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α-SCATTERED SPACES

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ABSTRACT. α-scattered spaces, spaces whose α-topology is scattered, are introduced and used to slightly extend a recent result of A. V. Arhangel'skii and P. J. Collins [1]. Strong irresolvability of J. Foran and P. Liebnitz [6], is also characterized in terms of the α-topology, and it is shown that a recent theorem of Julian Dontchev [5] in essence maximally extends the Arhangel'skii-Collins result

KEY WORDS AND PHRASES: Scattered space, α-scattered space, submaximality, irresolvable space, strong irresolvability, heredity irresolvability.
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1. INTRODUCTION AND DEFINITIONS

For any space $X = (X, \tau)$, the α-space of X is $X^\alpha = (X, \tau^\alpha)$ where $\tau^\alpha = \{U - N | U \in \tau$ and $Int\,Cl(N) = \emptyset\}$ is the α-topology for X, [9][11]. Certainly, $\tau \subseteq \tau^\alpha$and unless specifically notated, the interior and closure operators Int and Cl are with respect to the base topology τ. Int^α and Cl^α denote the interior and closure operators respectively relative to τ^α. A space X is an α-space if and only if $X = X^\alpha$. The α-topology for X is obviously a base for a topology since it is closed under finite intersection , and is really a topology on X, being also closed under arbitrary union of members This last fact can be verified as follows. For each indexed subfamily $\{U_\lambda | \lambda \in \Lambda\} \subseteq \tau^\alpha, \bigcup_\lambda (U_\lambda - N_\lambda) = (\bigcup_\lambda U_\lambda) - N$ where $N = (\bigcup_\lambda U_\lambda) - \bigcup_\lambda (U_\lambda - N_\lambda)$. It is enough to show that N is nowhere dense in X. Evidently, $\{U_\lambda | \lambda \in \Lambda\}$ is an open cover for N such that for each λ, $U_\lambda \cap N$ is nowhere dense since $U_\lambda \cap N \subseteq U_\lambda - (U_\lambda - N_\lambda) \subseteq N_\lambda$. So, N is locally nowhere dense The following proposition finishes the argument.

PROPOSITION 1. *Each locally nowhere dense set is nowhere dense.*

PROOF. Even though this result is well known, a simple argument is supplied as we use this result again later. Recall that a set N is nowhere dense if and only if for each nonempty open set U there exists a nonempty open subset $V \subseteq U$ such that $V \cap N = \emptyset$. Suppose that N has an open cover $\{U_\lambda | \lambda \in \Lambda\}$ such that for each λ, $U_\lambda \cap N$ is nowhere dense, and let U be a nonempty open set If $U \cap N = \emptyset$, set $V = U$. Otherwise, $U \cap U_\lambda \neq \emptyset$ for some $\lambda \in \Lambda$. In this case, set $V = (U \cap U_\lambda) - Cl(U_\lambda \cap N)$ In either case, $\emptyset \neq V \subseteq U$, and $V \cap N = \emptyset$. □

Let us say that a function $f : (X, \tau) \to (Y, \sigma)$ is π-continuous if τ is a π-base for the weak topology $f^{-1}(\sigma) = \{f^{-1}(V) | V \in \sigma\}$, i.e., for each open V, $Int\, f^{-1}(V) \neq \emptyset$ whenever $f^{-1}(V) \neq \emptyset$. It is evident that for any space X, the identity function $f : X \to X^\alpha$ is open and π-continuous, so that X and X^α share the same dense sets It follows that X and X^α also share the same nowhere dense sets, so that X^α is an α-space, i.e., $(X^\alpha)^\alpha = X^\alpha$.

Note that a space X is an α-space if and only if its nowhere dense sets are closed Recall that a space is submaximal if and only if its dense subsets are open, which is equivalent to having codense sets closed This immediately yields the following since nowhere dense sets are codense

PROPOSITION 2. *Every submaximal space is an α-space* □

Recall that a space X is scattered if every nonempty subspace has an isolated point. Let $I(X)$ denote the set of isolated points of X. Clearly, if X is scattered then $I(X)$ is the minimum dense subset of X, so that dense subsets of X have dense interiors

PROPOSITION 3. *For each space X, the following are equivalent:*

(a) Dense subsets of X have dense interiors,

(b) Codense subsets of X are nowhere dense,

(c) X^α is submaximal.

