Internat. J. Math. & Math. Sci.
VOL. 21 NO. 2 (1998) 235-238

SEPARABLE SUBALGEBRAS OF A CLASS OF AZUMAYA ALGEBRAS

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(Received July 15, 1996 and in revised form April 14, 1997)

ABSTRACT. Let S be a ring with 1, C the center of S, G a finite automorphism group of S of order n invertible in S, and S^G the subring of elements of S fixed under each element in G. It is shown that the skew group ring S*G is a G'-Galois extension of $(S*G)^{G'}$ that is a projective separable C^G-algebra where G' is the inner automorphism group of S*G induced by G if and only if S is a G-Galois extension of S^G that is a projective separable C^G-algebra. Moreover, properties of the separable subalgebras of a G-Galois H-separable extension S of S^G are given when S^G is a projective separable C^G-algebra.

KEY WORDS AND PHRASES: Azumaya algebras, Galois extensions, H-separable extensions, Skew group rings.
1991 AMS SUBJECT CLASSIFICATION CODES: 16S30; 16W20.

1. INTRODUCTION

DeMeyer [1] and Kanzaki [2] studied central Galois algebras and Galois extensions whose center is a Galois algebra with Galois group induced by and isomorphic with the group of the extension. These two types of Galois extensions were recently generalized to a bigger class of Galois Azumaya extensions [3] where S is called a G-Galois Azumaya extension of S^G if S is a G-Galois extension of S^G that is an Azumaya C^G-algebra where C is the center of S and S^G is the subring of elements fixed under each element of G. Sugano [4] investigated a G-Galois H-separable extension of S^G, and recently, Szeto [5] proved that a G-Galois H-separable extension S of S^G that is a projective separable C^G-algebra if and only if S is a C^G-Azumaya algebra. We call such an S a GHS-extension. It will be shown that the skew group ring S*G is a G'HS-extension if and only if S is a G-Galois extension of S^G that is a projective separable C^G-algebra, where G' is the inner automorphism group of S*G induced by G. Moreover, properties of some separable subalgebras of a GHS-extension are also given.

2. PRELIMINARIES

Throughout, S is a ring with 1, G a finite automorphism group of S of order n invertible in S, C the center of S, and S^G the subring of elements fixed under each element in G. S is called a separable extension of a subring T if there exist $\{a_i, b_i$ in $S / i = 1, 2, ..., m\}$ for some integer m such that $\sum a_i b_i = 1$ and $\sum s a_i \otimes b_i = \sum a_i \otimes b_i s$ for each s in S where $\otimes$ is over T. We call $\{a_i, b_i\}$ a separable system for S. S is called an H-separable extension of T if $S \otimes_T S$ is isomorphic with a direct summand of a finite direct sum of S as a bimodule over S. It is known that an H-separable extension is a separable extension and an Azumaya algebra is an H-separable extension. S is called a G-Galois extension of S^G, if there exist $\{c_i, d_i / i = 1, 2, ..., k\}$ in S for some integer k such that $\sum c_i d_i = 1$ and $\sum c_i g(d_i) = 0$ for each $g \neq 1$ in G. We call $\otimes\{c_i, d_i\}$ a G-Galois system for S.

3. SKEW GROUP RINGS

In this section, we shall show that S∗G is a G'HS-extension if and only if S is a G-Galois extensionof S^G that is a projective separable C^G-algebra, and give some properties of the separable subalgebras of an G'HS-extension skew group ring.

THEOREM 3.1. By keeping the notations of section 2, S∗G is a G'HS-extension if and only if S is a G-Galois extension of S^G that is a projective separable C^G-algebra, where G' is the inner automorphism group of S∗G induced by G.

PROOF. Let S be a G-Galois extension of S^G that is a projective separable C^G-algebra. Noting that S is a subring of S∗G, we have that S∗G is also a G'-Galois extension of $(S*G)^{G'}$ with a same Galois system as S where G' is the inner automorphism group of S∗G induced by G such that the restriction of G' to S is G. Hence S∗G is an H-separable extension of $(S*G)^{G'}$ ([4], Corollary 3). Moreover, since n is a unit in S, S∗G is a separable extension of S. But S∗G is a free module over S and S is a G-Galois extension of S^G that is a projective separable C^G-algebra by hypothesis, so S∗G is a projective separable C^G-algebra by the transitivity of projective separable extensions. Since the order of G' is n, it is easy to see that $(S*G)^{G'}$ is a direct summand of S∗G as a two sided $(S*G)^{G'}$-module. Noting that S∗G is finitely generated and projective module as a right $(S*G)^{G'}$-module or a left module, we have that $(S*G)^{G'}$ is a projective separable C^G-algebra by the same argument as given in the proof of Lemma 2 in [1]. This completes the sufficiency.

