Internat. J. Math. & Math. Sci.
VOL. 20 NO. 4 (1997) 673-680

FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS WITH APPLICATIONS TO THE SOLUTIONS OF FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMINGS

NAN-JING HUANG
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P.R CHINA

BYUNG SOO LEE
Department of Mathematics
Kyungsung University
Pusan 608-736
KOREA

MEE KWANG KANG
Department of Mathematics
Dongeui University
Pusan 614-714
KOREA

(Received October 4, 1995 and in revised form February 21, 1996)

ABSTRACT. Some common fixed point theorems for compatible mappings are shown As an application, the existence and uniqueness of common solutions for a class of functional equations arising in dynamic programmings are discussed.

KEY WORDS AND PHRASES: Common fixed point, compatible mapping, dynamic programming
1991 AMS SUBJECT CLASSIFICATION CODES: 54H25, 47H10

1. INTRODUCTION

In [1] the concept of compatible mappings was introduced as a generalization of commuting mappings and further investigation was given in [2-9]

The purpose of this paper is to prove some common fixed point theorems for compatible mappings, which generalized some recent results of [4, 10-13] As an application, we use the results presented to study the existence and uniqueness problem of a common solution for a class of functional equations arising in dynamic programmings, which generalized the corresponding results of [14,15].

2. FIXED POINT THEOREMS

DEFINITION 2.1. Self mappings A and S of a metric space (X,d) are called compatible, if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X

It is clear that commuting mappings and weakly commuting mappings are all compatible mappings, but the converse is false (see [1, 4]).

LEMMA 2.2 [1,4] If A and S are compatible self mappings of a metric space (X,d) and $\lim_n Sx_n = \lim_n Ax_n = t$ for some t in X, then $\lim_n ASx_n = St$ if S is continuous.

The following theorem can be obtained from Theorem 8 in [16].

THEOREM 2.3. Let (X,d) be a complete metric space and A, B, S and T are self mappings of X. Suppose that S and T are continuous, $A(X) \subset T(X)$, $B(X) \subset S(X)$, and that A, S and B, T are compatible and satisfy the following condition:

$$d(Ax,By) \le \Phi(\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{1}{2}[d(Sx,By)+d(Ty,Ax)]\}), \forall x,y \in X, \tag{2 1}$$

where $\Phi : [0,\infty) \to [0,\infty)$ is nondecreasing, upper semicontinuous and $\Phi(t) < t$ for all $t > 0$

Then A, B, S and T have a unique common fixed point in X

We merely state the proof for convenience

PROOF. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can choose a sequence $\{x_n\}$ in X such that $Sx_{2n} = Bx_{2n-1}$ and $Tx_{2n-1} = Ax_{2n-2}$ for all n in the set $\mathbb{N}$ of all positive integers Let

$$\left.\begin{array}{l} y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \\ y_{2n} = Sx_{2n} = Bx_{2n-1} \end{array}\right\} (n \in \mathbb{N}). \tag{2.2}$$

As in [10], we can prove that $\{y_n\}$ is a Cauchy sequence in X Letting $y_n \to y_* \in X\,(n \to \infty)$, we know that $\{y_{2n}\}$ and $\{y_{2n-1}\}$ converge to y_* too.

Since A and S, B and T are both compatible, it follows from the continuity of S and T, (2.2) and Lemma 2.2 that

$$Ty_{2n-1} \to Ty_*, \quad By_{2n-1} \to Ty_*, \quad Sy_{2n} \to Sy_*, \quad Ay_{2n} \to Sy_*. \tag{2.3}$$

By (2.1) and (2.2) we have

$$\begin{aligned} d(Ay_{2n}, By_{2n-1}) \leq \Phi(\max\{&d(Sy_{2n}, Ty_{2n-1}), d(Sy_{2n}, Ay_{2n}), \\ &d(Ty_{2n-1}, By_{2n-1}), \frac{1}{2}[d(Sy_{2n}, By_{2n-1}) + d(Ty_{2n-1}, Ay_{2n})]\}). \end{aligned}$$

By the upper semicontinuity of $\Phi(t)$ and (2.3) we have

$$\begin{aligned} d(Sy_*, Ty_*) &\leq \Phi(\max\{d(Sy_*, Ty_*), 0, 0, d(Sy_*, Ty_*)\}) \\ &= \Phi(d(Sy_*, Ty_*)). \end{aligned}$$

