

## ON GENERALIZATIONS OF THE POMPEIU FUNCTIONAL EQUATION

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(Received October 25, 1995 and in revised form January 15, 1997)

**ABSTRACT.** In this paper, we determine the general solution of the functional equations

$$f(x + y + xy) = p(x) + q(y) + g(x)h(y), \quad (\forall x, y \in \mathfrak{R}_*)$$

and

$$f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \quad (\forall x, y \in \mathfrak{R})$$

which are generalizations of a functional equation studied by Pompeiu. We present a method which is simple and direct to determine the general solutions of the above equations without any regularity assumptions.

**KEY WORDS AND PHRASES:** Pompeiu functional equation, multiplicative function, logarithmic function, exponential function.

**1991 AMS SUBJECT CLASSIFICATION CODES:** 39B22.

### 1. INTRODUCTION

Let  $\mathfrak{R}$  be the set of all real numbers and  $\mathfrak{R}_o$  denote the set of nonzero reals. Further, let  $\mathfrak{R}_* = \mathfrak{R} \setminus \{-1\}$ , that is the set of real numbers except negative one. A function  $M : \mathcal{D} \rightarrow \mathfrak{R}$  is said to be *multiplicative* if and only if  $M(xy) = M(x)M(y)$  for all  $x, y \in \mathcal{D}$ , where  $\mathcal{D} = \mathfrak{R}$  or  $\mathfrak{R}_o$ . A function  $E : \mathfrak{R} \rightarrow \mathfrak{R}$  is called *exponential* if and only if  $E(x + y) = E(x)E(y)$  for all  $x, y \in \mathfrak{R}$ . A function  $L : \mathfrak{R}_o \rightarrow \mathfrak{R}$  is said to be *logarithmic* if and only if  $L(xy) = L(x) + L(y)$  for all  $x, y \in \mathfrak{R}_o$ . A comprehensive treatment of these functions can be found in the book of Aczel and Dhombres [1].

If  $\mathbf{G} = \mathfrak{R}_*$ , then  $(\mathbf{G}, \circ)$  is an abelian group where the group operation is defined as

$$x \circ y = x + y + xy.$$

A characterization of the homomorphisms of the group  $(\mathbf{G}, \circ)$  can be obtained by solving the functional equation

$$f(x + y + xy) = f(x) + f(y) + f(x)f(y). \quad (\text{PE})$$

This functional equation is known as the *Pompeiu functional equation* [3,4].

Suppose that  $f : \mathfrak{K} \rightarrow \mathfrak{K}$  satisfies (PE). Then the only solution  $f$  of the Pompeiu equation (PE) is given by

$$f(x) = M(x + 1) - 1, \quad (1.1)$$

where  $M$  is multiplicative.

To see this, add 1 to both sides of (PE) and write  $F(x) = 1 + f(x)$ . Then (PE) reduces to  $F(x + y + xy) = F(x)F(y)$ . Now replacing  $x$  by  $x - 1$  and  $y$  by  $y - 1$ , we obtain  $M(xy) = M(x)M(y)$ , where  $M(x) = F(x - 1)$ . Thus,  $M$  is multiplicative and  $f(x) = F(x) - 1 = M(x + 1) - 1$ , which is (1.1).

In a special case,  $f$  is an automorphism of the field  $\mathfrak{K}$ . Suppose  $M$  is also additive. Then  $M$  is a ring homomorphism of  $\mathfrak{K}$ . If  $M$  is a nontrivial homomorphism, then  $f(x) = M(x) = x$ , that is,  $f$  is an automorphism of the field  $\mathfrak{K}$ .

In this paper, we determine the general solution of the functional equations

$$f(x + y + xy) = p(x) + q(y) + g(x)h(y), \quad (\forall x, y \in \mathfrak{K}_*) \quad (\text{FE1})$$

and

$$f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \quad (\forall x, y \in \mathfrak{K}) \quad (\text{FE2})$$

which are generalizations of the Pompeiu functional equation (PE). We present a method which is simple and direct to determine the general solutions of (FE1) and (FE2) without any regularity assumptions. For other related functional equations, the interested reader should refer to [2] and [5].

## 2. SOME PRELIMINARY RESULTS

The following two lemmas will be instrumental for establishing the main result of this paper.

