

## SUBCONTRA-CONTINUOUS FUNCTIONS

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**ABSTRACT.** A weak form of contra-continuity, called subcontra-continuity, is introduced. It is shown that subcontra-continuity is strictly weaker than contra-continuity and stronger than both subweak continuity and sub-LC-continuity. Subcontra-continuity is used to improve several results in the literature concerning compact spaces.

**KEY WORDS AND PHRASES:** subcontra-continuity, contra-continuity, subweak continuity, sub-LC-continuity.

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### 1. INTRODUCTION

In [1] Dontchev introduced the notion of a contra-continuous function. In this note we develop a weak form of contra-continuity, which we call subcontra-continuity. We show that subcontra-continuity implies both subweak continuity and sub-LC-continuity. We also establish some of the properties of subcontra-continuous functions. In particular it is shown that the graph of a subcontra-continuous function into a  $T_1$ -space is closed. Finally, we show that many of the applications of contra-continuous functions to compact spaces established by Dontchev [1] hold for subcontra-continuous functions. For example, we establish that the subcontra-continuous, nearly continuous image of an almost compact space is compact and that the subcontra-continuous,  $\beta$ -continuous image of an S-closed space is compact.

### 2. PRELIMINARIES

The symbols  $X$  and  $Y$  denote topological spaces with no separation axioms assumed unless explicitly stated. The closure and interior of a subset  $A$  of a space  $X$  are signified by  $Cl(A)$  and  $Int(A)$ , respectively. A set  $A$  is regular open (semi-open, nearly open) provided that  $A = Int(Cl(A))$  ( $A \subseteq Cl(Int(A))$ ,  $A \subseteq Int(Cl(A))$ ) and  $A$  is regular closed (semi-closed) if its complement is regular open (semi-open). A set  $A$  is locally closed provided that  $A = U \cap F$ , where  $U$  is an open set and  $F$  is a closed set.

**DEFINITION 1.** Dontchev [1]. A function  $f : X \rightarrow Y$  is said to be contra-continuous provided that for every open set  $V$  in  $Y$ ,  $f^{-1}(V)$  is closed in  $X$ .

**DEFINITION 2.** Rose [2]. A function  $f : X \rightarrow Y$  is said to be subweakly continuous if there is an open base  $\mathcal{B}$  for the topology on  $Y$  such that  $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$  for every  $V \in \mathcal{B}$ .

**DEFINITION 3.** Ganster and Reilly [3]. A function  $f : X \rightarrow Y$  is said to be sub-LC-continuous provided there is an open base  $\mathcal{B}$  for the topology on  $Y$  such that  $f^{-1}(V)$  is locally closed for every  $V \in \mathcal{B}$ .

**DEFINITION 4.** A function  $f : X \rightarrow Y$  is said to be semi-continuous (Levine [4]) (nearly continuous (Ptak [5]),  $\beta$ -continuous (Abd El-Monsef *et al.* [6])) if for every open set  $V$  in  $Y$ ,  $f^{-1}(V) \subseteq Cl(Int(f^{-1}(V)))$  ( $f^{-1}(V) \subseteq Int(Cl(f^{-1}(V)))$ ,  $f^{-1}(V) \subseteq Cl(Int(Cl(f^{-1}(V))))$ ).

**DEFINITION 5.** Gentry and Hoyle [7]. A function  $f : X \rightarrow Y$  is said to be c-continuous if, for every  $x \in X$  and every open set  $V$  in  $Y$  containing  $f(x)$  and with compact complement, there exists an open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subseteq V$ .

### 3. SUBCONTRA-CONTINUOUS FUNCTIONS

We define a function  $f : X \rightarrow Y$  to be subcontra-continuous provided there exists an open base  $\mathcal{B}$  for the topology on  $Y$  such that  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Obviously contra-continuity implies subcontra-continuity. The following example shows that the reverse implication does not hold.

**EXAMPLE 1.** Let  $X$  be a nondiscrete  $T_1$ -space and let  $Y$  be the set  $X$  with the discrete topology. Finally let  $f : X \rightarrow Y$  be the identity mapping. If  $\mathcal{B}$  is the collection of all singleton subsets of  $Y$ , then  $\mathcal{B}$  is an open base for the topology on  $Y$ . Since  $X$  is  $T_1$ ,  $f$  is subcontra-continuous with respect to  $\mathcal{B}$ . Obviously  $f$  is not contra-continuous.

Subcontra-continuity is independent of continuity. The function in Example 1 is subcontra-continuous but not continuous. The next example shows that continuity does not imply subcontra-continuity.

