPROXIMATIONS FOR A CLASS OF NON-LOCAL PARABOLIC EQUATIONS

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ABSTRACT. In this paper we study finite difference procedures for a class of parabolic equations with non-local boundary condition. The semi-implicit and fully implicit backward Euler schemes are studied. It is proved that both schemes preserve the maximum principle and monotonicity of the solution of the original equation, and fully-implicit scheme also possesses strict monotonicity. It is also proved that finite difference solutions approach to zero as $t \rightarrow \infty$ exponentially. The numerical results of some examples are presented, which support our theoretical justifications.

KEYWORDS AND PHRASES: Finite difference, non-local, monotonicity, decay, stability, maximum principle.

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1. INTRODUCTION.

In this paper we study finite difference approximations to the solution of the following parabolic equations with non-local boundary condition:

$$(1.1) \quad \begin{aligned} u_t - \Delta u &= 0 \text{ in } Q_T, \\ u(x, y, 0) &= \phi(x, y), & (x, y) \in \Omega, \\ u(x, y, t) &= \int_{\Omega} K(x, y, \xi, \eta) u(\xi, \eta, t) d\xi d\eta, & \text{on } \partial\Omega \times [0, T), \end{aligned}$$

where $Q_T = \Omega \times (0, T)$, $T > 0$, $\Omega = (0, 1) \times (0, 1)$, $\phi(x, y) \not\equiv 0$ and $K(x, y, \xi, \eta)$ are known functions. In addition, it is assumed that for some constant $0 < \rho < 1$ the kernel $K(x, y, \xi, \eta)$ satisfies

$$(1.2) \quad \int_{\Omega} |K(x, y, \xi, \eta)| d\xi d\eta \leq \rho < 1, \quad \forall (x, y) \in \partial\Omega.$$

In [4, 5] Day considered the one-dimensional problem on $(-L, L)$, $L > 0$, with the boundary conditions

$$(1.3) \quad u(-L, t) = \int_{-L}^L f_1(x)u(x, t)dx \quad \text{and} \quad u(L, t) = \int_{-L}^L f_2(x)u(x, t)dx,$$

and showed that if

$$(1.4) \quad \int_{-L}^L |f_1(x)|dx < 1 \quad \text{and} \quad \int_{-L}^L |f_2(x)|dx < 1,$$

then, for the solution u ,

$$(1.5) \quad U(t) = \max_{-L \leq x \leq L} |u(x, t)|$$

is decreasing in t . The solution u represents the entropy in a quasi-static theory of thermoelasticity [5, 6], so that Day's results shown that the maximum modulus of the entropy is decreasing in time. In [8] Friedman extended Day's results to a general parabolic equation in n -dimensions of the form

$$(1.6) \quad \frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{i,j}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x, t) \frac{\partial u}{\partial x_i} + au = 0$$

with $a(x, t) \geq 0$ and with the initial and boundary conditions as given in (1.1). Moreover, Friedman proved that there exists $C_0 > 0$ and $\lambda > 0$ such that

$$(1.8) \quad U(t) \leq C_0 e^{-\lambda t}, \quad t \geq 0,$$

i.e., $U(t)$ decays to zero exponentially as $t \rightarrow \infty$. Problems similar to the above also arise from the determination of the unknown source parameter [2, 9] and other related problems [10].

For physical applications of the problem (1.1), let us consider first the coupled partial differential equations

$$a\theta_{xx} = b\theta_t + \theta_0 Bv_{xxt}, \quad Av_{xxxx} = B\theta_{xx}$$

which describe the quasi-static flexure of a thermoelastic rod [5]. Here $\theta(x, t)$ is the temperature, θ_0 is a uniform reference temperature, $v(x, t)$ is the transverse displacement, a is the conductivity, b is the specific heat at the constant strain, the constant A is the flexure rigidity and the constant B is a measure of the cross-coupling between thermal and mechanical effects. We assume that the ends $x = -L$ and $x = L$ to be maintained at the reference temperture θ_0 and to be clamped, that is

$$\begin{aligned} \theta(-L, t) = \theta(L, t) = 0, \\ u(-L, t) = u_x(-L, t) = u(L, t) = u_x(L, t) = 0. \end{aligned}$$

Let

$$u = \frac{b}{\theta_0}(\theta - \theta_0) + Bu_{xx}$$

be the entropy, then after some mathematical manipulations [5] we obtain that u satisfies

$$au_{xx} = (b + \theta_0 \frac{B^2}{A}) u_t$$

with the boundary conditions

$$u(-L, t) = -\frac{\theta_0 B^2}{2bA^2 L^2} \int_{-L}^L (L - 3x)u(x, t)dx,$$

$$u(L, t) = -\frac{\theta_0 B^2}{2bA^2 L^2} \int_{-L}^L (L + 3x)u(x, t)dx$$

and an appropriate initial condition.

For the second example we consider the equations [1, 3, 5]

$$a\theta_{xx} = b\theta_t + \theta_0\alpha(3\lambda + 2\mu)v_{xt}$$

$$(\lambda + 2\mu)v_{xx} = \alpha(3\lambda + 2\mu)\theta_x$$

which describe the behavior of a slab $-L \leq x \leq L$ made of homogeneous and isotropic material. Here $\theta(x, t)$ is the temperature, $v(x, t)$ is the displacement component in the direction of the x -axis, θ_0 is a uniform reference temperature, α is the coefficient of expansion, and λ, μ are the elastic moduli. The boundary conditions are

$$\theta(-L, t) = \theta(L, t) = \theta_0, \quad v(-L, t) = v(L, t) = 0.$$

Let

$$u = \frac{b}{\theta_0} (\theta - \theta_0) + \alpha(3\lambda + 2\mu)v_x$$

be the entropy, one has [5] that u satisfies

$$au_{xx} = b^* u_t$$

with the boundary conditions

$$u(-L, t) = u(L, t) = -\frac{b^* - b}{b} \frac{1}{2L} \int_{-L}^L u(x, t)dx,$$

where

$$b^* = b + \theta_0\alpha^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu}.$$

For the detail derivations of the above equations we refer to [1, 3, 5].

The condition (1.2) implies for the first problem that $5\theta_0 B^2 < 3bA^2$ and for the second problem that $\frac{b^* - b}{b} < 1$ or $\theta_0\alpha^2(3\lambda - 2\mu)^2 < (\lambda + 2\mu)b$.