This implies that $Sy_* = Ty_*$

Similarly, from (2.1), (2.2) and (2.3) we can obtain

$$Sy_* = By_*, \quad Ty_* = Ay_*.$$

Hence we have

$$Ay_* = By_* = Sy_* = Ty_*. \tag{2 4}$$

From (2.1) and (2.2) we have

$$\begin{aligned} d(Ax_{2n}, By_*) \leq \Phi(\max\{&d(Sx_{2n}, Ty_*), d(Sx_{2n}, Ax_{2n}), \\ &d(Ty_*, By_*), \frac{1}{2}[d(Sx_{2n}, By_*) + d(Ty_*, Ax_{2n})]\}), \end{aligned}$$

and then

$$d(y_*, By_*) \leq \Phi(d(y_*, By_*)).$$

Hence we have $y_* = By_* = Ay_* = Sy_* = Ty_*$

The uniqueness is obvious. This completes the proof.

DEFINITION 2.4. A metric space (X, d) is (metrical) convex, if for each $x, y \in X$ with $x \neq y$, there exists a $z \in X$, $x \neq z \neq y$, such that

$$d(x, z) + d(z, y) = d(x, y).$$

LEMMA 2.5 [17]. Let K be a closed subset of a complete convex metric space X. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

DEFINITION 2.6. Let (X, d) be a metric space, $K \subset X$ and $A, S : K \to X$. The pair of mappings A and S is called compatible, if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in K such that $Ax_n, Sx_n \in K$ and $\lim_n Ax_n = \lim_n Sx_n = t \in K$

LEMMA 2.7. Let (X,d) be a metric space, $K \subset X$ and $A, S : K \to X$ If A and S are compatible mappings, $Ax_n, Sx_n \in K$ and $\lim_n Ax_n = \lim_n Sx_n = t$ for some $t \in K$, then $\lim_n ASx_n = St$ if S is continuous.

PROOF. It is obvious from Definition 2 6

THEOREM 2.8. Let (X,d) be a complete convex metric space and K a nonempty closed subset of X Suppose that S and T are continuous mappings from X into X with $\partial K \subset S(K) \cap T(K)$ and that $A, B : K \to X$ are continuous mappings with $A(K) \cap K \subset S(K)$, $B(K) \cap K \subset T(K)$ Suppose further that the pairs of mappings A, T and B, S are compatible and satisfying

$$d(Ax, By) \leq \Phi(d(Tx, Sy)), \forall x, y \in K, \tag{2 5}$$

where $\Phi : [0,\infty) \to [0,\infty)$ is nondecreasing upper semi-continuous and $\sum \Phi^n(t) < \infty$ for all $t \geq 0$

If for $x \in K$, $Tx \in \partial K$ implies $Ax, Bx \in K$ and $Sx \in \partial K$ implies $Ax, Bx \in K$, then there exists a $z \in K$ such that

$$z = Tz = Sz = Az = Bz.$$

If $Tv = Sv = Av = Bv$, then $Tz = Tv$

PROOF. Let $p \in \partial K$. Using Lemma 2 5 and the proof of [11] we can choose two sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{p'_n\}_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, $p_n \in K$, $p'_{2n-1} = Ap_{2n}$, $p'_{2n} = Bp_{2n-1}$ and the following implications hold·

(i) If $p'_{2n} \in K$, then $p'_{2n} = Tp_{2n}$, if $p'_{2n} \notin K$, then $Tp_{2n} \in \partial K$ and

$$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, Bp_{2n-1}) = d(Sp_{2n-1}, Bp_{2n-1})$$

(ii) If $p'_{2n+1} \in K$, then $p'_{2n+1} = Sp_{2n+1}$, if $p'_{2n+1} \notin K$, then $Sp_{2n+1} \in \partial K$ and

$$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, Ap_{2n}) = d(Tp_{2n}, Ap_{2n})$$

Further, as in [3] we can prove that

$$\left.\begin{array}{l} d(Tp_{2n}, Sp_{2n+1}) \leq \Phi^{n-1}(r) \\ d(Sp_{2n+1}, Tp_{2n+2}) \leq \Phi^n(r) \end{array}\right\} \quad (n \in \mathbb{N}), \tag{2.6}$$

where $r = \max\{d(Tp_2, Sp_3), d(Tp_2, Sp_1)\}$.

