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SOME RESULTS ON COMPACT CONVERGENCE SEMIGROUPS DEFINED BY FILTERS

PHOEBE HO, PAUL PLUMMER, and SHING SO

Central Missouri State University
Warrensburg, Missouri

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ABSTRACT. In this paper the concept of convergence defined by filters is used and applied in the study of semigroups. Special emphasis is placed on compact convergence semigroups and their properties.

KEY WORDS AND PHRASES: convergence semigroups, convergence groups, maximal subgroups, compact, pseudotopological, dense.

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1. INTRODUCTION.

Let $\dot{x}$ be the set of all subsets of a non-empty set S which contains $\{x\}$. Then $\dot{x}$ is an ultrafilter called *the principle ultrafilter* . In [1], Kent's approach to convergence was to set up a mapping q from F(X), the set of all filters on a set X, to P(X), the *power set* of X. Then a filter $\mathcal{F}$ on X is said to $q-$ *converge to* x in X, denoted by $\mathcal{F} \to x$, if $x \in q(\mathcal{F})$.

DEFINITION. A convergence space (X, q) is a non-empty set X and a mapping q between F(X) and P(X) which satisfy the following conditions:

i) $\dot{x} \to x$, for all $x \in X$;

ii) if $\mathcal{F} \to x$ and $\mathcal{F} \leq \mathcal{G}$, then $\mathcal{G} \to x$;

iii) if $\mathcal{F} \to x$ and $\mathcal{G} \to x$, then $\mathcal{F} \cap \mathcal{G} \to x$.

When these properties are satisfied q is known as a convergence structure. If the convergence structure is fixed for a specific discussion, as in reference to a general convergence space, we will refer to (X, q) as "the convergence space X". A convergence space X is Hausdorff if each filter converges to at most one point and X is compact if every ultrafilter converges in X.

DEFINITION. Let (X, q) and (Y, p) be convergence spaces and $f : X \to Y$. If $\mathcal{F}$ is a filter on S, then $f(\mathcal{F})$ will denote the filter on Y with $\{ f(F) \mid F \in \mathcal{F} \}$ as its base. The function f is said to be continuous at x if $f(\mathcal{F})$ p-converges to f(x) whenever $\mathcal{F}$ q-converges to x.

DEFINITION. A closure operation on a convergence space S is defined in the following way. If $A \subseteq S$ and $x \in S$, then $x \in \overline{A}$ if there exists a filter $\mathcal{F}$ such that $A \in \mathcal{F}$ and $\mathcal{F} \to x$.

A is defined to be closed if and only if $\overline{A} = A$.

The following five lemmas are immediate results from the definitions involved.

LEMMA 1.1. If A is a compact subset of X and f: (X, q) $\to$ (Y, p) is a continuous function, then f(A) is compact.

LEMMA 1.2. If X is a compact convergence space and T a closed subset of X, then T is compact.

LEMMA 1.3. If X is a compact space, and $\mathcal{D}$ is a descending family of non-empty closed subsets of X, then $\cap\mathcal{D} \neq \emptyset$.

LEMMA 1.4. If X is a Hausdorff convergence space and T a compact subset of X, then T is closed.

LEMMA 1.5. If A and B are compact subsets of a convergence space X, then A$\times$B is compact in the product convergence structure on X$\times$X.

2. MAIN RESULTS.

DEFINITION. A convergence semigroup is a convergence space S together with a continuous function m: S$\times$S $\to$ S such that:

i) S is Hausdorff;

ii) m is associative.

The following notations are useful in the discussion of convergence semigroups:

i) For a, b $\in$ S, ab = m((a, b)).

ii) For A, B $\subseteq$ S, AB = m(A$\times$B) = {ab | a $\in$ A and b $\in$ B}. In particular, A{b} will be denoted Ab.

iii) $\mathcal{F} \times \mathcal{G}$ is the filter on S$\times$S with {F $\times$ G | F $\in \mathcal{F}$ and G $\in \mathcal{G}$} as its base.

iv) $\mathcal{F} \cdot \mathcal{G}$ is the filter on S with $m(\mathcal{F}\times\mathcal{G})$ as its base.

v) $\mathcal{F}$a is the filter with {Fa | F $\in \mathcal{F}$} as its base.