**EXAMPLE 2** Let  $X = \{a, b\}$  be the Sierpinski space with the topology  $\mathcal{T} = \{X, \emptyset, \{a\}\}$  and let  $f : X \rightarrow X$  be the identity mapping. Obviously  $f$  is continuous. However, any open base for the topology on  $X$  must contain  $\{a\}$  and  $f^{-1}(\{a\})$  is not closed. It follows that  $f$  is not subcontra-continuous.

Since closed sets are locally closed, subcontra-continuity implies sub-LC-continuity. We see from the following theorem that subcontra-continuity also implies subweak continuity.

**THEOREM 1.** Every subcontra-continuous function is subweakly continuous.

**PROOF.** Assume  $f : X \rightarrow Y$  is subcontra-continuous. Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Then for  $V \in \mathcal{B}$ ,  $Cl(f^{-1}(V)) = f^{-1}(V) \subseteq f^{-1}(Cl(V))$  and hence  $f$  is subweakly continuous.  $\square$

Since a subweakly continuous function into a Hausdorff space has a closed graph (Baker [8]), a subcontra-continuous function into a Hausdorff space has a closed graph. However, the following stronger result holds for subcontra-continuous functions.

**THEOREM 2.** If  $f : X \rightarrow Y$  is a subcontra-continuous function and  $Y$  is  $T_1$ , then the graph of  $f$ ,  $G(f)$ , is closed.

**PROOF.** Let  $(x, y) \in X \times Y - G(f)$ . Then  $y \neq f(x)$ . Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Since  $Y$  is  $T_1$ , there exists  $V \in \mathcal{B}$  such that  $y \in V$  and  $f(x) \notin V$ . Then we see that  $(x, y) \in (X - f^{-1}(V)) \times V \subseteq X \times Y - G(f)$ . It follows that  $G(f)$  is closed.  $\square$

**COROLLARY 1.** If  $f : X \rightarrow Y$  is contra-continuous and  $Y$  is  $T_1$ , then the graph of  $f$  is closed.

Long and Hendrix [9] proved that the closed graph property implies c-continuity. Therefore we have the following corollary.

**COROLLARY 2.** If  $f : X \rightarrow Y$  is subcontra-continuous and  $Y$  is  $T_1$ , then  $f$  is c-continuous.

The next two results are also implied by the closed graph property (Fuller [10]).

**COROLLARY 3.** If  $f : X \rightarrow Y$  is subcontra-continuous and  $Y$  is  $T_1$ , then for every compact subset  $C$  of  $Y$ ,  $f^{-1}(C)$  is closed in  $X$ .

**COROLLARY 4.** If  $f : X \rightarrow Y$  is subcontra-continuous and  $Y$  is  $T_1$ , then for every compact subset  $C$  of  $X$ ,  $f(C)$  is closed.

For a function  $f : X \rightarrow Y$ , the graph function of  $f$  is the function  $g : X \rightarrow X \times Y$  given by  $g(x) = (x, f(x))$ . We shall see in the following example that the graph function of a subcontra-continuous function is not necessarily subcontra-continuous.

**EXAMPLE 3.** Let  $X = \{a, b\}$  be the Sierpinski space with the topology  $\mathcal{T} = \{X, \emptyset, \{a\}\}$  and let  $f : X \rightarrow X$  be given by  $f(a) = b$  and  $f(b) = a$ . Obviously  $f$  is subcontra-continuous, in fact contra-continuous. Let  $\mathcal{B}$  be any open base for the product topology on  $X \times Y$ . Then there exists  $V \in \mathcal{B}$  for which  $(a, b) \in V \subseteq \{(a, a), (a, b)\}$ . We see that  $V = \{(a, a), (a, b)\}$  and that, if  $g : X \rightarrow X \times X$  is the graph function for  $f$ , then  $g^{-1}(V) = \{a\}$  which is not closed. Thus the graph function of  $f$  is not subcontra-continuous.

However, the following result does hold for the graph function.

**THEOREM 3.** The graph function of a subcontra-continuous function is sub-LC-continuous.

**PROOF.** Assume  $f : X \rightarrow Y$  is subcontra-continuous and let  $g : X \rightarrow X \times Y$  be the graph function of  $f$ . Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Then  $\{U \times V : U \text{ is open in } X, V \in \mathcal{B}\}$  is an open base for the product topology on  $X \times Y$ . Since  $g^{-1}(U \times V) = U \cap f^{-1}(V)$ , we see that  $g$  is sub-LC-continuous.  $\square$

The graph function of a subweakly continuous function is subweakly continuous (Baker [8]) and the graph function of a sub-LC-continuous function is sub-LC-continuous (Ganster and Reilly [3]). It follows that the graph function in Example 3 is subweakly continuous and sub-LC-continuous but not subcontra-continuous. Therefore subcontra-continuity is strictly stronger than sub-LC-continuity and subweak continuity.