In this article we study finite difference schemes for (1.1). The finite difference procedures proposed below preserve monotonicity, the maximum principle and the exponential decay (if the

kernel is non-negative) of the solution for equation (1.1); therefore, they are considered as good numerical approximations.

Let $h = \Delta x = \Delta y = 1/N$ for some integer $N > 1$, and let $\tau > 0$ be a small step-size in time with $t_n = n\tau$, $n = 0, 1, \dots$. For a smooth function $v(x, y) \in C^2(\bar{\Omega})$ we assume that the following numerical integration formula is valid:

$$(1.9) \quad \int_{\Omega} K(x, y, \xi, \eta) v(\xi, \eta) d\xi d\eta = \sum_{m,l=0}^N w_{m,l} K(x, y, x_m, y_l) v_{m,l} + O(h^2),$$

where $w_{m,l} \geq 0$ are weights and $v_{m,l} = v(x_m, y_l)$ with $x_m = m\Delta x$, $y_l = l\Delta y$, $m, l = 0, 1, \dots, N$. For any $0 < \rho^* < 1$, we restrict h to be so small, say for some $h_0 > 0$, $0 < h \leq h_0$, that

$$(1.10) \quad \sum_{m,l=0}^N w_{m,l} |K(x, y, x_m, y_l)| \leq \rho^* = \frac{1+\rho}{2} < 1, \quad (x, y) \in \Omega.$$

Here and throughout this paper we assume that h is small enough so that (1.10) is satisfied. In fact (1.10) serves as a discrete version of (1.2). In order to obtain the numerical solution which preserves as many properties of the solution as possible, (1.10) is a necessary and cannot be considered as a constraint on space discretization. For example the weights can be chosen by using trapezoidal rule,

$$(1.11) \quad w_{m,l} = \begin{cases} \Delta x \Delta y, & m, l = 1, 2, \dots, N-1; \\ \frac{1}{4} \Delta x \Delta y, & m, l \in \{0, N\}; \\ \frac{1}{2} \Delta x \Delta y, & \text{otherwise.} \end{cases}$$

Define the following shorthand notations

$$\begin{aligned} \partial_t g^n &= \frac{g^n - g^{n-1}}{\tau}, \\ \Delta^2 g_{i,j} &= \frac{g_{i+1,j} + g_{i-1,j} + g_{i,j+1} + g_{i,j-1} - 4g_{i,j}}{h^2}. \end{aligned}$$

We now define our first numerical scheme: Find $\{U_{i,j}^n\}$ such that

$$(1.12) \quad \begin{aligned} \partial_t U_{i,j}^n - \Delta^2 U_{i,j}^n &= 0, \quad i, j = 1, 2, \dots, N-1, \quad n \geq 1, \\ U_{i,j}^0 &= \phi_{i,j}, \quad i, j = 0, 1, 2, \dots, N, \\ U_{i,j}^n &= K_{i,j}(\{U_{m,l}^n\}), \quad \{i, j\} \cap \{0, N\} \neq \emptyset, \quad n \geq 1, \end{aligned}$$

where

$$(1.13) \quad K_{i,j}(\{U_{m,l}^n\}) = \sum_{m,l=0}^N w_{m,l} K(x_i, y_j, x_m, y_l) U_{m,l}^n, \quad \{i, j\} \cap \{0, N\} \neq \emptyset, \quad n \geq 1.$$

It is clear that (1.12)-(1.13) is a fully-implicit scheme that requires a full-matrix to be solved at each time level due to the boundary integration. For small $h > 0$, the matrix will be diagonally-dominant and can be solved by Gaussian-elimination or any standard method.

Since (1.12)-(1.13) results in an error $O(h^2 + \tau)$, we may propose the following numerically economical semi-implicit scheme: Find $\{W_{i,j}^n\}$ such that

$$(1.14) \quad \begin{aligned} \partial_t W_{i,j}^n - \Delta^2 W_{i,j}^n &= 0, & i, j &= 1, 2, \dots, N-1, & n &\geq 1, \\ W_{i,j}^0 &= \phi_{i,j}, & i, j &= 0, 1, 2, \dots, N, \\ W_{i,j}^n &= K_{i,j} \left(\{W_{m,l}^{n-1}\} \right), & \{i, j\} \cap \{0, N\} &\neq \emptyset, & n &\geq 1. \end{aligned}$$

Clearly, (1.14) results in an error $O(h^2 + \tau)$ and is easy to implement numerically since only a pent-diagonal matrix system needs to be solved at each time level. Therefore, it is a very economical and fast algorithm. In addition, it is also unconditionally stable. Alternative methods, say ADI, may also be used to solve (1.14).

2. STABILITY, MONOTONICITY AND EXPONENTIAL DECAY.

In this section we prove some monotonicity, maximum principle and exponential decay properties for our numerical solutions $U_{i,j}^n$ and $W_{i,j}^n$. Define

$$(2.1) \quad U^n = \max_{0 \leq i, j \leq N} |U_{i,j}^n|, \quad W^n = 0 \leq i, j \leq N \rightarrow \max |W_{i,j}^n|.$$

THEOREM 2.1. Assume that $U_{i,j}^n$ is a solution of (1.12)-(1.13) and the initial approximation $U_{i,j}^0 \neq 0$ for $i, j = 1, 2, \dots, N-1$, then the following holds:

$$(2.2) \quad 0 < U^n < U^{n-1}, \quad \forall n \geq 1.$$

PROOF. We show that

$$(2.3) \quad 0 < U^n < U^{n-1}, \quad \forall n \geq 1.$$

We observe that $U^n \geq 0$ for all $n \geq 0$. Consider the first two levels $n = 0$ and $n = 1$. Assume to the contrary that $U^0 \leq U^1$, then $U^1 > 0$. If $U^1 = |U_{i_0, j_0}^1| = U_{i_0, j_0}^1 > 0$ (the case $U_{i_0, j_0}^1 < 0$ can be treated in a similar way) for some (i_0, j_0) , then it follows from the discrete maximum principle [7] that this maximum is attained at the boundary. Thus (i_0, j_0) can be selected to be a boundary point. Then, we see from (1.10), (1.12) and (1.13) that

$$(2.4) \quad U^1 = |K_{i_0, j_0}(\{U_{m,l}^1\})| \leq \rho^* U^1$$

which is impossible unless $U^1 = 0$ since $\rho^* < 1$. This contradicts $U^1 > 0$. By (1.12), $U^1 = 0$ will lead to $U_{i,j}^0 = 0$ for $i, j = 1, 2, \dots, N-1$ which is a contradiction. Thus, $0 < U^1 < U^0$.

