

SEPARABLE SUBALGEBRAS OF A CLASS OF AZUMAYA ALGEBRAS

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ABSTRACT. Let S be a ring with 1, C the center of S , G a finite automorphism group of S of order n invertible in S , and S^G the subring of elements of S fixed under each element in G . It is shown that the skew group ring $S*G$ is a G' -Galois extension of $(S*G)^{G'}$ that is a projective separable C^G -algebra where G' is the inner automorphism group of $S*G$ induced by G if and only if S is a G -Galois extension of S^G that is a projective separable C^G -algebra. Moreover, properties of the separable subalgebras of a G -Galois H -separable extension S of S^G are given when S^G is a projective separable C^G -algebra.

KEY WORDS AND PHRASES: Azumaya algebras, Galois extensions, H -separable extensions, Skew group rings.

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1. INTRODUCTION

DeMeyer [1] and Kanzaki [2] studied central Galois algebras and Galois extensions whose center is a Galois algebra with Galois group induced by and isomorphic with the group of the extension. These two types of Galois extensions were recently generalized to a bigger class of Galois Azumaya extensions [3] where S is called a G -Galois Azumaya extension of S^G if S is a G -Galois extension of S^G that is an Azumaya C^G -algebra where C is the center of S and S^G is the subring of elements fixed under each element of G . Sugano [4] investigated a G -Galois H -separable extension of S^G , and recently, Szeto [5] proved that a G -Galois H -separable extension S of S^G that is a projective separable C^G -algebra if and only if S is a C^G -Azumaya algebra. We call such an S a GHS-extension. It will be shown that the skew group ring $S*G$ is a G' HS-extension if and only if S is a G -Galois extension of S^G that is a projective separable C^G -algebra, where G' is the inner automorphism group of $S*G$ induced by G . Moreover, properties of some separable subalgebras of a GHS-extension are also given.

2. PRELIMINARIES

Throughout, S is a ring with 1, G a finite automorphism group of S of order n invertible in S , C the center of S , and S^G the subring of elements fixed under each element in G . S is called a separable extension of a subring T if there exist $\{a_i, b_i \text{ in } S \mid i = 1, 2, \dots, m\}$ for some integer m such that $\sum a_i b_i = 1$ and $\sum s a_i \otimes b_i = \sum a_i \otimes b_i s$ for each s in S where \otimes is over T . We call $\{a_i, b_i\}$ a separable system for S . S is called an H -separable extension of T if $S \otimes_T S$ is isomorphic with a direct summand of a finite direct sum of S as a bimodule over S . It is known that an H -separable extension is a separable extension and an Azumaya algebra is an H -separable extension. S is called a G -Galois extension of S^G , if there exist $\{c_i, d_i \mid i = 1, 2, \dots, k\}$ in S for some integer k such that $\sum c_i d_i = 1$ and $\sum c_i g(d_i) = 0$ for each $g \neq 1$ in G . We call $\otimes\{c_i, d_i\}$ a G -Galois system for S .

3. SKEW GROUP RINGS

In this section, we shall show that $S * G$ is a G'HS-extension if and only if S is a G-Galois extension of S^G that is a projective separable C^G -algebra, and give some properties of the separable subalgebras of an G'HS-extension skew group ring.

THEOREM 3.1. By keeping the notations of section 2, $S * G$ is a G'HS-extension if and only if S is a G-Galois extension of S^G that is a projective separable C^G -algebra, where G' is the inner automorphism group of $S * G$ induced by G .

PROOF. Let S be a G-Galois extension of S^G that is a projective separable C^G -algebra. Noting that S is a subring of $S * G$, we have that $S * G$ is also a G' -Galois extension of $(S * G)^{G'}$ with a same Galois system as S where G' is the inner automorphism group of $S * G$ induced by G such that the restriction of G' to S is G . Hence $S * G$ is an H-separable extension of $(S * G)^{G'}$ ([4], Corollary 3). Moreover, since n is a unit in S , $S * G$ is a separable extension of S . But $S * G$ is a free module over S and S is a G-Galois extension of S^G that is a projective separable C^G -algebra by hypothesis, so $S * G$ is a projective separable C^G -algebra by the transitivity of projective separable extensions. Since the order of G' is n , it is easy to see that $(S * G)^{G'}$ is a direct summand of $S * G$ as a two sided $(S * G)^{G'}$ -module. Noting that $S * G$ is finitely generated and projective module as a right $(S * G)^{G'}$ -module or a left module, we have that $(S * G)^{G'}$ is a projective separable C^G -algebra by the same argument as given in the proof of Lemma 2 in [1]. This completes the sufficiency.