PROOF. It is easy to see that (a) holds if and only if (b) holds. Now, if (b) holds and D is a dense subset of X^α, since X and X^α share dense, codense, and nowhere dense sets, $X - D$ is a closed subset of X^α so that $D \in \tau^\alpha$ and (c) holds. Conversely, if X^α is submaximal and $D \subseteq X$ is dense, then $D \in \tau^\alpha$ so that $D = U - N$ for some $U \in \tau$ and nowhere dense N Now, $U - Cl(N) \subseteq Int(D)$ implies $X = Cl(D) = Cl(U) \subseteq Cl\,Int(D)$ so that dense subsets of X have dense interiors □

An immediate consequence is that for each space X, if X is scattered then X^α is submaximal, and at once we have the following result of A. V. Arhangel'skii and P. J. Collins [1].

COROLLARY 1. *If X is scattered, then $X = X^\alpha$ if and only if X is submaximal.* □

PROBLEM. Find the largest class of spaces for X supporting the conclusion of Corollary 1

It is clear that we are looking for the largest class (if it exists) of spaces for X so that $X = X^\alpha$ implies X is submaximal. Enlarging the class of scattered spaces to contain the non-α-spaces only trivially extends the Arhangel'skii-Collins result. Perhaps a nontrivial strengthening of Corollary 1 results by assuming only that X^α is scattered. Certainly, the conclusion would be supported.

2. α-SCATTERED SPACES

DEFINITION 1. *A space X is an α-scattered space if X^α is scattered.*

Some characterizations of α-scattered spaces are stated in the following theorem.

THEOREM 1. *For a space X, the following are equivalent:*

(a) X^α is scattered,

(b) Every somewhere dense subspace of X has an isolated point.

(c) $I(X)$ is dense in X.

PROOF. Since X and X^α share the same somewhere dense subsets and since $\tau \subseteq \tau^\alpha$, (b) implies that each somewhere dense subspace of X^α has an isolated point. Consequently, (a) follows from (b) since the nowhere dense subspaces of X^α are discrete. To see that (a) implies (b), let A be a nonempty subspace of X having no isolated point. We will show that A is nowhere dense. Assuming that X^α is scattered, $I^\alpha(A) \neq \emptyset$ where $I^\alpha(A)$ denotes the set of isolated points in the subspace $(A, \tau^\alpha|A)$ For each $p \in I^\alpha(A)$, there exists $U - N \in \tau^\alpha$ (with the understanding that $U \in \tau$ and $Int\,Cl(N) = \emptyset$) such that $(U - N) \cap A = \{p\}$. Since $I(A) = \emptyset$, $p \in Cl(N)$. Thus, $U \cap I^\alpha(A) \subseteq U \cap A \subseteq N \cup \{p\} \subseteq Cl(N)$, a nowhere dense set, implies $U \cap I^\alpha(A)$ is a nowhere dense subset of X. This shows that $I^\alpha(A)$ is a locally nowhere dense subset of X and hence by Proposition 1, $I^\alpha(A)$ is nowhere dense. It follows that $A - I^\alpha(A) = (X - I^\alpha(A)) \cap A \in \tau^\alpha|A$. Thus, $I^\alpha(A - I^\alpha(A)) = \emptyset$ implies $A = I^\alpha(A)$, so that A is nowhere dense.

Clearly, (b) implies (c) since nonempty open sets are somewhere dense and for each open subspace U of X, $I(U) \subseteq I(X)$. To show that (c) implies (b), let $A \subseteq X$ be somewhere dense and choose $p \in I(X) \cap Int\,Cl(A)$. Then $\{p\} \in \tau$ and $p \in Cl(A)$ implies $p \in A$. Finally, this implies that $p \in I(A)$. □

Since the α-scattered spaces are precisely those having a dense set of isolated points, examples of such spaces can easily be found which are not scattered. One such example is offered.

EXAMPLE 1. *Let (X,τ) be the set of real numbers with the smallest expansion of the usual topology for which rational points are open. Then $I(X) = Q$ is dense so that X is α-scattered. But, the set $P = X - Q$ of irrationals is dense-in-itself so that $I(P) = \emptyset$ and X is not scattered.*

The following theorem decomposes scatteredness into two strictly weaker components

THEOREM 2. *A space X is scattered if and only if X is α-scattered and every nonempty nowhere dense subspace of X has an isolated point.* □

REMARK. Julian Dontchev has suggested that Theorem 2 might be rephrased more naturally as follows. A space X is scattered if and only if X is α-scattered and N-scattered where a space is N-scattered if every nowhere dense subset is scattered.