Now, we consider the levels $n = 1$ and $n = 2$. By repeating the above argument with $U_{i,j}^1$ as the initial data, we can show that $0 < U^2 < U^1$. Thus, (2.3) is proved by repeating the above argument for higher levels. Q.E.D.

THEOREM 2.2. Assume that $W_{i,j}^n$ is a solution of (1.14) and the initial approximation $W_{i,j}^0 \neq 0$ for $i, j = 1, 2, \dots, N-1$, then the following holds:

$$(2.5) \quad 0 < W^n < W^{n-1}, \quad \forall n \geq 1.$$

PROOF. We show first that $W^1 > 0$. If on the contrary $W^1 = 0$, we will get $W_{i,j}^0 = 0$ for $i, j = 1, 2, \dots, N-1$ by (1.14). This contradiction proves $W^1 > 0$. By (1.10), we get

$$(2.6) \quad |W_{i,j}^1| \leq \rho^* W^0 < W^0, \quad \{i, j\} \cap \{0, N\} \neq \emptyset.$$

According to the discrete maximum principle [7],

$$(2.7) \quad W^1 = \max_{\{i,j\} \cap \{0,N\} \neq \emptyset} |W_{i,j}^1| < W^0.$$

The remainder of the proof follows from an argument similar to the above and mathematical induction.

Q.E.D.

REMARK. Theorem 2.1 and Theorem 2.2 imply the unconditionally stability of numerical solutions $U_{i,j}^n$ and $W_{i,j}^n$, even though $W_{i,j}^n$ is the solution of semi-implicit finite difference scheme.

In [8] Friedman proved that $U(t)$ decays exponentially when (1.2) is satisfied. We have proved that both $\{U^n\}$ and $\{W^n\}$ possess the strict monotonicity. In fact numerically there exists $\lambda > 0$, as suggested in the examples of section 5, Figure 6 and Figure 7, such that

$$(2.8) \quad \log \frac{U^{n+1}}{U^n} \sim -\lambda \Delta t \quad \text{as } n \rightarrow \infty,$$

and same is true for W^n . This motivates the justifications of the exponential decay of U^n and W^n when the kernel is non-negative.

THEOREM 2.3. Under the assumption that $U_{i,j}^n$ is the solution of (1.12)-(1.13) and the kernel $K(x, y, \xi, \eta) \geq 0$, there exists a positive constant $\lambda > 0$ such that for $U^0 = \max_{i,j} |U_{i,j}^0|$,

$$(2.9) \quad U^n \leq U^0 e^{-\lambda t_n} \quad \text{for all } n \geq 0.$$

PROOF. Let $V(x, y, t) = e^{-\lambda t}(2U^0 - \frac{\epsilon}{4}(x^2 + y^2))$ where ϵ and λ are two positive constants to be chosen below. It follows easily that there exists $\epsilon_0 > 0$ such that

$$V(x, y, 0) = 2U^0 - \frac{\epsilon}{4}(x^2 + y^2) > U^0 \quad \text{on } \Omega \quad \text{if } 0 < \epsilon \leq \epsilon_0.$$

Also, since $K_{i,j}(\{1\}) \leq \rho^* < 1$, we find

$$K_{i,j}(\{2U^0\}) \leq \rho^* 2U^0 < 2U^0,$$

and then, there exists a positive constant $\epsilon_1 > 0$ small enough such that for all $0 < \epsilon \leq \epsilon_1$,

$$V_{i,j}^n > K_{i,j}(\{V_{m,l}^n\}), \quad \{i, j\} \cap \{0, N\} \neq \emptyset, \quad n \geq 1.$$

Thus, we choose $\epsilon = \min\{\epsilon_0, \epsilon_1\}$. It follows from a simple calculation that

$$\partial_t V_{i,j}^n - \Delta^2 V_{i,j}^n = e^{-\lambda t_{n+1}} \left(\epsilon - \lambda e^{\lambda \epsilon} (2U^0 - \frac{\epsilon}{4}(x_i^2 + y_j^2)) \right), \quad i, j = 1, 2, \dots, N-1.$$

where $\xi \in (0, \tau)$. As $\lambda e^{\lambda\tau} \rightarrow 0$ when $\lambda \rightarrow 0$, we have for some $\lambda_0 = \lambda_0(\epsilon) > 0$ (or $\lambda_0 = \min\{1/\tau, \epsilon/(2eU^0)\}$) such that for all $0 < \lambda \leq \lambda_0$,

$$\partial_t V_{i,j}^n - \Delta^2 V_{i,j}^n > 0, \quad i, j = 1, 2, \dots, N-1.$$

Now letting $Z_{i,j}^n = V_{i,j}^n - U_{i,j}^n$ with $\lambda = \lambda_0$ and ϵ chosen above, we see from (1.12) and the analysis above that

$$(2.10) \quad \begin{aligned} \partial_t Z_{i,j}^n - \Delta^2 Z_{i,j}^n &> 0, & i, j = 1, 2, \dots, N-1, & n \geq 1, \\ Z_{i,j}^0 &> 0, & j = 0, 1, 2, \dots, N, \\ Z_{i,j}^n &> K_{i,j}(\{Z_{m,t}^n\}), & \{i, j\} \cap \{0, N\} \neq \emptyset, & n \geq 1. \end{aligned}$$

We now show that $Z_{i,j}^n \geq 0$. Assume that n_0 is the first level that $Z_{i,j}^n$ may take the negative values, then we have (i_0, j_0) such that

$$Z_{i_0, j_0}^{n_0} = \min_{i,j} Z_{i,j}^{n_0} < 0.$$

It follows from the discrete maximum principle [5] that (i_0, j_0) must be the boundary point, otherwise $\partial_t Z_{i_0, j_0}^{n_0} - \Delta^2 Z_{i_0, j_0}^{n_0} \leq 0$ which is not possible. Thus we have from the positivity of kernel K and (2.10) that

$$-Z_{i_0, j_0}^{n_0} < K_{i,j}(\{-Z_{m,t}^{n_0}\}) \leq \rho^*(-Z_{i_0, j_0}^{n_0})$$

which implies that $Z_{i_0, j_0}^{n_0} = 0$, a contradiction. Hence, we have proved that $U_{i,j}^n \leq V_{i,j}^n$. Using a similar argument by treating $-U_{i,j}^n$ it can be shown that $U_{i,j}^n \geq -V_{i,j}^n$. This completes the proof. Q.E.D.