LEMMA 2.1. If $\mathcal{F}$ and $\mathcal{G}$ are filters on a convergence semigroup S such that $\mathcal{F} \to$ x and $\mathcal{G} \to$ y, then $\mathcal{F} \cdot \mathcal{G} \to$ xy.

LEMMA 2.2. If A and B are compact subsets of a convergence semigroup S, then AB is compact in S.

DEFINITION. If f: X $\to$ X, then {x $\in$ X | f(x) = x} is called the set of fixed points of f.

The next lemma shows two properties of fixed points in convergence spaces.

LEMMA 2.3. Let X be a Hausdroff convergence space and F the set of fixed points of a continuous function f: X $\to$ X. Then the following is true:

i) If F $\in \mathcal{F}$ a filter on X, then $f(\mathcal{F}) = \mathcal{F}$;

ii) If X is Hausdorff, then F is closed.

PROOF. i) If A $\in f(\mathcal{F})$, then there exists A* $\in \mathcal{F}$ such that f(A*) $\subseteq$ A. But F $\in \mathcal{F}$ implies that F$\cap$A* $\neq \emptyset$ and F$\cap$A* $\in \mathcal{F}$. Now F$\cap$A* $\subseteq$ F implies that F $\cap$ A* = f(F$\cap$A*) $\subseteq$

$f(A^*) \subseteq A$ which implies that $A \in \mathcal{F}$. Therefore, $f(\mathcal{F}) \subseteq \mathcal{F}$.

If $A \in \mathcal{F}$, then $A \cap F \in \mathcal{F}$ and $f(A \cap F) = A \cap F$. This implies that $A \cap F \in f(\mathcal{F})$. Now $A \cap F \subseteq A$ implies that $A \in f(\mathcal{F})$. Therefore, $\mathcal{F} \subseteq f(\mathcal{F})$.

ii) Let $y \in \overline{F}$. Then there exists a filter $\mathcal{F}$ such that $F \in \mathcal{F}$ and $\mathcal{F} \to y$. Since f is continuous, we know $f(\mathcal{F}) \to f(y)$, and from part i), $f(\mathcal{F}) = \mathcal{F}$, so $\mathcal{F} \to f(y)$. But X being Hausdorff implies that $f(y) = y$. So $y \in F$. Therefore, $\overline{F} \subseteq F$ so $\overline{F} = F$.

DEFINITION. An element e of a semigroup S is called an idempotent if $e^2 = e$. We use E(S) to denote the set of all idempotents in S.

DEFINITION. A nonempty subset T of a semigroup S is said to be a subsemigroup if $T \cdot T = T^2 \subseteq T$. A nonempty subset G of S is a subgroup of S if G is a group under the multiplication defined on S.

THEOREM 2.1. If a Hausdorff convergence semigroup S is compact , then S contains an idempotent.

PROOF. We will show that S contains a minimal closed subsemigroup and that every such subsemigroup consists of a single idempotent. Let $\mathcal{S}$ denote the set of closed subsemigroups of S. Note $S \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$. Partially order $\mathcal{S}$ by inclusion and let $\mathcal{C}$ be a maximal tower in $\mathcal{S}$ by use of the Hausdorff Maximality Principal. Let $T = \cap\mathcal{C}$. Then, from Lemma 1.3, $T \neq \emptyset$. Let $x \in T$. Since T is a non-empty closed subsemigroup of S, Lemma 1.2 implies that T is compact; hence, by Lemma 2.2, xT is compact. Therefore by Lemma 1.4, xT is a closed subsemigroup of S contained in T. Now, by the maximality of $\mathcal{C}$, we see that $xT = T$ and, similarly, $Tx = T$. Thus T is a subgroup of S. If e is the identity of T, the maximality of $\mathcal{C}$ ensures that $T = \{e\}$.

