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CALCULATING NORMS IN THE SPACES $l^\infty(\Gamma)/c_0(\Gamma)$ AND $l^\infty(\Gamma)/c(\Gamma)$

ROMAN SZNAJDER

Department of Mathematics
Bowie State University
Bowie, Maryland 20715, U.S.A.
E-mail: rsznajde@bowiestate.edu

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ABSTRACT. We explicitly compute norms in the quotient spaces $l^\infty(\Gamma)/c_0(\Gamma)$ and $l^\infty(\Gamma)/c(\Gamma)$.

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1. INTRODUCTION

It is interesting to have an explicit formula for quotient norms in concrete Banach spaces. In this short note we calculate the norms of elements in the real quotient Banach spaces $l^\infty(\Gamma)/c_0(\Gamma)$ and $l^\infty(\Gamma)/c(\Gamma)$ where Γ is an arbitrary nonempty set. These formulas seem not to appear (e.g., as an exercise) in texts on functional analysis, but might have been known a long time ago. We consider the following three real function spaces:

$$l^\infty(\Gamma) := \{(a_\gamma)_{\gamma\in\Gamma} : \exists_{M>0} \forall_{\gamma\in\Gamma} |a_\gamma| \le M \}, \tag{1.1}$$

$$c_0(\Gamma) := \{(a_\gamma)_{\gamma\in\Gamma} : \forall_{\varepsilon>0} \exists_{E\in\mathcal{F}} \forall_{\gamma\in\Gamma\setminus E} |a_\gamma| < \varepsilon \}, \tag{1.2}$$

$$c(\Gamma) := \{(a_\gamma)_{\gamma\in\Gamma} : \exists_{a\in R} (a_\gamma - a)_{\gamma\in\Gamma} \in c_0(\Gamma) \} \tag{1.3}$$

where $\mathcal{F} := \mathcal{F}(\Gamma)$ is the collection of all finite subsets of Γ. When $\Gamma = N$, these spaces are correspondingly, all bounded, null-convergent, and convergent real sequences. An element a appearing in the definition of $c(\Gamma)$ is called the limit of a function $(a_\gamma)_{\gamma\in\Gamma}$ and is defined uniquely. It is an immediate observation that $c_0(\Gamma) \subseteq c(\Gamma) \subseteq l^\infty(\Gamma)$. Moreover, the three above space become Banach spaces once equipped with the norm $||(a_\gamma)|| := \sup_{\gamma\in\Gamma} |a_\gamma|$. Also, $[c_0(\Gamma)]^* \simeq l^1(\Gamma)$, the space of all summable functions on Γ, and $[l^1(\Gamma)]^* \simeq l^\infty(\Gamma)$ (Day [2]). See more on this in Diestel [3], and Lindenstrauss and Tzafriri [4].

Before formulating the result, we introduce a necessary notation. For a function $(a_\gamma)_{\gamma\in\Gamma}$, let

$$\limsup_{\gamma\in\Gamma}(a_\gamma) := \inf_{E\in\mathcal{F}}\left[\sup_{\gamma\in\Gamma\setminus E}(a_\gamma)\right] \qquad \text{and} \qquad \liminf_{\gamma\in\Gamma}(a_\gamma) := \sup_{E\in\mathcal{F}}\left[\inf_{\gamma\in\Gamma\setminus E}(a_\gamma)\right].$$

As in the case of countable sequences, $\liminf_{\gamma\in\Gamma}(a_\gamma) \le \limsup_{\gamma\in\Gamma}(a_\gamma)$, and when the equality holds, the function $(a_\gamma)_{\gamma\in\Gamma}$ becomes an element of $c(\Gamma)$. For two Banach spaces $Y \subseteq X$, the quotient linear space X/Y is equipped with the norm

$$||[x]||_{X/Y} := \inf_{y\in Y} ||x - y||.$$

Theorem 1.1. The following formulas hold true:

$$||(a_\gamma)||_{l^\infty(\Gamma)/c_0(\Gamma)} = \limsup_{\gamma\in\Gamma} |a_\gamma|, \tag{1.4}$$

$$||(a_\gamma)||_{l^\infty(\Gamma)/c(\Gamma)} = \frac{1}{2}\left[\limsup_{\gamma\in\Gamma}(a_\gamma) - \liminf_{\gamma\in\Gamma}(a_\gamma)\right]. \tag{1.5}$$

2. PROOFS

The proofs of the above formulas are standard and make use of basic properties of least upper bound (supremum) and greatest lower bound (infimum) of a set.