For the necessity, S∗G is a projective separable C^G-algebra by the transitivity of projective separable extensions because S∗G is a G'-Galois extension of $(S*G)^{G'}$ that is a projective separable algebra of C^G. Hence S∗G is an Azumaya algebra of its center Z. But S is a free module over S and n is a unit in S, so S is a finitely generated and projective left S∗G-module by the proof of Proposition 2.3 in [6] where gs = g(s) for each s in S and g in G. Thus S is a finitely generated and projective Z-module by the transitivity of finitely generated and projective modules. Noting that 1 is in C^G and that C^G is contained in Z, we have that S is a faithful Z-module. Thus S is a progenerator over Z. Since S∗G is an Azumaya Z-algebra, S is a progenerator over S∗G. Therefore, S is a G-Galois extension of S^G. Moreover, since S is a direct summand of S∗G as a C^G-module and S∗G is a finitely generated and projective C^G-module, S is also a finitely generated and projective C^G-module. Now n is a unit in S, so S^G is a S^G-direct summand of S. This implies that S^G is a finitely generated and projective C^G-module. So, it suffices to show that S^G is a separable C^G-algebra. In fact, since S is a progenerator over C^G (for 1 is in C^G), $Hom_{C^G}(S,S)$ is an Azumaya C^G-algebra. But $S^G \approx Hom_{S*G}(S,S)$ = the commutator of S∗G in $Hom_{C^G}(S,S)$, so S^G is a separable C^G-algebra (for so is S∗G) by the commutant theorem for Azumaya algebras ([6], Theorem 4.3).

Next we give some properties of the separable subalgebras of S∗G.

COROLLARY 3.2. If S∗G is a G'HS-extension, then, for any subgroup K of G, S∗K is a K'-Galois extension of $(S*K)^{K'}$ that is a separable C^G-algebra where K' is the inner automorphism group of S∗K induced by K.

PROOF. By Theorem 3.1, S is a G-Galois extension of S^G that is a projective C^G-algebra, so S is a K-Galois extension of S^K. Hence S∗K is a K'-Galois of $(S*K)^{K'}$. Noting that the order of K' is a unit in S, we have that $(S*K)^{K'}$ is a direct summand of S∗K as a $(S*K)^{K'}$-module. But S∗K is a projective separable C^G-algebra, so $(S*K)^{K'}$ is a separable C^G-algebra by the same argument as given in the proof of Lemma 2 in [1].

Let $V_S(T)$ be the commutator subring of the subring T in S, and Z the center of S∗G. We give an expression of the commutator subring of $(S*G)^{K'}$ in S∗G.

THEOREM 3.3. If S∗G is a G'HS-extension, then (1) for any subgroup K fo G, $V_{S*G}((S*G)^{K'})$ = ZK, and (2) ZK is an Azumaya algebra over its center D such that $D = D^{K'}$.

PROOF. (1) By Theorem 3.1, S∗G is a separable C^G-algebra, so S∗G is an Azumaya Z-algebra. Since n is a unit in S, the order of K is a unit in S; and so ZK is a separable Z-algebra contained in S∗G. Noting that $V_{S*G}(ZK) = (S*G)^{K'}$, we have that $(S*G)^{K'}$ is a separable Z-subalgebra of S∗G such that ZK $= V_{S*G}((S*G)^{K'})$ by the commutant theorem for Azumaya algebras ([6], Theorem 4.3).

(2) Since S∗G is a separable C^G-algebra, Z is a separable C^G-algebra. Hence ZK is a separable C^G-algebra (for the order of K is a unit in Z). Thus ZK is an Azumaya D-algebra. It remains to show that $D = D^{K'}$. Clearly, $D^{K'} \subset D$. Conversely, let d be an element in D. Then gd = dg for each g in K, so $gdg^{-1} = d$ for each g in K. Hence d is in $D^{K'}$.

The follwoing consequences are immediate.

COROLLARY 3.4. Let S∗G be an G'HS-extension. If K is an abelian subgroup of G, then $(S*G)^{K'}$ is an Azumaya ZK-algebra.

PROOF. By the proof of Corollary 3.3, $(S*G)^{K'}$ and ZK are separable subalgebras of the Azumaya Z-algebra S∗G such that $V_{S*G}(ZK) = (S*G)^{K'}$ and $V_{S*G}((S*G)^{K'}) = ZK$, so ZK is contained in the center of $(S*G)^{K'}$ and the center of $(S*G)^{K'}$ is contained in ZK. Thus ZK is the center of $(S*G)^{K'}$.

COROLLARY 3.5. Let S∗G be a G'HS-extension. Then (1) if $(S*G)^{K'}$ is a commutative ring, then ZK is an Azumaya $(S*G)^{K'}$-algebra, and (2) if $(S*G)^{K'}$ and ZK are commutative, then ZK is a splitting ring for the Azumaya Z-algebra S∗G.

PROOF. (1) It is immediate by the same argument of Corollary 3.4-(1). (2) Since $(S*G)^{K'}$ and ZK are separable subalgebras of the Azumaya Z-algebra S∗G such that $V_{S*G}((S*G)^{K'}) = ZK$ and $V_{S*G}(ZK) = (S*G)^{K'}$, $(S*G)^{K'} = ZK$ (for $(S*G)^{K'}$ and ZK are commutative) such that $V_{S*G}(ZK) = ZK$. Hence ZK is a maximal commutative separable subalgebra of S∗G. Thus ZK is a splitting ring for the Azumaya algebra S∗G ([6], Theorem 5.5).