Let f be a mapping from from the semigroup S into S×S by $f(x) = (x, x)$ for all $x \in S$, and let $\mathcal{F}$ be a filter on S such that $\mathcal{F} \to x$. If $F \in \mathcal{F}$, then $f(F) = \{(x,x) \mid x \in F\}$. Now $\pi_i(f(F)) = F$ for i = 1, 2, so $f(\mathcal{F}) \to (x, x)$. Therefore f is continuous. Next, consider the composite function $f \circ m$. Since f and m are both continuous, we know $f \circ m$ is continuous and $f \circ m$: $S \to S$ by $f \circ m(x) = x^2$. Then, by Lemma 2.2 the set of fixed points of $f \circ m$ is closed. But the set of fixed points of $f \circ m$ is E(S), the set of idempotents in the semigroup. Therefore, we have the following theorem.

THEOREM 2.2. The set E(S), of all idempotents of a convergence semigroup S, is closed.

THEOREM 2.3. Let S be a convergence semigroup and G a subgroup of S. Then $\overline{G}$ is a subsemigroup of S with identity.

PROOF. Let $x, y \in \overline{G}$. This implies that there exists $\mathcal{F}$ and $\mathcal{G}$ such that G is contained in $\mathcal{F}$ and $\mathcal{G}$, and $\mathcal{F} \to x$, $\mathcal{G} \to y$. Since G is a group, $G \cdot G = G$ so $G \in \mathcal{F} \cdot \mathcal{G}$ and $\mathcal{F} \cdot \mathcal{G} \to xy$. Therefore, $xy \in \overline{G}$ so $\overline{G}^2 \subseteq \overline{G}$.

Now if e is the identity of G and $x \in \overline{G}$, then there exists $\mathcal{F}$ such that $G \in \mathcal{F}$ and $\mathcal{F} \to x$. But since $\dot{e} \to e$, we know $\dot{e} \cdot \mathcal{F} \to ex$. Also since $\{e\} \in \dot{e}$ and $eG = G$, we have $G = eG \in \dot{e} \cdot \mathcal{F}$. Let $F \in \mathcal{F}$. Then $G, F \in \mathcal{F}$ which implies that $G \cap F \in \mathcal{F}$ and since $G \cap F \subseteq G$ we know $e(G \cap F) = G \cap F$ so $G \cap F \in \dot{e} \cdot \mathcal{F}$. But $G \cap F \subseteq F$ implies $F \in \dot{e} \cdot \mathcal{F}$ which means $\mathcal{F} \leq \dot{e} \cdot \mathcal{F}$.

So $\dot{e} \cdot \mathcal{F} \to x$. Now S being Hausdorff implies that $ex = x$, similarly $xe = x$. Therefore, e is the identity for $\overline{G}$.

THEOREM 2.4. If S is a compact convergence semigroup and G a subgroup of S, then $\overline{G}$ is also a subgroup.

PROOF. By Theorem 2.3, it suffices to show if $x \in \overline{G}$ then x has an inverse which is equivalent to the existence of an $x^{-1} \in \overline{G}$ such that $x^{-1}x = xx^{-1} = e$. Let $y \in \overline{G}$. Now if $y \in G$. G is a group so y has an inverse and we are done. If $y \notin G$, then there exists a filter $\mathcal{F}$ such that $G \in \mathcal{F}$ and $\mathcal{F} \to y$. For each $F \subseteq G$, let $F^{-1} = \{x^{-1} \mid x \in F\}$ and $\mathcal{F}^{-1}$ the filter with $\{F^{-1} \mid F \in \mathcal{F} \text{ and } F \subseteq G\}$ as its base. Now there exists an ultrafilter $\mathcal{H}$ such that $\mathcal{F}^{-1} \leq \mathcal{H}$ and, since S is compact, $\mathcal{H} \to h$ for some $h \in S$. Now since $G \in \mathcal{F}$ and $G^{-1} = G$, $G \in \mathcal{F}^{-1}$ so $G \in \mathcal{H}$ and $h \in \overline{G}$. By Lemma 2.1, $\mathcal{H} \cdot \mathcal{F} \to hy$ so consider $HF \in \mathcal{H} \cdot \mathcal{F}$. Since $\mathcal{F}^{-1} \leq \mathcal{H}$ and $F \cap G \in \mathcal{F}$, $(F \cap G)^{-1} \in \mathcal{H}$ which implies $H \cap (F \cap G)^{-1} \in \mathcal{H}$ and $H \cap (F \cap G)^{-1} \neq \emptyset$. Thus there exists $x^{-1} \in (F \cap G)^{-1}$ such that $x^{-1} \in H$ and $x \in (F \cap G) \subseteq F$ so $x^{-1}x = e \in HF$, so $e \in HF$ for all $HF \in \mathcal{H} \cdot \mathcal{F}$ which implies $\mathcal{H} \cdot \mathcal{F} \leq \dot{e}$. Therefore, $\dot{e} \to hy$ and we know $\dot{e} \to e$, but the fact that S is Hausdorff implies that $hy = e$. Similarly, we can show $yh = e$. Thus, $h = y^{-1} \in G$.