For the necessity, $S * G$ is a projective separable C^G -algebra by the transitivity of projective separable extensions because $S * G$ is a G' -Galois extension of $(S * G)^{G'}$ that is a projective separable algebra of C^G . Hence $S * G$ is an Azumaya algebra of its center Z . But S is a free module over S and n is a unit in S , so S is a finitely generated and projective left $S * G$ -module by the proof of Proposition 2.3 in [6] where $gs = g(s)$ for each s in S and g in G . Thus S is a finitely generated and projective Z -module by the transitivity of finitely generated and projective modules. Noting that 1 is in C^G and that C^G is contained in Z , we have that S is a faithful Z -module. Thus S is a progenerator over Z . Since $S * G$ is an Azumaya Z -algebra, S is a progenerator over $S * G$. Therefore, S is a G-Galois extension of S^G . Moreover, since S is a direct summand of $S * G$ as a C^G -module and $S * G$ is a finitely generated and projective C^G -module, S is also a finitely generated and projective C^G -module. Now n is a unit in S , so S^G is a S^G -direct summand of S . This implies that S^G is a finitely generated and projective C^G -module. So, it suffices to show that S^G is a separable C^G -algebra. In fact, since S is a progenerator over C^G (for 1 is in C^G), $\text{Hom}_{C^G}(S, S)$ is an Azumaya C^G -algebra. But $S^G \approx \text{Hom}_{S * G}(S, S) =$ the commutator of $S * G$ in $\text{Hom}_{C^G}(S, S)$, so S^G is a separable C^G -algebra (for so is $S * G$) by the commutant theorem for Azumaya algebras ([6], Theorem 4.3).

Next we give some properties of the separable subalgebras of $S * G$.

COROLLARY 3.2. If $S * G$ is a G'HS-extension, then, for any subgroup K of G , $S * K$ is a K' -Galois extension of $(S * K)^{K'}$ that is a separable C^G -algebra where K' is the inner automorphism group of $S * K$ induced by K .

PROOF. By Theorem 3.1, S is a G-Galois extension of S^G that is a projective C^G -algebra, so S is a K -Galois extension of S^K . Hence $S * K$ is a K' -Galois of $(S * K)^{K'}$. Noting that the order of K' is a unit in S , we have that $(S * K)^{K'}$ is a direct summand of $S * K$ as a $(S * K)^{K'}$ -module. But $S * K$ is a projective separable C^G -algebra, so $(S * K)^{K'}$ is a separable C^G -algebra by the same argument as given in the proof of Lemma 2 in [1].

Let $V_S(T)$ be the commutator subring of the subring T in S , and Z the center of $S * G$. We give an expression of the commutator subring of $(S * G)^{K'}$ in $S * G$.

THEOREM 3.3. If S^*G is a GHS-extension, then (1) for any subgroup K of G , $V_{S^*G}((S^*G)^K)$ is ZK , and (2) ZK is an Azumaya algebra over its center D such that $D = D^{K'}$.

PROOF. (1) By Theorem 3.1, S^*G is a separable C^G -algebra, so S^*G is an Azumaya Z -algebra. Since n is a unit in S , the order of K is a unit in S ; and so ZK is a separable Z -algebra contained in S^*G . Noting that $V_{S^*G}(ZK) = (S^*G)^{K'}$, we have that $(S^*G)^{K'}$ is a separable Z -subalgebra of S^*G such that $ZK = V_{S^*G}((S^*G)^{K'})$ by the commutant theorem for Azumaya algebras ([6], Theorem 4.3).

(2) Since S^*G is a separable C^G -algebra, Z is a separable C^G -algebra. Hence ZK is a separable C^G -algebra (for the order of K is a unit in Z). Thus ZK is an Azumaya D -algebra. It remains to show that $D = D^{K'}$. Clearly, $D^{K'} \subset D$. Conversely, let d be an element in D . Then $gd = dg$ for each g in K , so $gdg^{-1} = d$ for each g in K . Hence d is in $D^{K'}$.

The following consequences are immediate.

COROLLARY 3.4. Let S^*G be an GHS-extension. If K is an abelian subgroup of G , then $(S^*G)^{K'}$ is an Azumaya ZK -algebra.

PROOF. By the proof of Corollary 3.3, $(S^*G)^{K'}$ and ZK are separable subalgebras of the Azumaya Z -algebra S^*G such that $V_{S^*G}(ZK) = (S^*G)^{K'}$ and $V_{S^*G}((S^*G)^{K'}) = ZK$, so ZK is contained in the center of $(S^*G)^{K'}$ and the center of $(S^*G)^{K'}$ is contained in ZK . Thus ZK is the center of $(S^*G)^{K'}$.

