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ON A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS NEAR CRITICAL GROWTH

J.V. GONCALVES [*]
Departamento de Matemática
Universidade de Brasilia
70.910-900 Brasilia, DF, Brasil

S. MEIRA [*]
Departamento de Matemática
Unesp - Presidente Prudente
Presidente Prudente, SP, Brasil

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ABSTRACT. We use Minimax Methods and explore compact embedddings in the context of Orlicz and Orlicz-Sobolev spaces to get existence of weak solutions on a class of semilinear elliptic equations with nonlinearities near critical growth. We consider both biharmonic equations with Navier boundary conditions and Laplacian equations with Dirichlet boundary conditions.

KEY WORDS AND PHRASES: Elliptic Equations, Variational Methods, Orlicz Spaces.
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1. INTRODUCTION

Our concern in this paper is on finding weak solutions for the problem

$$(-1)^m \Delta^m u = f(x,u) \ \ in \ \Omega, \ \ B_m(u) = 0 \ \ on \ \partial\Omega \tag{1.1}$$

where Δ^m is the elliptic operator

$$\Delta^m \equiv \sum_{i=1}^{N} \frac{\partial^{2m}}{\partial x_i^{2m}} + (m-1) \sum_{\substack{i,j=1, \\ i \neq j}}^{N} \frac{\partial^{2m}}{\partial x_i^m \partial x_j^m} \qquad m = 1,2,$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and the boundary operator B_m is given by

$$B_m(u) = (u, (m-1)\Delta u),$$

that is, $B_m(u) = 0$ means either the Dirichlet or the Navier boundary conditions according to $m = 1$ or $m = 2$.

By a weak solution of (1.1) we mean an element $u \in H_m \equiv H_o^1(\Omega) \cap H^m(\Omega)$ satisfying

$$\langle u, v \rangle_m = \int_\Omega f(x,u)v, \ \ v \in H_m$$

with $\Delta u = 0$ on $\partial\Omega$ when $m = 2$, where

$$\langle u, v \rangle_m \equiv (m-1) \int_\Omega \Delta u \Delta v + (2-m) \int_\Omega \nabla u \nabla v \ \ u, v \in H_m.$$

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By the way $\langle .,. \rangle_m$ is an inner product in H_m, we denote by $\|.\|_m$ its corresponding norm and we remark that H_m is a Hilbert space.

Now let $a : [0,\infty) \to I\!R$ be a right continuous, nondecreasing function satisfying the following conditions

$$a(0) = 0, \quad a(t) > 0 \text{ for } t > 0, \quad a(t) \to \infty \text{ as } t \to \infty \tag{1.2}$$

and let

$$A(t) = \int_0^t a(|s|)ds \text{ and } p^* = \frac{2N}{(N-2m)}.$$

We shall assume that both

$$|f(x,t)| \le C_1 + C_2\, a(|t|), \quad (x,t) \in \Omega \times I\!R \tag{1.3}$$

for some $C_1 \ge 0$, $C_2 > 0$ and

$$A(t) = o(t^{p^*}) \text{ as } t \to \infty. \tag{1.4}$$

Now consider the functional

$$I_m(u) = \frac{1}{2}\|u\|_m^2 - \int_\Omega F(x,u)dx, \quad u \in H_m$$

where $F(x,t) = \int_0^t f(x,s)ds$. It follows under conditions (1.2)(1.3)(1.4) and condition (1.5) below that $I_m \in C^1(H_m, I\!R)$ and its derivative is given by

$$\langle I_m'(u), v\rangle = \langle u,v\rangle_m - \int_\Omega f(x,u)v \quad u,v \in H_m.$$

We shall look for weak solutions of (1.1) by finding critical points of I_m. Our main result is the following.

THEOREM 1. Assume (1.2)(1.3)(1.4). Assume in addition that

$$a(|t|) \le |t|^{(p^*-1)} \quad t \in I\!R, \tag{1.5}$$

$$f(x,t) = o(t) \quad t \to 0, \text{ uniformly } x \in \Omega \tag{1.6}$$

$$0 < \theta F(x,t) \le t f(x,t) \text{ a.e. } x \in \Omega \quad |t| \ge M \tag{1.7}$$

for some $M > 0$, $\theta > 2$.

Then (1.1) has a non zero weak solution.

Our Theorem improves results by Rabinowitz [15], Gu [7], deFigueiredo, Clement & Mitidieri [3] in the sense that we allow less restrictive growth on $f(x,t)$. It is also related to some results in Brézis & Nirenberg [14], Pucci & Serrin [12], van der Vorst [13].

We employ the Ambrosetti & Rabinowitz Mountain Pass Theorem as in some of the above mentioned papers and the main point here is the use of Orlicz and Orlicz-Sobolev spaces to overcome compactness difficulties.

2. PRELIMINARIES

We shall apply the following variant of the Ambrosetti & Rabinowitz [2] Mountain Pass Theorem (see Mawhin & Willem [6]).

THEOREM 2. Let X be a Banach space and let $I \in C^1(X, I\!R)$ with $I(0) = 0$. Assume in addition that

$$I(u) \geq r \quad \text{when} \quad \|u\| = \rho, \quad \text{for some} \quad r, \rho > 0 \tag{2.1}$$

$$I(e) \leq 0, \quad \text{for some} \quad e \in X \quad \text{with} \quad \|e\| > \rho. \tag{2.2}$$

Then there is a sequence $u_n \in X$ such that

$$I(u_n) \to c \ \text{ and } \ I'(u_n) \to 0$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad c \geq r$$

and

$$\Gamma = \{\gamma \in C([0,1], X) \ \mid \ \gamma(0) = 0 \ \ \gamma(1) = e\}.$$

We shall apply theorem 2 with $I = I_m$ and $X = H_m$. The two lemmas below are crucial in applying theorem 2 to prove theorem 1.