If X is any dense-in-itself space and hence far from being scattered, by a result in [13] X^α is dense-in-itself and yet nowhere dense subspaces of X^α are discrete.

We now offer the following slight improvement of the Arhangel'skii-Collins result.

THEOREM 3. *If X is α-scattered, then X is submaximal if and only if X is an α-space.* □

3. STRONGLY IRRESOLVABLE SPACES

Let us call a space *crowded* if it is dense-in-itself. Edwin Hewitt in [8] showed the existence of crowded submaximal spaces of arbitrary infinite cardinality. Such a space X is of course an α-space but is far from being α-scattered since $I(X) = \emptyset$. Can the Arhangel'skii-Collins result be extended to a class of spaces not requiring the existence of isolated points? Yes Julian Dontchev lifted the A-C result to the class of strongly irresolvable spaces in [5] Recall that a space is strongly irresolvable [6] if each nonempty open subset is irresolvable. A space is irresolvable, i.e., not resolvable, if it cannot be expressed as the union of two disjoint dense (or codense) subsets. We state the Dontchev result formally and then show that in essence, it gives the best possible extension of the class of spaces supporting the A-C result.

THEOREM 4. (Dontchev [5]) *If X is strongly irresolvable, then X is submaximal if and only if X is an α-space.*

Actually, Theorem 4 is a corollary of the following observation.

THEOREM 5. *A space X is strongly irresolvable if and only if X^α is submaximal.*

PROOF. By Proposition 3, X^α is submaximal if and only if every codense subset of X is nowhere dense. Such a space X must be strongly irresolvable. Otherwise, there is a nonempty open set U which is a union of two disjoint nonempty codense subsets. But since codense subsets of an open set are codense in X, and hence nowhere dense, this forces the contradiction that U is nowhere dense.

Conversely, if a nonempty space X is strongly irresolvable and if $D \subseteq X$ is a dense subset, $Int(D) \neq \emptyset$ since X is irresolvable. Further, $Int(D)$ is dense for otherwise, $D - Cl\,Int(D)$ would be a dense and codense set in the open subspace $X - Cl\,Int(D)$. Certainly, if $\emptyset \neq U \subseteq X - Cl\,Int(D)$ is open, $U \in \tau$, the topology on X, implies $U \cap D \neq \emptyset$, and also $U - D \neq \emptyset$ since $U \cap Int(D) = \emptyset$. □

A decomposition of submaximality now follows.

COROLLARY 2. *A space X is submaximal if and only if X is strongly irresolvable and $X = X^\alpha$.* □

It is easy to produce irresolvable spaces which are not strongly irresolvable, showing that strong irresolvability is strictly stronger than irresolvability. For example, if Y is any nonempty irresolvable space and Z is any nonempty resolvable space, the free join $X = Y \sqcup Z$ cannot be strongly irresolvable. But, X is irresolvable since a dense and codense subset D of X makes $D \cap Y$ to be a dense and codense subset of Y. On the other hand, strong irresolvability is not as strong as hereditary irresolvability. To see this, consider the space (X,τ) of Example 1. It is clearly irresolvable since it is α-scattered but it fails to be hereditarily irresolvable since the subspace P of irrationals is resolvable. The countable set of irrational algebraic numbers and its complement the set of transcendental numbers form a resolution of P as a disjoint union of two dense subsets. Evidently, strong irresolvability lies strictly between the properties of irresolvability and hereditary irresolvability Of course, strong irresolvability is strictly weaker than submaximality since submaximality implies hereditary irresolvability. To see that strong irresolvability is also strictly weaker than the α-scattered condition, consider the crowded infinite submaximal spaces of Hewitt. Having no isolated points, they cannot be α-scattered, and yet being submaximal α-spaces, they are strongly irresolvable.

We now want to note some similarities in behavior for submaximality, hereditary irresolvability, and scatteredness in contrast to corresponding behaviors for α-scatteredness and strong irresolvability Recall that a property is semitopological [3], if it is preserved by semihomeomorphisms, bijections which are both irresolute and have an irresolute inverse, i.e., bijections for which images and inverse images of semiopen sets are semiopen. A set A is semiopen if and only if $A \subseteq Cl\,Int(A)$ It was later shown implicitly in [2] and later independently and explicitly in [7] that semitopological properties are precisely the α-topological properties. The same result had essentially been stated by O. Njåstad [12] as a consequence of the fact observed in a former paper [11], that in any topological space, the class of semiopen sets determined the α-topology and vice-versa. Recall that a property P is α-topological if both X and X^α have P when either has P (see also [14]). Example 1 above shows that scatteredness, submaximality, and hereditary irresolvability are not semitopological properties. It was noted in [13] that crowdedness is semitopological. Moreover, having isolated points is semitopological since one can show that for any space X, $I(X) = I(X^\alpha)$. In fact, since X and X^α share the same dense sets, α-scatteredness is semitopological. Also, strong irresolvability is semitopological for a space X is strongly irresolvable if and only if $X^\alpha = (X^\alpha)^\alpha$ is submaximal which occurs if and only if X^α is strongly irresolvable. As a side remark, it was noted in [7] that resolvability is semitopological. Thus, irresolvability is also semitopological.