This implies that for any $n \in \mathbb{N}$,

$$d(Tp_{2n}, Tp_{2n+2}) \leq \Phi^{n-1}(r) + \Phi^n(r).$$

Hence the sequence $\{Tp_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete and K is closed, it follows that there exists a $z \in K$ such that $z = \lim_n Tp_{2n}$ From (2 6) we have

$$z = \lim_n Tp_{2n} = \lim_n Sp_{2n+1}.$$

Now we prove that $z = Tz = Sz = Az = Bz$ It is obvious that there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that $Tp_{2n_k} = Bp_{2n_k-1}$, or $Sp_{2n_k-1} = Ap_{2n_k-2}, \forall k \in \mathbb{N}$. Without loss of generality, we can suppose that $Tp_{2n_k} = Bp_{2n_k-1}, \forall k \in \mathbb{N}$ From (2.5) we have

$$\begin{aligned} d(STp_{2n_k}, Az) &\leq d(SBp_{2n_k-1}, BSp_{2n_k-1}) + d(BSp_{2n_k-1}, Az) \\ &\leq d(SBp_{2n_k-1}, BSp_{2n_k-1}) + \Phi(d(SSp_{2n_k-1}, Tz)). \end{aligned}$$

Since B, S are compatible and S is continuous, we have

$$d(Sz, Az) \leq \Phi(d(Sz, Tz)). \tag{2 7}$$

From (2.5) we have

$$d(Ap_{2n_k}, Tp_{2n_k}) = d(Ap_{2n_k}, Bp_{2n_k-1}) \leq \Phi(d(Sp_{2n_k-1}, Tp_{2n_k})).$$

By the upper semi-continuity of $\Phi(t)$, it follows that

$$\lim_k Ap_{2n_k} = z. \tag{2 8}$$

Using (2.5) we have

$$d(Ap_{2n_k}, BSp_{2n_k-1}) \leq \Phi(d(Tp_{2n_k}, SSp_{2n_k-1}).$$

Since B, S are compatible and S is continuous, it follows from (2.8) and Lemma 2 7 that

$$d(z, Sz) \leq \Phi(d(z, Sz)).$$

This implies that $d(z, Sz) = 0$, i e. $z = Sz$.

Since A, T are compatible and A and T are continuous, from (2.8) and Lemma 2.7 we have

$$Az = \lim_k ATp_{2n_k} = Tz.$$

In view of (2.7) we have

$$d(Sz, Tz) \leq \Phi(d(Sz, Tz))$$

and so

$$z = Sz = Tz = Az.$$

Besides, from (2.5) we have

$$d(Az, Bz) \leq \Phi(d(Sz, Tz)) = \Phi(0) = 0.$$

Hence

$$z = Tz = Sz = Az = Bz.$$

Finally, if $Tv = Sv = Av = Bv$, then

$$d(Tv, Sz) = d(Av, Bz) \leq \Phi(d(Sz, Tv))$$

and so $Tv = Sz = Tz$.

This completes the proof of Theorem 2 8.

As an immediate consequence we can obtain the following result.

THEOREM 2.9. Let (X, d) be a complete convex metric space, K a nonempty closed subset of X, and S and T continuous mappings from X into X such that $\partial K \subset S(K) \cap T(K)$. Suppose that for every $n \in \mathbb{N}$, $A_n : K \to X$ is a continuous mapping with $A_{2n}(K) \cap K \subset T(K)$ and $A_{2n-1}(K) \cap K \subset S(K)$, and that the pairs of mappings A_{2n-1}, T and A_{2n}, S are compatible such that for any $n \in \mathbb{N}$

$$d(A_n x, A_{n+1} y) \leq \Phi(d(Tx, Sy)), \forall x, y \in K,$$

where $\Phi(t)$ is the same as in Theorem 2.8.

If for every $n \in \mathbb{N}$ and $x \in K$,

$$Tx \in \partial K \text{ implies } A_n x \in K \text{ and } Sx \in \partial K \text{ implies } A_n x \in K,$$

then there exists a $z \in K$ such that

$$z = Tz = Sz = A_n z, \quad \forall n \in \mathbb{N},$$

and if $Tv = Sv = A_n v$ for every $n \in \mathbb{N}$, then $Tz = Tv$.