THEOREM 2.4. Assume that $W_{i,j}^n$ is the solution of (1.14), then there exists a positive constant $\lambda > 0$ such that $W^0 = \max_{i,j} |W_{i,j}^0|$,

$$(2.11) \quad W^n \leq W^0 e^{-\lambda t_n} \quad \text{for all } n \geq 0.$$

PROOF. The proof consists of an argument similar to that given in the proof of Theorem 2.3, we therefore only give the outline.

Let $V(x, y, t) = e^{-\lambda t}(2W^0 - \frac{\epsilon}{4}(x^2 + y^2))$ and as before, let ϵ_0 be chosen so small that $V(x, y, 0) > W^0$ for $0 < \epsilon \leq \epsilon_0$. Because the numerical integration uses the data on the previous level for the boundary condition, we need to first select a $\lambda_0 > 0$ such that $0 < \lambda \leq \lambda_0$,

$$e^{-\lambda\tau} > \rho^*, \quad \text{i.e.} \quad e^{-\lambda\tau} 2W^0 > \rho^* 2W^0.$$

Since $e^{-\lambda\tau} \rightarrow 1$ when $\lambda \rightarrow 0$, the existence of such a λ_0 is not a problem. With ϵ_0 and λ_0 chosen as above, we then select $\epsilon_1 > 0$ so small that for $0 < \epsilon \leq \epsilon_1$

$$V_{i,j}^n > K_{i,j}(\{V_{m,t}^{n-1}\}), \quad \{i, j\} \cap \{0, N\} \neq \emptyset, \quad n \geq 1.$$

Using $\epsilon = \min\{\epsilon_0, \epsilon_1\}$ and the λ_0 selected above, we select $\lambda_1 >$ such that for $0 < \lambda \leq \lambda_1$,

$$\partial_t V_{i,j}^n - \Delta^2 V_{i,j}^n > 0, \quad i, j = 1, 2, \dots, N-1.$$

We take $\lambda = \min\{\lambda_0, \lambda_1\}$. We omit the reminder of the proof which is the same as that given in the proof of Theorem 2.1 with the ϵ and λ as chosen here. Q.E.D.

3. CONVERGENCE AND ERROR ESTIMATES.

In this section we study the convergence and error estimates of the numerical procedures proposed in section 1. First, we show the following result.

THEOREM 3.1. Assume that in addition to (1.2), the kernel K in problem (1.1) satisfies $K \in C^2(\overline{\partial\Omega \times \Omega})$ and

$$(3.1) \quad K(x, y, \xi, \eta) \geq 0, \quad \forall (x, y, \xi, \eta) \in \partial\Omega \times \Omega.$$

If the solutions u of (1.1) is known a priori to be smooth enough, $u \in C^{4,2}(\bar{Q}_T)$, then there exists a positive constant $C = C(\|u\|_{C^{4,2}}, \|K\|_{C^2}) > 0$ such that the solution $U_{i,j}^n$ of (1.12) satisfies

$$(3.2) \quad \max_{i,j,n} |U_{i,j}^n - u(x_i, y_j, t_n)| \leq C(h^2 + \tau).$$

PROOF. Let $e_{i,j}^n = U_{i,j}^n - u(x_i, y_j, t_n)$ for all i, j, n , then we see from (1.10) and (1.12) that $e_{i,j}^n$ satisfies

$$(3.3) \quad \begin{aligned} \partial_t e_{i,j}^n - \Delta^2 e_{i,j}^n &= \tau_{i,j}^n, & i, j &= 1, 2, \dots, N-1, \quad n \geq 1, \\ e_{i,j}^0 &= 0, & i, j &= 0, 1, 2, \dots, N, \\ e_{i,j}^n &= K_{i,j} \left(\{e_{m,l}^n\} \right) + \epsilon_{i,j}^n, & \{i, j\} \cap \{0, N\} &\neq \emptyset, \quad n \geq 1. \end{aligned}$$

Here $\tau_{i,j}^n$ and $\epsilon_{i,j}^n$ are the truncation errors induced by the discretization of differential equation and numerical integration respectively. Then there exists $L_0 > 0$ such that

$$(3.4) \quad \max_{i,j,n} |\tau_{i,j}^n| \leq L_0(h^2 + \tau), \quad \max_{i,j,n} |\epsilon_{i,j}^n| \leq L_0(h^2 + \tau).$$

We now define an auxiliary function $\theta(x, y)$ by

$$(3.5) \quad \theta(x, y) = \frac{1 - x^2 - y^2}{4} L_0(h^2 + \tau)$$

then it is easy to verify that

$$(3.6) \quad -\Delta^2 \theta_{i,j} = L_0(h^2 + \tau) \quad \text{and} \quad 0 \leq \theta_{i,j} \leq \frac{L_0}{2} (h^2 + \tau).$$

Let $Z_{i,j}^n = e_{i,j}^n - \theta_{i,j}$ for all i, j, n , we find from (3.3) and (3.6) that

$$(3.7) \quad \begin{aligned} \partial_t Z_{i,j}^n - \Delta^2 Z_{i,j}^n &\leq 0, & i, j &= 1, 2, \dots, N-1, \quad n \geq 1, \\ Z_{i,j}^0 &= -\theta_{i,j} \leq 0, & i, j &= 0, 1, 2, \dots, N, \\ Z_{i,j}^n &= K_{i,j} \left(\{Z_{m,l}^n\} \right) + K_{i,j} (\{\theta_{m,l}\}) - \theta_{i,j} + \epsilon_{i,j}^n, & \{i, j\} \cap \{0, N\} &\neq \emptyset, \quad n \geq 1. \end{aligned}$$

We now show that there exists $C > 0$ such that $Z_{i,j}^n \leq C(h^2 + \tau)$ for all i, j, n . If $Z_{i,j}^n$ has a positive maximum, then according to the discrete maximum principle then it must be attained at a boundary point. Assume that $M = Z_{i_0, j_0}^{n_0} > 0$ with $n_0 \geq 1$, is the positive maximum. From the boundary condition in (3.7) we see that

$$(3.8) \quad \begin{aligned} M &\leq \rho^* M + (\rho^* + 1) \max_{i,j} |\theta_{i,j}| + \max_{i,j,n} |\epsilon_{i,j}^n| \\ &\leq \rho^* M + \frac{\rho^* + 3}{2} L_0(h^2 + \tau), \end{aligned}$$

which implies

$$(3.9) \quad M \leq \frac{\rho^* + 3}{2(1 - \rho^*)} L_0(h^2 + \tau).$$

Hence, we have proved that

$$(3.10) \quad e_{i,j} \leq \theta_{i,j} + \frac{\rho^* + 3}{2(1 - \rho^*)} L_0(h^2 + \tau).$$

If instead $Z_{i,j}^n = e_{i,j}^n + \theta_{i,j}$, then a similar argument gives

$$(3.11) \quad e_{i,j} \geq -\theta_{i,j} + \frac{\rho^* + 3}{2(1 - \rho^*)} L_0(h^2 + \tau).$$

Therefore, we find from (3.10)-(3.11) that

$$(3.12) \quad \begin{aligned} |e_{i,j}| &\leq |\theta_{i,j}| + \frac{\rho^* + 3}{2(1 - \rho^*)} L_0(h^2 + \tau) \\ &\leq \frac{3L_0}{1 - \rho^*} (h^2 + \tau), \end{aligned}$$

which is (3.1). The proof is complete. Q.E.D.