**LEMMA 1.** Let  $g, h : \mathfrak{K}_o \rightarrow \mathfrak{K}$  satisfy the functional equation

$$g(xy) = g(y) + g(x)h(y) \quad (2.1)$$

for all  $x, y \in \mathfrak{K}_o$ . Then for all  $x, y \in \mathfrak{K}_o$ ,  $g(x)$  and  $h(y)$  are given by

$$g(x) = 0, \quad h(y) = \text{arbitrary}; \quad (2.2)$$

$$g(x) = L(x), \quad h(y) = 1; \quad (2.3)$$

$$g(x) = \alpha [M(x) - 1], \quad h(y) = M(y), \quad (2.4)$$

where  $M : \mathfrak{K}_o \rightarrow \mathfrak{K}$  is a multiplicative map not identically one,  $L : \mathfrak{K}_o \rightarrow \mathfrak{K}$  is a logarithmic function not identically zero and  $\alpha$  is an arbitrary nonzero constant.

**PROOF.** If  $g \equiv 0$ , then  $h$  is arbitrary and they satisfy the equation (2.1). Hence we have the solution (2.2). We assume hereafter that  $g \not\equiv 0$ .

Interchanging  $x$  with  $y$  in (2.1) and comparing the resulting equation to (2.1), we get

$$g(y)[h(x) - 1] = g(x)[h(y) - 1]. \quad (2.5)$$

Suppose  $h(x) = 1$  for all  $x \in \mathfrak{K}_o$ . Then (2.1) yields  $g(xy) = g(y) + g(x)$  and hence the function  $g : \mathfrak{K}_o \rightarrow \mathfrak{K}$  is logarithmic. This yields the solution (2.3).

Finally, suppose  $h(y) \neq 1$  for some  $y$ . Then from (2.5), we have

$$g(x) = \alpha [h(x) - 1], \tag{2.6}$$

where  $\alpha$  is a nonzero constant, since  $g \neq 0$ . Using (2.6) in (2.1), and simplifying, we obtain

$$h(xy) = h(x) h(y). \tag{2.7}$$

Hence,  $h : \mathfrak{R}_o \rightarrow \mathfrak{R}$  is a multiplicative function. This gives the asserted solution (2.4) and the proof of the lemma is now complete.

**LEMMA 2.** The general solutions  $f, g, h : \mathfrak{R}_o \rightarrow \mathfrak{R}$  of the functional equation

$$f(xy) = f(x) + f(y) + \alpha g(x) + \beta h(y) + g(x)h(y) \quad (\forall x, y \in \mathfrak{R}_o) \tag{2.8}$$

where  $\alpha$  and  $\beta$  are apriori chosen constants, have values  $f(x), g(x)$  and  $h(y)$  given, for all  $x, y \in \mathfrak{R}_o$ , by

$$\left. \begin{aligned} f(x) &= L(x) + \alpha\beta \\ g(x) &\text{ is arbitrary} \\ h(y) &= -\alpha; \end{aligned} \right\} \tag{2.9}$$

$$\left. \begin{aligned} f(x) &= L(x) + \alpha\beta \\ g(x) &= -\beta \\ h(y) &\text{ is arbitrary;} \end{aligned} \right\} \tag{2.10}$$

$$\left. \begin{aligned} f(x) &= L_o(x) + \frac{1}{2} c L_1^2(x) + \alpha\beta \\ g(x) &= c L_1(x) - \beta \\ h(y) &= L_1(y) - \alpha; \end{aligned} \right\} \tag{2.11}$$

$$\left. \begin{aligned} f(x) &= L(x) + \gamma\delta [M(x) - 1] + \alpha\beta \\ g(x) &= \gamma [M(x) - 1] - \beta \\ h(y) &= \delta [M(y) - 1] - \alpha, \end{aligned} \right\} \tag{2.12}$$

where  $M : \mathfrak{R}_o \rightarrow \mathfrak{R}$  is a multiplicative map not identically one,  $L_o, L_1, L : \mathfrak{R}_o \rightarrow \mathfrak{R}$  are logarithmic functions with  $L_1$  not identically zero, and  $c, \delta, \gamma$  are arbitrary nonzero constants.