REMARK 2.10. Theorem 2.9 is a generalization of Theorem 1 in [11]

3. APPLICATIONS

Throughout this section we assume that X and Y are Banach spaces, $S \subset X$ is a state space, $D \subset Y$ a decision space and $\mathbb{R} = (-\infty, +\infty)$ We denote by $B(S)$ the set of all bounded real-valued functions defined on S.

As suggested in Bellman and Lee [18], the basic form of the functional equations of dynamic programming is

$$f(x) = \text{opt}_y H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return function with initial state x (here opt denotes max or min)

In this section, we shall study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programmings

$$f(x) = \sup_{y \in D} H_1(x, y, f(T(x, y))), \quad x \in S, \tag{3 1}$$

$$g(x) = \sup_{y \in D} H_2(x, y, g(T(x, y))), \quad x \in S, \tag{3 2}$$

$$p(x) = \sup_{y \in D} F_1(x, y, p(T(x, y))), \quad x \in S, \tag{3 3}$$

$$q(x) = \sup_{y \in D} F_2(x, y, q(T(x, y))), \quad x \in S, \tag{3 4}$$

where $T : S \times D \to S$, H_i and $F_i : S \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$.

THEOREM 3.1. Suppose that the following conditions are satisfied:

(i) H_i and F_i are bounded, $i = 1, 2$.

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(\max\{|T_1 h(t) - T_2 k(t)|,$
$|T_1 h(t) - A_1 h(t)|, |T_2 k(t) - A_2 k(t)|, \frac{1}{2}[|T_1 h(t) - A_2 k(t)| + |T_2 k(t) - A_1 h(t)|]\})$,

for all $(x, y) \in S \times D, h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 2 3, and the mappings A_i and T_i are defined as follows:

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \quad x \in S, \quad h \in B(S) \quad \text{and}$$

$$T_i k(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \quad x \in S, \quad k \in B(S), \quad i = 1, 2.$$

(iii) For any $\{k_n\} \subset B(S)$ and $k \in B(S)$,

$$\limsup_n \sup_{x \in S} |k_n(x) - k(x)| = 0 \quad \text{implies} \quad \limsup_n \sup_{x \in S} |T_i k_n(x) - T_i k(x)| = 0, \quad i = 1, 2.$$

(iv) For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1 h(x) = T_2 k_1(x), \quad A_2 h(x) = T_1 k_2(x), \quad x \in S.$$

(v) For any $\{k_n\} \subset B(S)$, if there exists $h \in B(S)$ such that

$$\limsup_n \sup_{x \in S} |A_i k_n(x) - h(x)| = \limsup_n \sup_{x \in S} |T_i k_n(x) - h(x)| = 0,$$

then

$$\limsup_n \sup_{x \in S} |A_i T_i k_n(x) - T_i A_i k_n(x)| = 0, \quad i = 1, 2.$$

Then the system of functional equations (3.1)-(3.4) has a unique common solution in $B(S)$.

PROOF. For any $h, k \in B(S)$, let

$$d(h,k) = \sup\{|h(x) - k(x)| : x \in S\},$$

then $(B(S), d)$ is a complete metric space From (i)-(v) we know that A_i and T_i are self mappings of $B(S), T_i$ are continuous, $A_1(B(S)) \subset T_2(B(S)), A_2(B(S)) \subset T_1(B(S))$, and the pair of mappings A_i, T_i are compatible, $i = 1, 2$.

Let h_1, h_2 be any two points of $B(S)$, let $x \in S$ and η be any positive number, there exist y_1, and y_2 in D such that

$$\left.\begin{array}{l} A_i h_i(x) < H_i(x, y_i, h_i(x_i)) + \eta \\ \text{where } x_i = T(x, y_i) \end{array}\right\} \quad (i = 1, 2). \tag{3 5}$$