THEOREM 3.2. Under the same assumptions of Theorem 3.1, let $W_{i,j}^n$ be the solution of (1.14). Then for some positive constant $C > 0$, independent of h and τ , we have

$$(3.13) \quad \max_{i,j,n} |W_{i,j}^n - u(x_i, y_j, t_n)| \leq C(h^2 + \tau).$$

PROOF. It follows by a similar argument to that given in the proof of Theorem 3.1. Q.E.D.

REMARK. The error estimates in (3.2) and (3.13) are uniform for all $0 \leq t < \infty$ if $T = \infty$, which is guaranteed by the condition (1.2).

4. GENERAL SMOOTH KERNEL $K(x, y, \xi, \eta)$.

In this section we consider the effect on the original problem (1.1) when the kernel condition (1.2) is replaced by:

$$(4.1) \quad 0 \leq K(x, y, \xi, \eta) \leq K_0, \quad \forall (x, y, \xi, \eta) \in \partial\Omega \times \Omega.$$

In general if the condition (1.2) is not satisfied, then the numerical procedure of (1.12) or (1.14) may not be stable uniformly for $0 < t < \infty$. This will be demonstrated in both theoretically and through numerical examples below. For these kernels, the stability will depend upon K_0 and $T > 0$. Here we consider a class of kernels which satisfy (4.1) but not (1.2).

We first consider the continuous problem. Let $w(x, y)$ be an auxiliary function defined by

$$(4.2) \quad 1 \leq w(x, y) = 1 + M ((x - 1/2)^d + (y - 1/2)^d), \quad d > 0,$$

where M and d (even) are two positive constants to be chosen. Clearly, we have

$$(4.3) \quad \min_{(x,y) \in \partial\Omega} w(x, y) = 1 + 2M \left(\frac{1}{2}\right)^d$$

Let $u(x, y, t)$ be a solution of (1.1) with K satisfying (4.1), and set $v(x, y, t) = \frac{u(x,y,t)}{w(x,y)}$ and find that v satisfies

$$(4.4) \quad \begin{aligned} v_t &= \Delta v + 2 \frac{\nabla w \cdot \nabla v}{w} + \frac{\Delta w}{w} v, & \text{in } Q_T, \\ v(x, y, 0) &= \frac{\phi(x,y,t)}{w(x,y)}, & (x, y) \in \Omega \\ v(x, y, t) &= \int_{\Omega} R(x, y, \xi, \eta) v(\xi, \eta) d\xi d\eta, & (x, t) \in \partial\Omega, \quad t \geq 0, \end{aligned}$$

where

$$(4.5) \quad R(x, y, \xi, \eta) = K(x, y, \xi, \eta) \frac{w(\xi, \eta)}{w(x, y)}.$$

Thus, we have from (4.1) that

$$(4.6) \quad \int_{\Omega} |R(x, y, \xi, \eta)| d\xi d\eta \leq \frac{K_0}{1 + M(1/2)^{d-1}} \int_{\Omega} w(\xi, \eta) d\xi d\eta.$$

A simple calculation shows that if d is an even integer,

$$(4.7) \quad \int_{\Omega} w(\xi, \eta) d\xi d\eta = 1 + M \frac{(1/2)^{d-1}}{d+1}.$$

Then it follows that

$$(4.8) \quad \int_{\Omega} |R(x, y, \xi, \eta)| d\xi d\eta \leq K_0 \frac{1 + M \frac{(1/2)^{d-1}}{d+1}}{1 + M(1/2)^{d-1}} \rightarrow \frac{K_0}{d+1} \quad \text{as } M \rightarrow \infty.$$

Hence, taking $d = 2K_0$ and $M = M(K_0) > 0$ large enough, we can achieve

$$(4.9) \quad \int_{\Omega} |R(x, y, \xi, \eta)| d\xi d\eta \leq \frac{K_0}{d+1} \leq \frac{2K_0}{2K_0+1} < 1, \quad \forall (x, y) \in \partial\Omega.$$

For $w(x, y)$ chosen in this way, we have for some $K_1 = K_1(K_0) > 0$ that $|\Delta w/w| \leq K_1$.

Now consider the transformation $v(x, y, t) = e^{\lambda t} Y(x, y, t)$ with $\lambda \geq K_1$, we find that Y satisfies

$$(4.10) \quad \begin{aligned} Y_t &= \Delta Y + 2 \frac{\nabla w \cdot \nabla Y}{w} + \left(\frac{\Delta w}{w} - \lambda\right) Y & \text{in } Q_T, \\ Y(x, y, 0) &= \frac{\phi(x,y,t)}{w(x,y)}, & (x, y) \in \Omega \\ Y(x, y, t) &= \int_{\Omega} R(x, y, \xi, \eta) Y(\xi, \eta) d\xi d\eta, & (x, t) \in \partial\Omega, \quad t \geq 0. \end{aligned}$$

REMARK. We now see from [5] that Y possesses the maximum principle, monotonicity and exponential decay properties, which in turn results in monotonic and stable numerical schemes if it is discretized as (1.12) or (1.14) in an appropriate way.

Turning to numerical approximations for (1.1) with condition (4.1), we let $\tau = T/N_1$ where N_1 is a positive integer. Numerical solutions to the problem, $U_{i,j}^n$ or $W_{i,j}^n$, are defined as in (1.12) or (1.14). We cannot expect that these two schemes have the monotonic properties as described in Theorem 2.1 and Theorem 2.2 when (1.2) is not satisfied. However, we have the following local stability estimates.