**PROOF.** Interchanging  $x$  with  $y$  in (2.8) and comparing the resulting equation to (2.8), we obtain

$$[\alpha + h(y)][\beta + g(x)] = [\alpha + h(x)][\beta + g(y)]. \tag{2.13}$$

Now we consider several cases.

**Case 1.** Suppose  $h(y) = -\alpha$  for all  $y \in \mathfrak{R}_o$ . Then (2.8) yields

$$f(xy) = f(x) + f(y) - \alpha\beta. \tag{2.14}$$

Hence

$$f(x) = L(x) + \alpha\beta, \tag{2.15}$$

where  $L : \mathfrak{R}_o \rightarrow \mathfrak{R}$  is a logarithmic function. Hence we have the asserted solution (2.9).

**Case 2.** Suppose  $g(x) = -\beta$  for all  $x \in \mathfrak{R}_o$ . Then (2.8) yields

$$f(xy) = f(x) + f(y) - \alpha\beta.$$

Hence, as before,

$$f(x) = L(x) + \alpha\beta,$$

where  $L : \mathfrak{X}_o \rightarrow \mathfrak{X}$  is a logarithmic function. Thus we have the asserted solution (2.10).

**Case 3.** Now we assume  $h(x) \neq -\alpha$  for some  $x \in \mathfrak{X}_o$  and  $g(x) \neq -\beta$  for some  $x \in \mathfrak{X}_o$ . From (2.13), we get

$$\beta + g(y) = c[\alpha + h(y)], \quad (2.16)$$

where  $c$  is a nonzero constant.

Using (2.8), we compute

$$\begin{aligned} f(x \cdot yz) &= f(x) + f(y) + f(z) + \alpha g(y) + \beta h(z) \\ &\quad + g(y)h(z) + \alpha g(x) + \beta h(yz) + g(x)h(yz). \end{aligned} \quad (2.17)$$

Again, using (2.8), we have

$$\begin{aligned} f(xy \cdot z) &= f(x) + f(y) + f(z) + \alpha g(x) + \beta h(y) \\ &\quad + g(x)h(y) + \alpha g(xy) + \beta h(z) + g(xy)h(z). \end{aligned} \quad (2.18)$$

From (2.17) and (2.18), we obtain

$$[\alpha + h(z)][g(y) - g(xy)] = [\beta + g(x)][h(y) - h(yz)], \quad \forall x, y \in \mathfrak{X}_o. \quad (2.19)$$

Since  $g(x) \neq -\beta$  for some  $x \in \mathfrak{X}_o$ , there exists a  $x_o \in \mathfrak{X}_o$  such that  $g(x_o) + \beta \neq 0$ . Letting  $x = x_o$  in (2.19), we have

$$h(yz) = h(y) + [\alpha + h(z)]k(y), \quad (2.20)$$

where

$$k(y) = \frac{g(yx_o) - g(y)}{g(x_o) + \beta}. \quad (2.21)$$

The general solution of (2.20) can be obtained from Lemma 1 (add  $\alpha$  to both sides). Hence, taking into consideration that  $h(y) + \alpha \neq 0$ , we have

$$h(y) = L_1(y) - \alpha. \quad (2.22)$$

or

$$h(y) = \delta [M(y) - 1] - \alpha, \quad (2.23)$$

where  $L_1$  is logarithmic not identically zero,  $M$  is multiplicative not identically one, and  $\delta$  is an arbitrary constant.

Now we consider two subcases.

**Subcase 3.1.** From (2.22) and (2.16), we have

$$g(y) = cL_1(y) - \beta. \quad (2.24)$$

Using (2.22) and (2.24) in (2.8), we get

$$f(xy) = f(x) + f(y) + cL_1(x)L_1(y) - \alpha\beta. \quad (2.25)$$

Defining

$$L_o(x) := f(x) - \frac{1}{2}cL_1^2(x) - \alpha\beta, \quad (2.26)$$

we see that (2.25) reduces to

$$L_o(xy) = L_o(x) + L_o(y)$$

for all  $x, y \in \mathfrak{R}_o$ , that is,  $L_o$  is logarithmic and from (2.26), we have

$$f(x) = L_o(x) + \frac{1}{2} c L_1^2(x) + \alpha\beta. \tag{2.27}$$

Hence (2.27), (2.24) and (2.22) yield the asserted solution (2.11).