COROLLARY 3.5. Let S^*G be a GHS-extension. Then (1) if $(S^*G)^{K'}$ is a commutative ring, then ZK is an Azumaya $(S^*G)^{K'}$ -algebra, and (2) if $(S^*G)^{K'}$ and ZK are commutative, then ZK is a splitting ring for the Azumaya Z -algebra S^*G .

PROOF. (1) It is immediate by the same argument of Corollary 3.4-(1). (2) Since $(S^*G)^{K'}$ and ZK are separable subalgebras of the Azumaya Z -algebra S^*G such that $V_{S^*G}((S^*G)^{K'}) = ZK$ and $V_{S^*G}(ZK) = (S^*G)^{K'}$, $(S^*G)^{K'}$ and ZK are commutative) such that $V_{S^*G}(ZK) = ZK$. Hence ZK is a maximal commutative separable subalgebra of S^*G . Thus ZK is a splitting ring for the Azumaya algebra S^*G ([6], Theorem 5.5).

4. SEPARABLE ALGEBRAS

In this section, we shall give a property of a separable subalgebra of any GHS-extension similar to Theorem 3.3. Let T be a subalgebra of S over C^G . The commutator subring of T in S is denoted by T' .

THEOREM 4.1. Let S be a GHS-extension and T a separable C^G -subalgebra of S , and $K = \{g \text{ in } G / g(t) = t \text{ for each } t \text{ in } T\}$. Then T' is invariant under K and an Azumaya $D^{K''}$ -algebra, where D is the center of T' and K'' is the restriction of K to T' .

PROOF. Since $tt' = t't$ for each t in T and t' in T' , $tg(t) = g(t)t$ for each g in K . Hence T' is invariant under K .

Next, since S is a GHS-extension, $V_S(V_S(S^G)) = S^G$ ([4], Proposition 4-1). Hence $C = V_S(S) \subset V_S(V_S(S^G)) = S^G$. This implies $C = C^G$. Noting that S is a separable C^G -algebra (for S is a GHS-extension), we have that S is an Azumaya C^G -algebra. But T is a separable subalgebra of S , so T' is also a separable subalgebra of S such that $V_S(T') = T$ by the commutant theorem for Azumaya algebras ([6], Theorem 4.3). Let D be the center of T' . Then $D \subset V_S(T') = T \subset S^K$. Thus $D = D^{K''}$ where K'' is the restriction of K to T' . The proof is complete.

COROLLARY 4.2. Let S be a GHS-extension, T a separable C^G -algebra of S , and $N = \{g \text{ in } G / g(t') = t' \text{ for each } t' \text{ in } V_S(T)\}$. Then T is invariant under N and an Azumaya $E^{N''}$ -algebra, where E is the center of T and N'' is the restriction of N to T .

PROOF. By the proof of Theorem 4.1, T and T' ($= V_S(T)$) are separable subalgebras of the Azumaya C^G -algebra S such that $T = V_S(T')$, so the corollary is immediate by Theorem 4.1.

By Theorem 3.1 in [5], Corollary 4.2 implies the following consequence.

COROLLARY 4.3. By keeping the notations and hypotheses of Corollary 4.2, T is a N'' HS-extension.

We conclude the paper with two examples: (1) S is a G -Galois extension of S^G that is a projective separable C^G -algebra, but not an H -separable extension of S^G , and (2) S is a GHS-extension.

Example 1. Let Q be the rational field, $Q[\sqrt{2}]$ the G -Galois extension of Q with Galois group $G = \{1, g\}$ where $g(\sqrt{2}) = -\sqrt{2}$ and $S = M_2(Q[\sqrt{2}])$, the matrix ring of order 2 over $Q[\sqrt{2}]$.

Let $G' = \{1, g'\}$ where $g'([a_{ij}]) = [g(a_{ij})]$ for all $[a_{ij}]$ in S . Then

- (1) S is a G' -Galois extension of $S^{G'}$,
- (2) $S^{G'} = M_2(Q)$, the matrix ring of order 2 over Q ,
- (3) $S^{G'}$ is a projective separable Q -algebra,
- (4) the center C of S is $Q[\sqrt{2}]$ and $C^G = Q$, and
- (5) S is not an H -separable extension of $S^{G'}$ because $C \neq C^G$.

Example 2. Let S and G' be given by Example 1. Then S is a G' -Galois extension of $S^{G'}$ that is a projective separable C^G -algebra by properties (1) through (4) in Example 1. Then the skew group ring $S * G'$ is a GHS-extension by Theorem 3.1.

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