Proof of the formula (1.4). Let $(a_\gamma)_{\gamma\in\Gamma} \in l^\infty(\Gamma)$ and $(b_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma)$. By the triangle inequality for any $E \in \mathcal{F}$, $\sup_{\gamma\in\Gamma\setminus E}(|a_\gamma| - |b_\gamma|) \leq \sup_{\gamma\in\Gamma\setminus E}|a_\gamma - b_\gamma|$. Hence,

$$\limsup_{\gamma\in\Gamma}|a_\gamma| = \inf_{E\in\mathcal{F}}\left[\sup_{\gamma\in\Gamma\setminus E}|a_\gamma|\right] = \inf_{E\in\mathcal{F}}\left[\sup_{\gamma\in\Gamma\setminus E}|a_\gamma - b_\gamma|\right] \leq \sup_{\gamma\in\Gamma}|a_\gamma - b_\gamma|.$$

Taking infimum over all $(b_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma)$, we get

$$\limsup_{\gamma\in\Gamma}|a_\gamma| \leq \inf_{(b_\gamma)\in c_0(\Gamma)}\left[\sup_{\gamma\in\Gamma}|a_\gamma - b_\gamma|\right] = ||(a_\gamma)||_{l^\infty(\Gamma)/c_0(\Gamma)}.$$

To prove the converse, for a fixed $E \in \mathcal{F}$ consider the following sequence:

$$b_\gamma^{(E)} := \begin{cases} a_\gamma & \text{for } \gamma \in E \\ 0 & \text{for } \gamma \notin E \end{cases}$$

Obviously $\left(b_\gamma^{(E)}\right)_{\gamma\in\Gamma} \in c_0(\Gamma)$, moreover,

$$\sup_{\gamma\in\Gamma}|a_\gamma - b_\gamma^{(E)}| = \sup_{\gamma\in\Gamma\setminus E}|a_\gamma| \qquad \text{and} \qquad \inf_{(b_\gamma)\in c_0(\Gamma)}\left[\sup_{\gamma\in\Gamma}|a_\gamma - b_\gamma|\right] \leq \sup_{\gamma\in\Gamma\setminus E}|a_\gamma|,$$

so

$$||(a_\gamma)||_{l^\infty(\Gamma)/c_0(\Gamma)} \leq \inf_{E\in\mathcal{F}}\left[\sup_{\gamma\in\Gamma\setminus E}|a_\gamma|\right] = \limsup_{\gamma\in\Gamma}|a_\gamma|.$$

The formula (1.4) is proved.

Proof of the formula (1.5). For a given $(a_\gamma)_{\gamma\in\Gamma} \in l^\infty(\Gamma)$, in the following sequence of inequalities we make use of the formula (1.4):

$$||(a_\gamma)||_{l^\infty(\Gamma)/c(\Gamma)} = \inf_{(b_\gamma)\in c(\Gamma)}\left[\sup_{\gamma\in\Gamma}|a_\gamma - b_\gamma|\right] = \inf_{b\in R}\left[\inf_{(b_\gamma - b)\in c_0(\Gamma)}\left[\sup_{\gamma\in\Gamma}|(a_\gamma - b) - (b_\gamma - b)|\right]\right] =$$

$$\inf_{b\in R}\left[\inf_{(b_\gamma)\in c_0(\Gamma)}\left[\sup_{\gamma\in\Gamma}[\,|(a_\gamma - b) - b_\gamma)|\,]\right]\right] = \inf_{b\in R}||(a_\gamma - b)||_{l^\infty(\Gamma)/c_0(\Gamma)} = \inf_{b\in R}\left[\limsup_{\gamma\in\Gamma}|a_\gamma|\right] =$$

$$\inf_{b\in R}\left[\max\{\,|\limsup_{\gamma\in\Gamma}(a_\gamma) - b|,\ |\liminf_{\gamma\in\Gamma}(a_\gamma) - b|\,\}\right] = \frac{1}{2}\left[\limsup_{\gamma\in\Gamma}(a_\gamma) - \liminf_{\gamma\in\Gamma}(a_\gamma)\right].$$

The proof of the formula (1.5) is complete.

3. REMARKS

Dr. Thomas Armstrong of the University of Maryland Baltimore County has informed us that our Theorem has an analogue for spaces of measurable functions. More precisely, let $(\Omega, \mathcal{M}, \mu)$ be a measure space with a σ-finite measure μ. Define $L^\infty(\mu)$ as the space of all μ-essentially bounded measurable functions defined on Ω and

$$c_0(\mu) := \{ f : (\Omega, \mathcal{M}, \mu) \to R,\ f \text{ is } \mu-\text{measurable and } \lim_{\substack{A \uparrow \Omega \\ \mu(A) < \infty}} ||I_{A^c} f||_\infty = 0 \}. \tag{3.1}$$

We define the space $c(\mu)$ in a similar manner. All these spaces are equipped with the μ-essup norm. The formulas (1.4) and (1.5) are valid for quotient norms corresponding to the spaces $L^\infty(\mu)/c_0(\mu)$ and $L^\infty(\mu)/c(\mu)$. For more on the duality properties of the spaces of the above type we refer to Armstrong [1].

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