**Subcase 3.2.** Finally, from (2.23) and (2.16), we obtain

$$g(y) = \delta c [M(y) - 1] - \beta. \tag{2.28}$$

With (2.23) and (2.28) in (2.8), we have

$$f(xy) = f(x) + f(y) - \alpha\beta + c\delta^2 [M(x) - 1][M(y) - 1]. \tag{2.29}$$

Defining

$$L(x) := f(x) - c\delta^2 [M(x) - 1] - \alpha\beta, \tag{2.30}$$

we see that (2.29) reduces to

$$L(xy) = L(x) + L(y)$$

for all  $x, y \in \mathfrak{R}_o$ , that is,  $L$  is a logarithmic function. Using (2.30), we have

$$f(x) = L(x) + \gamma\delta [M(x) - 1] + \alpha\beta, \tag{2.31}$$

where  $\gamma = c\delta$ . Hence (2.31), (2.28) and (2.23) yield the asserted solution (2.12). This completes the proof of the lemma.

### 3. SOLUTION OF THE FUNCTIONAL EQUATION (FE1)

Now we are ready to present the general solution of (FE1) using Lemma 2.

**THEOREM 1.** The functions  $f, p, q, g, h : \mathfrak{R}_* \rightarrow \mathfrak{R}$  satisfy the functional equation

$$f(x + y + xy) = p(x) + q(y) + g(x)h(y) \tag{FE1}$$

for all  $x, y \in \mathfrak{R}_*$  if and only if, for all  $x, y \in \mathfrak{R}_*$ ,

$$\left. \begin{aligned} f(x) &= L(x + 1) + \alpha\beta + a + b \\ p(x) &= L(x + 1) + b \\ q(y) &= L(y + 1) + \alpha\beta + a + \beta h(y) \\ g(x) &= -\beta \\ h(y) &\text{ is arbitrary;} \end{aligned} \right\} \tag{3.1}$$

$$\left. \begin{aligned} f(x) &= L(x + 1) + \alpha\beta + a + b \\ p(x) &= L(x + 1) + \alpha\beta + b + \alpha g(x) \\ q(y) &= L(y + 1) + a \\ g(x) &\text{ is arbitrary} \\ h(y) &= -\alpha; \end{aligned} \right\} \tag{3.2}$$

$$\left. \begin{aligned} f(x) &= L(x+1) + \gamma\delta [M(x+1) - 1] + \alpha\beta + a + b \\ p(x) &= L(x+1) + (\delta + \alpha)\gamma [M(x+1) - 1] + b \\ q(y) &= L(y+1) + (\gamma + \beta)\delta [M(y+1) - 1] + a \\ g(x) &= \gamma [M(x+1) - 1] - \beta \\ h(y) &= \delta [M(y+1) - 1] - \alpha; \end{aligned} \right\} \tag{3.3}$$

$$\left. \begin{aligned} f(x) &= L_o(x+1) + \frac{1}{2} c L_1^2(x+1) + \alpha\beta + a + b \\ p(x) &= L_o(x+1) + \frac{1}{2} c L_1^2(x+1) + \alpha c L_1(x+1) + b \\ q(y) &= L_o(y+1) + \frac{1}{2} c L_1^2(y+1) + \beta L_1(y+1) + a \\ g(x) &= c L_1(x+1) - \beta \\ h(y) &= L_1(y+1) - \alpha, \end{aligned} \right\} \tag{3.4}$$

where  $M : \mathfrak{R}_o \rightarrow \mathfrak{R}$  is a multiplicative function not identically one,  $L_o, L_1, L : \mathfrak{R}_o \rightarrow \mathfrak{R}$  are logarithmic maps with  $L_1$  not identically zero, and  $\alpha, \beta, \gamma, \delta, a, b, c$  are arbitrary real constants.