**THEOREM 4.** If  $Y$  is a  $T_1$ -space and  $f : X \rightarrow Y$  is a subcontra-continuous injection, then  $X$  is  $T_1$ .

**PROOF.** Let  $x_1$  and  $x_2$  be distinct points in  $X$ . Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Since  $Y$  is  $T_1$  and  $f(x_1) \neq f(x_2)$ , there exists  $V \in \mathcal{B}$  such that  $f(x_1) \notin V$  and  $f(x_2) \in V$ . Then  $x_1 \in X - f^{-1}(V)$  which is open and  $x_2 \notin X - f^{-1}(V)$ .  $\square$

**THEOREM 5.** Let  $A \subseteq X$  and  $f : X \rightarrow X$  be a subcontra-continuous function such that  $f(X) = A$  and  $f|_A$  is the identity on  $A$ . Then, if  $X$  is  $T_1$ ,  $A$  is closed in  $X$ .

**PROOF.** Suppose  $A$  is not closed. Let  $x \in Cl(A) - A$ . Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed for every  $V \in \mathcal{B}$ . Since  $x \notin A$ , we have that  $x \neq f(x)$ . Since  $X$  is  $T_1$ , there exists  $V \in \mathcal{B}$  such that  $x \in V$  and  $f(x) \notin V$ . Let  $U$  be an open set containing  $x$ . Then  $x \in U \cap V$  which is open. Since  $x \in Cl(A)$ ,  $(U \cap V) \cap A \neq \emptyset$ . Let  $y \in (U \cap V) \cap A$ . Since  $y \in A$ ,  $f(y) = y \in V$ . So  $y \in f^{-1}(V)$ . Thus  $y \in U \cap f^{-1}(V)$  and hence  $U \cap f^{-1}(V) \neq \emptyset$ . We see that  $x \in Cl(f^{-1}(V)) = f^{-1}(V)$  which is a contradiction. Therefore  $A$  is closed.  $\square$

The next result follows easily for the definition.

**THEOREM 6.** If  $f : X \rightarrow Y$  is subcontra-continuous, then for every open set  $V$  in  $Y$ ,  $f^{-1}(V)$  is a union of closed sets in  $X$ .

Obviously every function with a  $T_1$ -domain satisfies the above condition. However, as we see in the following example, a function with a  $T_1$ -domain can fail to be subcontra-continuous. It follows that the converse of Theorem 6 does not hold..

**EXAMPLE 4.** Let  $X = \mathbb{R}$  with the usual topology and let  $f : X \rightarrow X$  be the identity mapping. Since  $X$  is connected,  $f$  is not subcontra-continuous. However, since  $X$  is  $T_1$ ,  $f$  has the property that the inverse image of every (open) set is a union of closed sets.

#### 4. APPLICATIONS TO COMPACT SPACES

In [1] Dontchev establishes that the image of an almost compact space under a contra-continuous, nearly continuous mapping is compact and that the contra-continuous image of a strongly S-closed space is compact. In this section, we strengthen both of these results by replacing contra-continuity with subcontra-continuity. The proofs mostly follow Dontchev's.

**DEFINITION 6.** Dontchev [1]. A space  $X$  is almost compact provided that every open cover of  $X$  has a finite subfamily the closures of whose members cover  $X$ .

**THEOREM 7.** The image of an almost compact space under a subcontra-continuous, nearly continuous mapping is compact.

**PROOF.** Let  $f : X \rightarrow Y$  be subcontra-continuous and nearly continuous and assume that  $X$  is almost compact. Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  closed in  $X$  for every  $V \in \mathcal{B}$ . Let  $\mathcal{C}$  be an open cover of  $f(X)$ . For each  $x \in X$ , let  $C_x \in \mathcal{C}$  such that  $f(x) \in C_x$ . Then let  $V_x \in \mathcal{B}$  for which  $f(x) \in V_x \subseteq C_x$ . Now  $f^{-1}(V_x)$  is closed and nearly open. It follows that  $f^{-1}(V_x)$  is clopen and hence that  $\{f^{-1}(V_x) : x \in X\}$  is a clopen cover of  $X$ . Since  $X$  is almost compact, there is a finite subfamily  $\{f^{-1}(V_{x_i}) : i = 1, \dots, n\}$  for which  $X = \bigcup_{i=1}^n Cl(f^{-1}(V_{x_i})) = \bigcup_{i=1}^n f^{-1}(V_{x_i}) \subseteq \bigcup_{i=1}^n f^{-1}(C_{x_i})$ . Thus we have that  $f(X) \subseteq \bigcup_{i=1}^n C_{x_i}$  and therefore that  $f(X)$  is compact.  $\square$

**DEFINITION 7.** Dontchev [1]. A space  $X$  is strongly S-closed provided that every closed cover of  $X$  has a finite subcover.