For a further comparison-contrast, unlike scatteredness, submaximality, and hereditary irresolvability, evidently, α-scatteredness, strong irresolvability, and irresolvability are not hereditary properties. Hereditary α-scatteredness is scatteredness and hereditary strong irresolvability is hereditary irresolvability which is strictly weaker than submaximality. Perhaps, the property strong irresolvability could be renamed α-submaximality in light of Theorem 5. However, note that in this sense, α-submaximality, α-hereditary irresolvability, and α-strong irresolvability are pairwise equivalent.

It might also be observed that unlike scatteredness, submaximality, hereditary irresolvability, and irresolvability, the α-scatteredness and strong irresolvability properties are not generally preserved by open surjections.

EXAMPLE 2. *Let (Y, σ) be any infinite α-scattered space and let Z be any countably infinite set disjoint from Y endowed with the cofinite topology. Let $X = Y \cup Z$ be the space with topology $\tau = \sigma \cup \{U \subseteq X \mid X - U$ is a finite subset of $Z\}$ and let $Y \sqcup Z$ be the free join of Y and Z. Then the identity function $f : X \to Y \sqcup Z$ is an open surjection from the α-scattered and hence strongly irresolvable space X onto a space having a nonempty resolvable open subspace Z. Hence, $Y \sqcup Z$ is neither strongly irresolvable nor α-scattered.*

REMARK. The space X of Example 2 has T_0 separation but fails to be a T_1 space since points of Y are not closed. In fact, the space X of the example may be replaced by a T_1 regular space without disturbing the validity of the example. Let $Y = Q$ be the set of rational real numbers and let T be the set of rational numbers having terminating decimal expansions. Then with respect to the usual subspace (order) topology v on Q, T is both dense and codense so that (Q, v) is resolvable. Let $Z = Y \times \{1\}$ have the usual product topology. Let $X = Y \sqcup Z$ have a topology τ defined via an open base. Let η be a fixed positive irrational real number. If $z = (y, 1) \in Z$, for each positive rational number r, let

$B_r(z) = ([(y-r, y+r) \cap Q] \times \{1\}) \cup [(y+\eta-r, y+\eta+r) \cap Q]$ be a basic open neighborhood of z For each $y \in Y - T$, and each positive rational number r, let $B_r(y) = (y-r, y+r) \cap Q$ be a basic open neighborhood of y. Finally, if $y \in T$, let $\{y\} \in \tau$ Since $y + \eta$ is irrational for each $y \in Q$, it follows that (X, τ) is T_1 and regular. Also, X is α-scattered since Y is an open and α-scattered subspace and Z is nowhere dense. The identity map $f : X \to Y \sqcup Z$ is an open bijection yet the T_1 and regular free join $Y \sqcup Z$ is not strongly irresolvable since the open subspace Z is homeomorphic to (Q, v)

However, the properties α-scatteredness and strong irresolvability are preserved by open π-continuous surjections. In particular, if a product space is either scattered, hereditarily irresolvable, submaximal, α-scattered, or strongly irresolvable, then so is each factor On the other hand, an example is given in [10] of a submaximal space X whose square X^2 is not submaximal so that generally, submaximality is not productive. Also, (α-)scatteredness is not infinitely productive since a countably infinite power of a two point discrete space is homeomorphic to the usual crowded and compact Cantor set. This also shows that strong irresolvability is not infinitely productive since E Hewitt [8], showed that locally compact crowded Hausdorff spaces are resolvable. However, every finite product of α-scattered spaces is α-scattered. For if X and Y are spaces with dense sets of isolated points $I(X)$ and $I(Y)$ respectively, then $I(X \times Y) = I(X) \times I(Y)$ is dense in $X \times Y$. Is scatteredness finitely productive? Is strong irresolvability finitely productive? Both of these questions are answered affirmatively in a sequel to this paper [4].

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