**PROOF.** First, we substitute  $y = 0$  in (FE1) and then we put  $x = 0$  in (FE1) to obtain

$$p(x) = f(x) - a + \alpha g(x) \tag{3.5}$$

and

$$q(y) = f(y) - b + \beta h(y), \tag{3.6}$$

where  $a := q(0), b := p(0), \alpha := -h(0), \beta := -g(0)$ . Using (3.5) and (3.6) in (FE1), we have

$$f(x + y + xy) = f(x) + f(y) - a - b + \alpha g(x) + \beta h(y) + g(x)h(y) \tag{3.7}$$

for  $x, y \in \mathfrak{R}_*$ . Replacing  $x$  by  $u - 1$  and  $y$  by  $v - 1$  in (3.7) and then defining

$$F(u) := f(u - 1) - a - b, \quad G(u) := g(u - 1), \quad H(u) := h(u - 1) \tag{3.8}$$

for all  $u \in \mathfrak{R}_o$ , we obtain

$$F(uv) = F(u) + F(v) + \alpha G(u) + \beta H(v) + G(u)H(v) \tag{3.9}$$

for all  $u, v \in \mathfrak{R}_o$ . The general solution of (3.9) can now be obtained from Lemma 2. The first two solutions of Lemma 2 (see (2.9) and (2.10)) together with (3.5) and (3.6) yield the solutions (3.1) and (3.2). The next two solutions of Lemma 2 (that is, solution (2.11) and (2.12)) yield together with (3.5) and (3.6) the asserted solutions (3.3) and (3.4). This completes the proof of the theorem.

#### 4. SOLUTION OF THE FUNCTIONAL EQUATION (FE2)

Let  $a, b$  and  $c$  be real parameters. We consider the functional equation

$$f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \quad \forall x, y \in \mathfrak{R}. \tag{FE2}$$

The only constant solutions of (FE2) are  $f \equiv 0$  and  $f \equiv -1$ . So we look for nonconstant solutions of the functional equation (FE2).

Substitution of  $x = 0 = y$  in (FE2) yields  $f(0)[f(0) + 1] = 0$ . Hence, either  $f(0) = 0$  or  $f(0) = -1$ . Now we consider two cases.

**Case 1.** Suppose  $f(0) = -1$ . Then  $x = 0$  in (FE2) gives  $f(by) = f(0)$ , so that when  $b \neq 0$ ,  $f$  is a constant which is not the case. Similarly by putting  $y = 0$  in (FE2), we get  $f$  is a constant when  $a \neq 0$ .

Suppose  $a = 0 = b$ . If  $c$  is also zero, then (FE2) is  $[1 + f(x)][1 + f(y)] = 0$  since  $f(0) = -1$ . That is  $f$  is a constant. So, assume  $c \neq 0$ . Then replacing  $x$  by  $\frac{x}{c}$  and  $y$  by  $\frac{y}{c}$  in (FE2), we obtain

$$M(xy) = M(x)M(y), \tag{4.1}$$

where  $M : \mathfrak{K} \rightarrow \mathfrak{K}$  is a multiplicative map with  $M(x) = 1 + f\left(\frac{x}{c}\right)$ . Hence

$$f(x) = M(cx) - 1 \tag{4.2}$$

is a solution of (FE2) with  $f(0) = -1$ ,  $a = 0 = b$ ,  $c \neq 0$ .

**Case 2.** Suppose  $f(0) = 0$ . Let  $a = 0$ . Then  $y = 0$  in (FE2) gives  $f \equiv 0$  which is not the case. So,  $a \neq 0$ . Similarly  $b \neq 0$ . Setting  $x = 0$  and  $y = 0$  separately in (FE2), we get

$$f(by) = f(y) \quad \text{and} \quad f(ax) = f(x) \tag{4.3}$$

so that (FE2) becomes

$$f(ax + by + cxy) = f(ax) + f(by) + f(ax)f(by). \tag{4.4}$$

Suppose  $c = 0$ . Then replacing  $x$  by  $\frac{x}{a}$  and  $y$  by  $\frac{y}{b}$  in (4.4) we have

$$E(x + y) = E(x)E(y)$$

where  $E : \mathfrak{K} \rightarrow \mathfrak{K}$  given by

$$E(x) = 1 + f(x) \tag{4.5}$$

is an exponential map. Further, from (4.3) and (4.5), we get

$$E(ax) = E(x) = E(bx)$$

and since  $E(x)E(-x) = 1$ , so we get

$$E((a - b)x) = 1 = E((a - 1)x). \tag{4.6}$$

If  $a \neq b$ , then  $E$  is a constant map and so  $f$  is also a constant function. If  $a \neq 1$ , then  $E$  and so  $f$  is a constant. Hence  $a = 1 = b$ . Thus by (4.5)

$$f(x) = E(x) - 1$$

is a solution of (FE2) with  $a = b = 1$ ,  $c = 0$ .