Also we have

$$A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)), \tag{3 6}$$

$$A_2 h_2(x) \geq H_2(x, y_1, h_2(x_1)). \tag{3 7}$$

From (3.5), (3 7) and (ii) we have

$$\begin{aligned} A_1 h_1(x) - A_2 h_2(x) &< H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\ &\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta \\ &\leq \Phi(\max\{|T_1 h_1(x_1) - T_2 h_2(x_1)|, |T_1 h_1(x_1) - A_1 h_1(x_1)|, \\ &\qquad |T_2 h_2(x_1) - A_2 h_2(x_1)|, \frac{1}{2}[|T_1 h_1(x_1) - A_2 h_2(x_1)| \\ &\qquad + |T_2 h_2(x_1) - A_1 h_1(x_1)|]\}) + \eta \\ &\leq \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &\qquad d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) \\ &\qquad + d(T_2 h_2, A_1 h_1)]\}) + \eta. \end{aligned}$$

Similarly from (3.5), (3.6) and (ii) we have

$$\begin{aligned} A_1 h_1(x) - A_2 h_2(x) \geq\ &- \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) \\ &+ d(T_2 h_2, A_1 h_1)]\}) - \eta. \end{aligned}$$

Hence we have

$$\begin{aligned} |A_1 h_1(x) - A_2 h_2(x)| \leq\ &\Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) \\ &+ d(T_2 h_2, A_1 h_1)]\}) + \eta. \end{aligned} \tag{3 8}$$

Since (3.8) is true for any $x \in S$ and η is any positive number, we have

$$\begin{aligned} d(A_1 h_1, A_2 h_2) \leq\ &\Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2), \\ &+ d(T_2 h_2, A_1 h_1)]\}). \end{aligned}$$

Therefore by Theorem 2.3, A_1, A_2, T_1 and T_2 have a unique common fixed point $h^* \in B(S)$, i.e $h^*(x)$ is a unique common solution of functional equations (3.1) - (3.4). This completes the proof.

The following result is an immediate consequence of Theorem 2.3 and Theorem 3 1

THEOREM 3.2. Suppose that the following conditions are satisfied:

(i) H_i is bounded, $i = 1, 2$;

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(\max\{|h(t) - k(t)|, |h(t) - A_1 h(t)|,$
$|k(t) - A_2 k(t)|, \frac{1}{2}[|h(t) - A_2 k(t)| + |k(t) - A_1 h(t)|]\})$

for all $(x, y) \in S \times D, h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 2 3 and A_i is defined by

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \quad x \in S, \quad h \in B(S), \quad i = 1, 2.$$

Then the functional equations (3.1) and (3.2) have a unique common solution in $B(S)$

REMARK 3.3. Theorem 3.2 is a generalization of Theorem 2.1 in [15].

THEOREM 3.4. Suppose that the following conditions are satisfied.

(i) H_i and F_i are bounded, $i = 1, 2$,

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(|T_1 h(t) - T_2 k(t)|)$ for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 2.8 and T_i is defined as in Theorem 3 1, $i = 1, 2$;

(iii) For any $\{k_n\} \subset B(S)$ and $k \in B(S)$,

$$\lim_n \sup_{x \in S} |k_n(x) - k(x)| = 0 \text{ implies } \lim_n \sup_{x \in S} |T_i k_n(x) - T_i k(x)| = 0$$

and

$$\lim_n \sup_{x \in S} |A_i k_n(x) - A_i k(x)| = 0, \quad i = 1, 2,$$

where A_i is defined as in Theorem 3.1, $i = 1, 2$;

(iv) For any $h \in B(S)$ such that $\sup_{x \in S} |h(x)| = 1$, there exist $k_1, k_2 \in B(S)$ such that

$$\sup_{x \in S} |k_i(x)| \leq 1 \quad \text{and} \quad T_i k_i(x) = h(x), \quad x \in S, \quad i = 1, 2;$$

(v) For any $h \in B(S)$ such that $\sup_{x \in S} |h(x)| \leq 1$, there exist $k_1, k_2 \in B(S)$ such that

$$\sup_{x \in S} |k_i(x)| \leq 1, \quad i = 1, 2, \quad A_1 h(x) = T_2 k_1(x) \quad \text{and} \quad A_2 h(x) = T_1 k_2(x), \quad x \in S;$$

(vi) For any $h \in B(S)$ such that $\sup_{x \in S} |h(x)| \leq 1$,

$$\sup_{x \in S} |T_i h(x)| = 1 \quad \text{implies} \quad \sup_{x \in S} |A_j h(x)| \leq 1, \quad i, j = 1, 2;$$