THEOREM 4.1. Assume that $U_{i,j}^n$ is defined as in (1.12) or (1.14) for the problem (1.1) with $K \geq 0$ and (4.1) satisfied. If the solution u of (1.1) is known apriori to be smooth enough, $u \in C^{4,2}(\bar{Q}_T)$, then there is some constant $C^* = C^*(\|u\|_{C^{4,2}}, \|K\|_{C^2}, K_0, T) > 0$ such that

$$(4.11) \quad \max_{i,j,n} |U_{i,j}^n - u(x_i, y_j, t_n)| \leq C^*(h^2 + \tau).$$

PROOF. The proof is similar to that given in section 3, so is outlined as below. For (4.11), we let $U_{i,j}^n = e^{\lambda t_n} w_{i,j} Y_{i,j}^n$, where λ and $w(x, y)$ are defined as above. Thus, it follows from a simple calculation that $Y_{i,j}^n$ satisfies a difference equation which is the discrete version of the equation (4.10). Thus it follows from Theorem 3.1 and Theorem 3.2 (The proof needs only minor modifications from that given in Section 3, we therefore omit.) that there exists a positive constant $C > 0$ such that

$$(4.12) \quad \max_{i,j,n} |Y_{i,j}^n - Y(x_i, y_j, t_n)| \leq C(h^2 + \tau),$$

where C is independent of K_0 and $T > 0$, and then, we obtain that

$$(4.13) \quad |U_{i,j}^n - u(x_i, y_j, t_n)| \leq e^{\lambda t_n} w_{i,j} |Y_{i,j}^n - u(x_i, y_j, t_n)| \leq C^*(h^2 + \tau),$$

which completes the proof.

Q.E.D.

REMARK. The constant C^* above can be very large if K_0 and $T > 0$ are very large. This can be seen from the choices of d and K_1 in the above analysis, and also is demonstrated in the examples in section 5. In another words although h and τ are small, the error could be very big, even approaching ∞ as $n \rightarrow \infty$.

5. NUMERICAL EXAMPLES.

We shall report several numerical examples which support our theoretical justifications in the previous sections, i.e., stability, monotonicity and exponential decay as $t \rightarrow \infty$. Both semi-implicit and fully explicit schemes using trapezoidal rule for numerical integration are used in our computations.

EXAMPLE 1. In order to demonstrate the error analysis and stability, we select $\Omega = [0, 2\pi] \times [0, 2\pi]$, $\phi(x, y) = \sin(x)\sin(y)$ and $K(x, y, \xi, \eta) = \frac{k}{4\pi}$. Thus, for any real constant $k > 0$, $u(x, y, t) = \sin(x)\sin(y)e^{-2t}$ is the solution with $\int_{\Omega} |K(x, y, \xi, \eta)| d\xi\eta = k$. Figure 1 and Figure 2 show by using semi-implicit scheme that the error distributions of u (the maximum error on each level via the time) with the various parameter k from 0.1 to 4. Clearly, for $k = 0.1, 0.3, 0.5$

and $k = 0.8$, even $k = 1.0$, the errors are under control as predicted by Theorem 3.1. On the other hand, for $k = 1.5, 2.5, 3$ and $k = 4$, it is seen that the errors are under control only for a short period of time, and then divergent to ∞ as $n \rightarrow \infty$. This is the exact same result as predicted by Theorem 4.1, i.e., the numerical schemes are stable locally dependent upon $K_0 > 0$ and $T > 0$. Figure 3 shows the error distribution of u by using fully implicit scheme. For $0 < k < 1$ the error distributions of u in this example are almost identical to the case of $k = 1$. Also we noticed that the fully implicit scheme is more stable than the semi-implicit scheme.

EXAMPLE 2. We now take a simple model problem with the same spatial domain and kernel as in example 1, $\phi(x, y) = \sin(xy)$ and $k = 0.8$. Figure 4 and Figure 5, by using semi-implicit and fully implicit schemes respectively, shows the distribution of U^n via the time t , which decrease to zero exponentially as $t \rightarrow \infty$. If we assume roughly that for some $\lambda(t), C(t)$ such that

$$U(t) \sim C(t)e^{\lambda(t)t} \quad \text{as } t \rightarrow \infty,$$

then $\lambda(t)$ can be calculated by the following formula

$$\lambda^n \sim \frac{1}{\Delta t} \log\left(\frac{U^{n+1}}{U^n}\right) \quad \text{as } t \rightarrow \infty,$$

Figure 6 and Figure 7, by using semi-implicit and fully implicit schemes respectively, show the distributions of $\lambda(t)$ proposed above, and it is seen that λ^n approaches to a negative constant as expected. For semi-explicit scheme we find $\lambda^n \sim -0.145$, and fully explicit $\lambda^n \sim -0.1336$, thus the difference is 1.2×10^{-2} which is within the rate of the truncation error of the discretization.

With λ^n calculated above we then can compute $C(t)$ by

$$C^n \sim U^n e^{-\lambda^n \Delta t^n} \quad \text{as } n \rightarrow \infty.$$

Figure 8 shows the distribution of $C(t)$ computed by semi-implicit scheme according the above assumption. In this example we see that $C(t)$ also approaches to a constant. Figure 9 and Figure 10 are the numerical solutions of u at $t = 0.5$ and $t = 1.0$ with $h = \pi/20$ and $\tau = 0.01$.

EXAMPLE 3. Taking the same model problem as in example 2 except that the initial data $\phi(x, y) = (\pi - x)(\pi - y)$ and $k = 0.4$. Figure 11, Figure 12 and Figure 13 show the distributions of $U(t), \lambda(t)$ and $C(t)$ using the semi-implicit scheme. It is noticed that $U(t)$ goes exponentially to zero very rapidly as $t \rightarrow \infty$ compared to that in example 2, this is due to that $C(t)$ also approaches to zero, not a fixed constant as in example 2.

From these examples We have a rough idea how $U(t)$ will behave as the time advances, i.e., we can at least by using numerical methods, semi-implicit or fully implicit scheme, to estimate the parameter λ mentioned in Section 1.