**THEOREM 8.** The subcontra-continuous image of a strongly S-closed space is compact.

**PROOF.** Let  $f : X \rightarrow Y$  be subcontra-continuous and assume that  $X$  is strongly S-closed. Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Let  $\mathcal{C}$  be an open cover of  $f(X)$ . For each  $x \in X$ , let  $C_x \in \mathcal{C}$  with  $f(x) \in C_x$ . Then let  $V_x \in \mathcal{B}$  for which  $f(x) \in V_x \subseteq C_x$ . Since  $\{f^{-1}(V_x) : x \in X\}$  is a closed cover of  $X$  and  $X$  is strongly S-closed, there is a finite subcover  $\{f^{-1}(V_{x_i}) : i = 1, \dots, n\}$  of  $X$ . Then we see that  $f(X) = f\left(\bigcup_{i=1}^n f^{-1}(V_{x_i})\right) = \bigcup_{i=1}^n f(f^{-1}(V_{x_i})) \subseteq \bigcup_{i=1}^n V_{x_i} \subseteq \bigcup_{i=1}^n C_{x_i}$  and hence that  $f(X)$  is compact.  $\square$

In [1] Dontchev also shows that the contra-continuous,  $\beta$ -continuous image of an S-closed space is compact. We extend this result by replacing contra-continuity with subcontra-continuity. The proof parallels that of Dontchev's.

**DEFINITION 8.** Mukherjee and Basu [11]. A space  $X$  is S-closed provided that every semi-open cover of  $X$  has a finite subfamily the closures of whose members covers  $X$ .

From Herrmann [12], a space  $X$  is S-closed if and only if every regular closed cover of  $X$  has a finite subcover.

**THEOREM 9.** The subcontra-continuous,  $\beta$ -continuous image of an S-closed space is compact.

**PROOF.** Assume that  $f : X \rightarrow Y$  is subcontra-continuous and  $\beta$ -continuous and that  $X$  is S-closed. Let  $\mathcal{B}$  be an open base for the topology on  $Y$  for which  $f^{-1}(V)$  is closed in  $X$  for every  $V \in \mathcal{B}$ . Let  $\mathcal{C}$  be an open cover of  $f(X)$ . Then for each  $x \in X$  there exists  $C_x \in \mathcal{C}$  for which  $f(x) \in C_x$ . For each  $x \in X$ , let  $V_x \in \mathcal{B}$  such that  $f(x) \in V_x \subseteq C_x$ . Since  $f$  is subcontra-continuous,  $\{f^{-1}(V_x) : x \in X\}$  is a closed cover of  $X$ . The  $\beta$ -continuity of  $f$  implies that  $f^{-1}(V_x) \subseteq Cl(Int(Cl(f^{-1}(V_x))))$  and therefore we see that  $f^{-1}(V_x) = Cl(Int(f^{-1}(V_x)))$  or that

$f^{-1}(V_x)$  is regular closed. Since  $X$  is S-closed, the regular closed cover  $\{f^{-1}(V_x) : x \in X\}$  has a finite subcover  $\{f^{-1}(V_{x_i}) : i = 1, \dots, n\}$ . Then we have  $f(X) = f\left(\bigcup_{i=1}^n f^{-1}(V_{x_i})\right) \subseteq \bigcup_{i=1}^n V_{x_i} \subseteq \bigcup_{i=1}^n C_{x_i}$  and therefore  $f(X)$  is compact.  $\square$

In the above proof we showed that, if  $f : X \rightarrow Y$  is subcontra-continuous and  $\beta$ -continuous, then there exists an open base  $\mathcal{B}$  for the topology on  $Y$  such that for every  $V \in \mathcal{B}$ ,  $f^{-1}(V)$  is regular closed and hence semi-open. Since unions of semi-open sets are semi-open (Arya and Bhamini [13]), it follows that inverse images of open sets are semi-open. Therefore we have the following theorem which strengthens the corresponding result for contra-continuous functions established by Dontchev [1].

**THEOREM 10.** Every subcontra-continuous,  $\beta$ -continuous function is semi-continuous.

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