Finally, let  $a \neq 0$ ,  $b \neq 0$  and  $c \neq 0$ . Set  $\alpha = \frac{c}{ab}$ . Replacing  $x$  by  $\frac{x}{a\alpha}$  and  $y$  by  $\frac{y}{b\alpha}$  in (4.4), we obtain

$$F(x + y + xy) = F(x)F(y), \tag{4.7}$$

where

$$F(x) = 1 + f\left(\frac{x}{\alpha}\right). \tag{4.8}$$

Changing  $x$  to  $x - 1$  and  $y$  to  $y - 1$  in (4.7) we have

$$M(xy) = M(x)M(y),$$

where  $M : \mathfrak{X} \rightarrow \mathfrak{X}$  is multiplicative and

$$M(x) = F(x - 1). \tag{4.9}$$

Thus by (4.8) and (4.9), we have

$$f(x) = F(\alpha x) - 1 = M(1 + \alpha x) - 1. \tag{4.10}$$

If we use (4.10) in (4.3), and recall that  $\alpha = \frac{c}{ba}$ , we get

$$M\left(1 + \frac{c}{a}x\right) = M\left(1 + \frac{c}{b}x\right) = M\left(1 + \frac{c}{ab}x\right). \tag{4.11}$$

Recall that, since  $M$  is multiplicative,  $M(x)M\left(\frac{1}{x}\right) = 1$  (otherwise if  $M(1) = 0$ , then  $M \equiv 0$  so that  $f \equiv -1$ ). Changing separately  $x$  to  $\frac{ax}{c}$  and  $x$  to  $\frac{bx}{c}$  in (4.11), we obtain

$$M(1 + x) = M\left(1 + \frac{x}{b}\right) = M\left(1 + \frac{x}{a}\right). \tag{4.12}$$

Similarly, replacing  $x$  by  $\frac{abx}{c}$  in (4.11), we have

$$M(1 + x) = M(1 + ax) = M(1 + bx). \tag{4.13}$$

Replacing  $x$  by  $x - 1$  in (4.13), we obtain  $M(x) = M(1 + a(x - 1))$  which yields

$$M\left(\frac{1 - a + ax}{x}\right) = 1 \quad \text{if } x \neq 0.$$

Suppose  $a \neq 1$ . Changing  $x$  to  $(1 - a)x$ , we have  $M\left(a + \frac{1}{x}\right) = 1$  and thus (again replacing  $x$  by  $\frac{1}{x-a}$ ) we have  $M(x) = 1$  when  $x \neq a$ . Similarly, if  $b \neq 1$ , we get  $M(x) = 1$  when  $x \neq 0, b$ .

Hence,  $M(x) = 1$  for all  $x$  which leads to  $f$  is a constant. Therefore  $a = 1 = b$ . Then from (4.10), we obtain

$$f(x) = M(1 + cx) - 1 \tag{4.14}$$

where  $M : \mathfrak{X} \rightarrow \mathfrak{X}$  is multiplicative. Thus we have proved the following theorem.

**THEOREM 2.** The function  $f : \mathfrak{X} \rightarrow \mathfrak{X}$  is a solution of (FE2) if and only if  $f(x)$ , for every  $x \in \mathfrak{X}$ , is given by

$$f(x) = \begin{cases} M(cx) - 1 & \text{if } a = 0 = b, c \neq 0 \\ E(x) - 1 & \text{if } a = 1 = b, c = 0 \\ M(cx + 1) - 1 & \text{if } a = 1 = b, c \neq 0 \\ k & \text{otherwise,} \end{cases}$$

where  $M : \mathfrak{X} \rightarrow \mathfrak{X}$  is multiplicative,  $E : \mathfrak{X} \rightarrow \mathfrak{X}$  is exponential, and  $k$  is a constant satisfying  $k(k + 1) = 0$ .

**ACKNOWLEDGMENTS.** We are thankful to the referee for suggestions that improved the presentation of this paper. This research is partially supported by a grant from the Graduate Programs and Research of the University of Louisville.

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Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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Manuscript Due	February 1, 2009
First Round of Reviews	May 1, 2009
Publication Date	August 1, 2009

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