4. SEPARABLE ALGEBRAS

In this section, we shall give a property of a separable subalgebra of any GHS-extension similar to Theorem 3.3. Let T be a subalgebra of S over C^G. The commutator subring of T in S is denoted by T'.

THEOREM 4.1. Let S be a GHS-extension and T a separable C^G-subalgebra of S, and K = {g in G / g(t) = t for each t in T}. Then T' is invariant under K and an Azumaya $D^{K''}$-algebra, where D is the center of T' and K" is the restriction of K to T'.

PROOF. Since tt' = t't for each t in T and t' in T', tg(t') = g(t')t for each g in K. Hence T' is invariant under K.

Next, since S is a GHS-extension, $V_S(V_S(S^G)) = S^G$ ([4], Proposition 4-1). Hence $C = V_S(S) \subset V_S(V_S(S^G)) = S^G$. This implies $C = C^G$. Noting that S is a separable C^G-algebra (for S is a GHS-extension), we have that S is an Azumaya C^G-algebra. But T is a separable subalgebra of S, so T' is also a separable subalgebra of S such that $V_S(T') = T$ by the commutant theorem for Azumaya algebras ([6], Theorem 4.3). Let D be the center of T'. Then $D \subset V_S(T') = T \subset S^K$. Thus $D = D^{K''}$ where K" is the restriction of K to T'. The proof is complete.

COROLLARY 4.2. Let S be a GHS-extension, T a separable C^G-algebra of S, and N = {g in G / g(t') = t' for each t' in $V_S(T)$}. Then T is invariant under N and an Azumaya $E^{N''}$-algebra, where E is the center of T and N" is the restriction of N to T.

PROOF. By the proof of Theorem 4.1, T and T' (= $V_S(T)$) are separable subalgebras of the Azumaya C^G-algebra S such that $T = V_S(T')$, so the corollary is immediate by Theorem 4.1.

By Theorem 3.1 in [5], Corollary 4.2 implies the following consequence.

COROLLARY 4.3. By keeping the notations and hypotheses of Corollary 4.2, T is a N"HS-extension.

We conclude the paper with two examples: (1) S is a G-Galois extension of S^G that is a projective separable C^G-algebra, but not an H-separable extension of S^G, and (2) S is a GHS-extension.

Example 1. Let Q be the rational field, $Q[\sqrt{2}]$ the G-Galois extension of Q with Galois group $G = \{1, g\}$ where $g(\sqrt{2}) = -\sqrt{2}$ and $S = M_2(Q[\sqrt{2}])$, the matrix ring of order 2 over $Q[\sqrt{2}]$. Let $G' = \{1, g'\}$ where $g'([a_{ij}]) = [g(a_{ij}]$ for all $[a_{ij}]$ in S. Then

(1) S is a G'-Galois extension of $S^{G'}$,

(2) $S^{G'} = M_2(Q)$, the matrix ring of order 2 over Q,

(3) $S^{G'}$ is a projective separable Q-algebra,

(4) the center C of S is $Q[\sqrt{2}]$ and $C^G = Q$, and

(5) S is not an H-separable extension of $S^{G'}$ because $C \neq C^G$.

Example 2. Let S and G' be given by Example 1. Then S is a G'-Galois extension of $S^{G'}$ that is a projective separable C^G-algebra by properties (1) through (4) in Example 1. Then the skew group ring S*G' is a G'HS-extension by Theorem 3.1.

ACKNOWLEDGMENT. This work was done under the support of a Caterpillar Fellowship at Bradley University. The paper is revised under the suggestions of the referee. The author would like to thank him for his valuable suggestions.

REFERENCES

[1] DEMEYER, F. R., Some Notes On The General Galois Theory Of Rings, *Osaka J. Math.* **2** (1965), 117-127.

[2] KANZAKI, T., On Galois Algebra Over A Commutative Ring, *Osaka J. Math.* **2** (1965), 309-317.

[3] ALFARO, R. and SZETO, G., Skew Group Rings Which Are Azumaya, *Comm. In Algebra* **23** (6), (1995), 2255-2261.

[4] SUGANO, K., On A Special Type Of Galois Extensions, *Hokkaido J. Math.* **9** (1980), 123-128.

[5] SZETO, G. , On A Class Of Azumaya Algebras, Math. Japon. (to appear).

[6] DEMEYER, F. R. and INGRAHAM, E., Separable Algebras Over Commutative Rings **181**, Springer-Verlag, Berlin,1971.

[7] IKEHATA, S., Note On Azumaya Algebras And H-Separable Extensions, *Math. J. Okayama Univ.* **23** (1981), 17-18.

[8] OKAMOTO, H., On Projective H-separable Extensions Of Azumaya Algebras, *Results In Mathematics* **14** (1988), 330-332.