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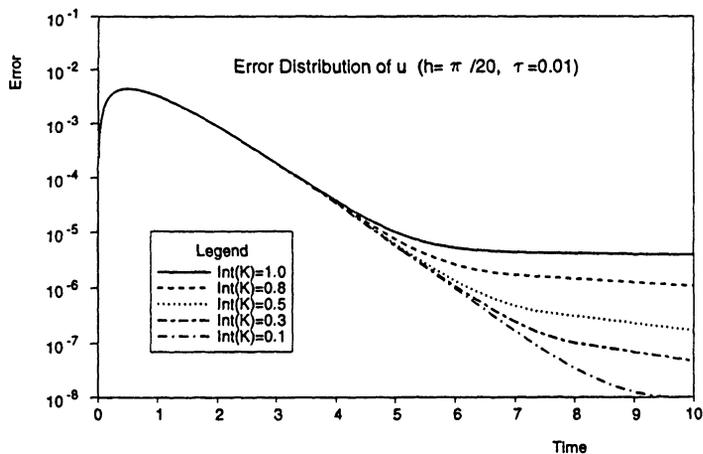


Figure 1: The semi-implicit scheme

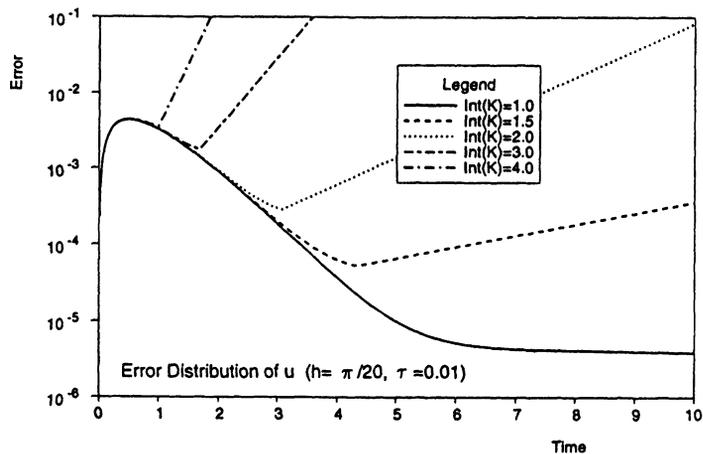


Figure 2: The semi-implicit scheme

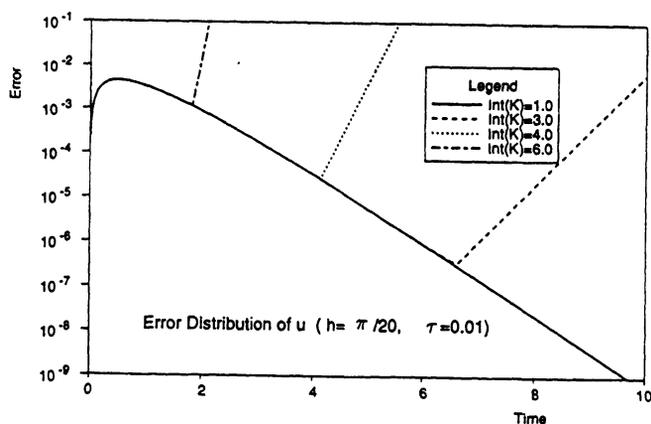


Figure 3: The fully-implicit scheme

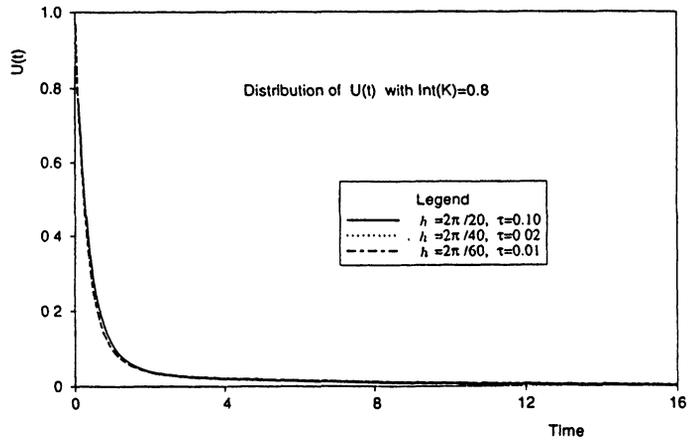


Figure 4: The semi-implicit scheme

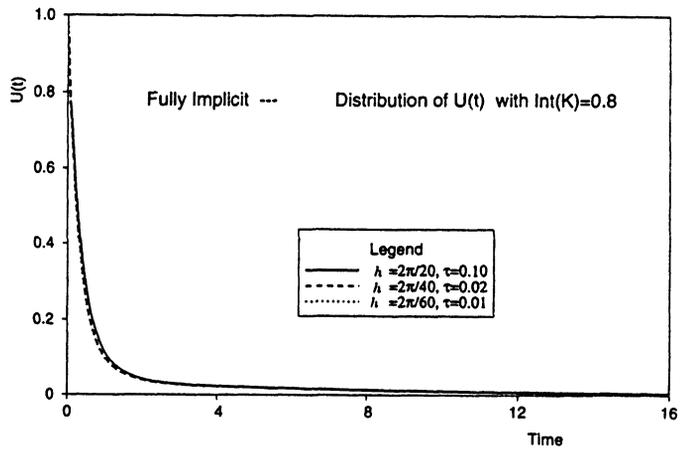


Figure 5: The fully-implicit scheme

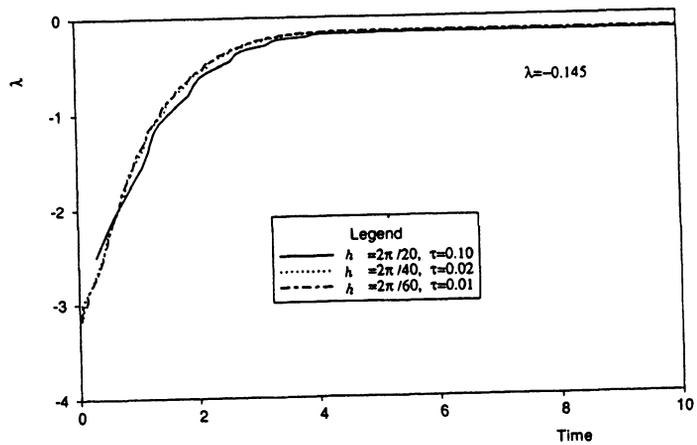


Figure 6: The semi-implicit scheme

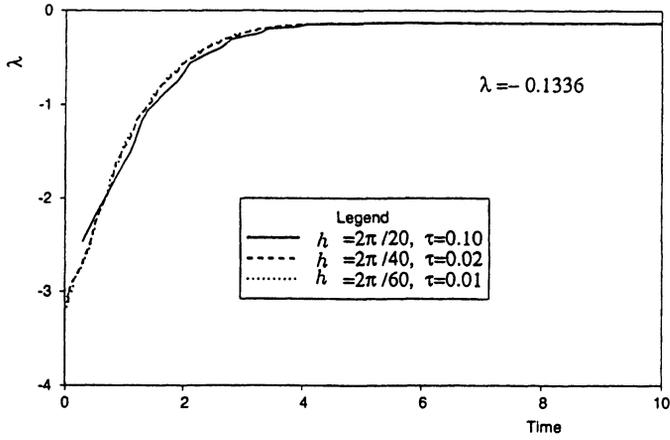