(vii) For any $\{k_n\} \subset B(S)$, if there exists $h \in B(S)$ such that $\sup_{x \in S} |T_i k_n(x)| \leq 1$ and

$$\lim_n \sup_{x \in S} |A_i k_n(x) - h(x)| = \lim_n \sup_{x \in S} |T_i k_n(x) - h(x)| = 0,$$

then

$$\lim_n \sup_{x \in S} |A_i T_i k_n(x) - T_i A_i k_n(x)| = 0, \quad i = 1, 2.$$

Then the system of functional equations (3 1) - (3.4) have a unique common solution $h^* \in B(S)$ and $\sup_{x \in S} |h^*(x)| \leq 1$.

PROOF. Let us consider $B(S)$ as a Banach space of all bounded real-valued functions defined on S with a supremum norm, and K the closed unit ball in $B(S)$. By conditions (i)-(vii) we know that $A_i : K \to B(S)$ and $T_i : B(S) \to B(S)$, $i = 1, 2$, satisfy all of the conditions of Theorem 2.8 and have a unique common fixed point $h^* \in K$, i.e , $h^*(x)$ is a unique common solution of functional equations (3.1) - (3 4).

REMARK 3.5. Theorem 3 4 is a generalization of Theorem 3.2 in [14].

ACKNOWLEDGMENT. The authors are indebted to the referees for their helpful comments The second author was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation

REFERENCES

[1] JUNGCK, G., Compatible mappings and common fixed points, *Internat. J. Math. & Math. Sci.* **9** (1986), 771-779.

[2] JUNGCK, G., Common fixed points for compatible maps on the unit interval, *Proc. Amer. Math. Soc.* **115** (1992), 495-499

[3] JUNGCK, G., Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.* **103** (1988), 977-983

[4] JUNGCK, G., Compatible mappings and common fixed points (2), *Internat. J. Math. & Math. Sci.* **11** (1988), 285-288.

[5] JUNGCK, G, MURTHY, P.P and CHO, Y.J., Compatible mappings of type (A) and common fixed points, *Math. Japonica* **38** (1993), 381-390.

[6] JUNGCK, G. and RHOADES, B.E., Some fixed point theorems for compatible maps, *Internat. J. Math. & Math. Sci.* **16** (1993), 417-428.

[7] KANG, S.M., CHO, Y.J and JUNGCK, G, Common fixed points of compatible mappings, *Internat. J. Math. & Math. Sci.* **13** (1990), 61-65

[8] KANG, S.M. and PYU, J W., A common fixed point theorem for compatible mappings, *Math. Japonica* **35** (1990), 153-157.

[9] MURTHY, P.P., CHANG, S S., CHO, Y.J and SHARMA, B.K., Compatible mappings of type (A) and common fixed point theorems, *Kyungpook Math. J.* **32** (1992), 203-216.

[10] CHANG, S.S., On common fixed point theorem for a family of Φ-contraction mappings, *Math. Japonica* **29** (1984), 527-536.

[11] HADZIC, O, On coincidence theorems for a family of mappings in convex metric spaces, *Internat. J. Math. & Math. Sci.* **10** (1987), 453-460.

[12] HADZIC, O., Common fixed point theorems for a family of mappings in complete metric spaces, *Math. Japonica* **29** (1984), 127-134

[13] SINGH, S.L. and SINGH, S.P., A fixed point theorem, *Indian J. Pure Appl. Math.* **11** (1980), 1584-1586.

[14] BASKARAN, R and SUBRAHMANYAM, P.V., A note on the solution of a class functional equation, *Applicable Anal.* **22** (1986), 235-241.

[15] BHAKTA, P.C. and MITRA, S., Some existence theorems for functional equations arising in dynamic programming, *J. Math. Anal. Appl.* **98** (1984), 348-362.

[16] JACHVMSKI, J., Common fixed point theorems for some families of maps, *Indian J. Pure Appl. Math.* **25** (9) (1994), 927-937.

[17] ASSAD, N.A. and KIRK, W A., Fixed point theorems for set-valued mappings of contractive type, *Pacific J. Math.* **43** (1972), 553-562.

[18] BELLMAN, R. and LEE, E.S., Functional equations arising in dynamic programming, *Aequationes Math.* **17** (1978), 1-18.