DEFINITION. Let G be a subgroup of a convergence semigroup S. If the inversion mapping $f : G \to G$ defined by $f(x) = x^{-1}$ is continuous, then G is called a convergence semigroup.

DEFINITION. A convergence space X is said to be pseudotopological if a filter $\mathcal{F}$ converges to some point x in X if and only if all ultrafilters finer than $\mathcal{F}$ converge to x.

THEOREM 2.5. If S is a compact pseudotopological semigroup, which is algebraically a group, then S is a convergence group.

PROOF. Let $f : S \to S$ defined by $f(x) = x^{-1}$ and $\mathcal{F}$ a filter such that $\mathcal{F} \to x$. Then f($\mathcal{F}$) is a filter in S. Let $\mathcal{H}$ be a ultrafilter such that $\mathcal{H} \geq f(\mathcal{F})$. Since S is compact, $\mathcal{H} \to y$ for some y in S. Since for every $H \in \mathcal{H}$, $H \cap f(\mathcal{F}) \neq \emptyset$ for all $F \in \mathcal{F}$. Then $e \in HF$ for all $H \in \mathcal{H}$ and $F \in \mathcal{F}$ where e is the identity of the group S. Now $\mathcal{F}\mathcal{H} \to y$, $\mathcal{F}\mathcal{H} \leq \dot{e}$, and $\dot{e} \to e$ imply that $xy = e$.

Similarly, $yx = e$. Thus $y = x^{-1}$. Since S is pseudotopological, $f(\mathcal{F}) \to x^{-1}$ so f is continuous. Therefore, S is a convergence group.

DEFINITION. A subset D of a convergence space X is said to be dense in X if $\overline{D} = X$.

DEFINITION. If S is a convergence semigroup and $e \in E(S)$, then the largest proper subgroup of S which contains e is called the maximal subgroup of S containing e.

THEOREM 2.6. i) If S is a compact convergence semigroup, then each maximal subgroup is either closed or dense in S.

ii) If S is a compact pseudotopological semigroup, then either every maximal subgroup of S is closed or S is a convergence group.

iii) If S is a compact pseudotopological semigroup, then every maximal subgroup of S is a convergence group.

PROOF. i) By Theorem 2.4, if G is a subgroup then so is $\overline{G}$. Since $G \subset \overline{G} \subset S$, $\overline{G}$ is a group, and G is maximal, either $\overline{G} = G$ and G is closed, or $\overline{G} = S$ and G is dense.

ii) If S does have a dense subgroup, then $\overline{G} = S$ implies S is a group. Since S is compact and pseudotopological, S is a convergence semigroup by Theorem 2.5.

iii) Let G be a maximal subgroup of S. Then G is either closed or dense in S.

If G is closed, then G satisfies all the hypothesis of Theorem 2.5. so G is a convergence group.

If G is dense in S, then by ii) S is a convergence group. Since the inversion mapping on G is a restriction of the inversion mapping on S, it must be continuous. Therefore G is a convergence group.

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