Figure 7: The fully-implicit scheme

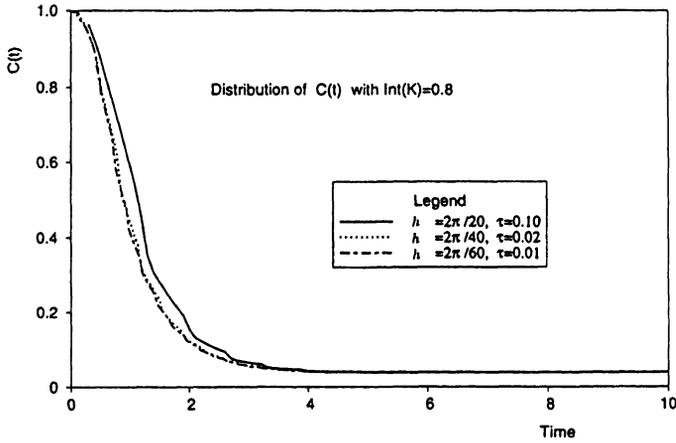


Figure 8: The semi-implicit scheme

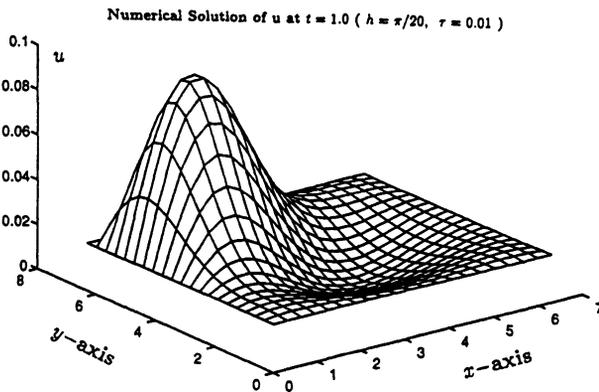


Figure 9: The semi-implicit scheme

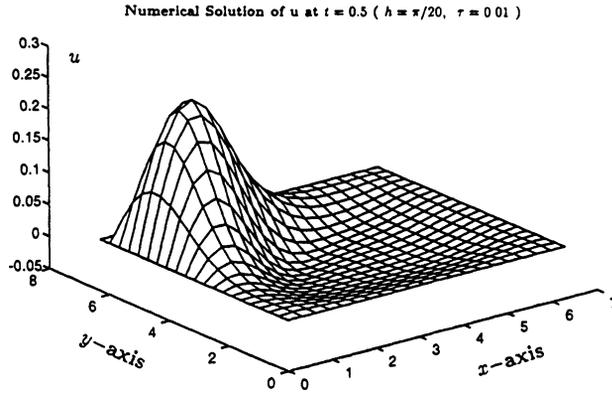


Figure 10: The semi-implicit scheme

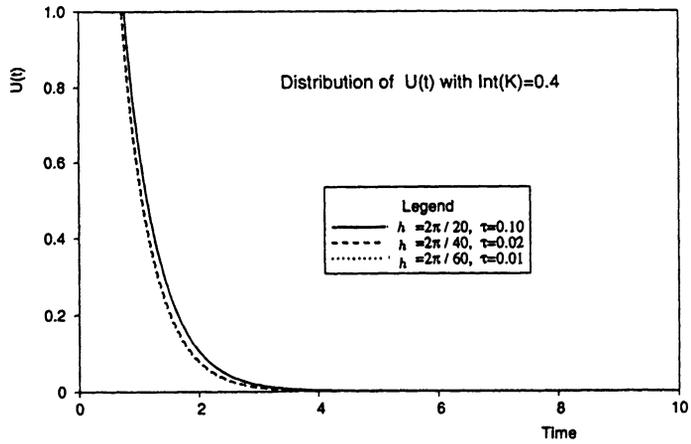


Figure 11: The semi-implicit scheme

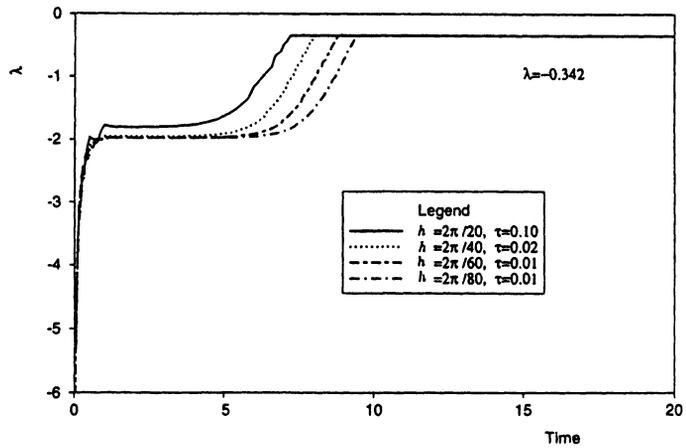


Figure 12: The semi-implicit scheme

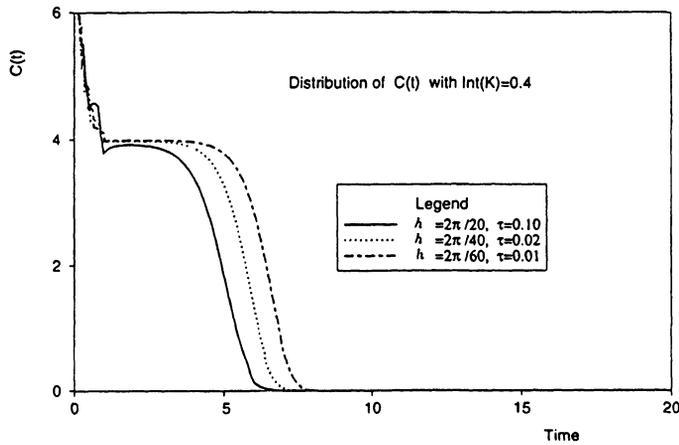


Figure 13: The semi-implicit scheme

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