LEMMA 3. (The Mountain Pass Geometry) Assume (1.2)-(1.7). Then (2.1)-(2.2) hold true.

We remark that by lemma 3 there is a sequence $u_n \in H_m$ such that

$$I_m(u_n) \to c \ \text{ and } \ I'_m(u_n) \to 0.$$

Such a sequence is called a $(PS)_c$ sequence.

We are going to show, (see lemma 5 below), that u_n has a convergent subsequence. The proof of lemma 5 uses a crucial compactness type result (see lemma 4 below).

Prior to stating lemma 4 we shall recall some notations and basic results on Orlicz and Orlicz-Sobolev spaces. We refer the reader to Krasnosels'kii & Rutickii [5], Gossez [4], Adams [1] for an accounting on the subject. In this regard a function A satisfying the set of conditions:

$$A \ \textit{is convex}, \ \textit{even}, \ \textit{continuous} \tag{2.3}$$

$$A(t) = 0 \ \ iff \ \ t = 0 \tag{2.4}$$

$$\frac{A(t)}{t} \ \to \ \begin{cases} 0 & \text{when } t \to 0 \\ \infty & \text{when } \ t \to \infty \end{cases} \tag{2.5}$$

is referred to in the literature on Orlicz Spaces as an N-function. An Orlicz space is defined by

$$L_A(\Omega) \equiv \{u : \Omega \to I\!R \mid u \text{ is measurable and } \int_\Omega A(l|u|) < \infty \ \textit{forsome} \ l > 0\}$$

and the norm given by

$$|u|_A \equiv \inf_{\alpha \in I\!R} \left\{ \alpha > 0 \mid \int_\Omega A\left(\frac{|u|}{\alpha}\right) \leq 1 \right\}$$

turns it into a (not necessarily reflexive) Banach space and as a matter of fact $L_A(\Omega) \to L^1(\Omega)$.

Corresponding to A there is an N-function labeled $\overline{A}$ called the conjugate function of A which satisfies the so called Young's inequality

$$st \leq A(t) + \overline{A}(s)$$

and in addition

$$ta(t) = A(t) + \overline{A}(a(t))$$

where

$$A(t) = \int_0^t a(|s|)ds$$

and a satisfies (1.2).

Moreover one also has a Hölder inequality namely

$$\int u \cdot v \leq 2|u|_{L_A}|v|_{L_{\overline{A}}}.$$

Now the Orlicz-Sobolev space is defined by

$$W^m L_A(\Omega) = \{u \in L_A(\Omega) \mid D^\alpha u \in L_A(\Omega), \ |\alpha| \leq m\}$$

and the norm

$$\|u\| \equiv \left[\sum_{|\alpha|\leq m} |D^\alpha u|^2_{L_A}\right]^{\frac{1}{2}}$$

turns it into a Banach space.

LEMMA 4. Assume (1.4). Then $H_m \hookrightarrow L_A(\Omega)$, $m = 1, 2$

LEMMA 5. Assume (1.2) – (1.7). Then the sequence u_n has a convergent subsequence.

3. PROOFS.

PROOF OF LEMMA 3.

At first given $\epsilon > 0$ there is by (1.6) some $\delta > 0$ such that

$$\frac{f(x,t)}{t} \leq \epsilon, \quad |t| < \delta \quad a.e. \ x \in \Omega$$

so that

$$F(x,t) \leq \frac{\epsilon}{2}t^2, \quad |t| < \delta \quad a.e. \ x \in \Omega.$$

On the other hand from (1.3), (1.5) we have

$$|f(x,t)| \leq C_1 + C_2|t|^{(p^*-1)}$$

so that

$$F(x,t) \leq C_1|t| + \frac{C_2}{p^*}|t|^{p^*}, \quad \text{a.e.} \ x \in \Omega, \ t \in \mathbb{R}.$$

Hence

$$F(x,t) \leq \tfrac{\epsilon}{2}t^2 + C_\delta|t|^{p^*}, \quad \text{a.e.} \ x \in \Omega, \ t \in \mathbb{R}. \tag{3.1}$$

Now observing that

$$(m-1)\int_\Omega |\Delta u|^2 + (2-m)\int |\nabla u|^2 \geq \lambda_{1m}\int_\Omega u^2$$

where λ_{1m} is the first eigenvalue of

$$\begin{cases} (-1)^m \Delta^m u = \lambda u \text{ in } \Omega \\ B_m(u) = 0 \text{ on } \partial\Omega \end{cases}$$

and using (3.1) we get

$$\int_\Omega F(x,u) \leq \frac{\epsilon}{2\lambda_{1m}} |u|_m^2 + C_\delta \int_\Omega |u|^{p^*}$$

CLAIM 1. $|u|_{L^{p^*}} \leq C\|u\|_m,\ u \in H_m$.

Using CLAIM 1, we get

$$\int_\Omega F(x,u) \leq \frac{\epsilon}{2\lambda_{1m}} \|u\|_m^2 + C\|u\|_m^{p^*}$$

so that

$$I_m(u) \geq (\frac{1}{2} - \frac{\epsilon}{2\lambda_{1m}})\|u\|_m^2 - C\|u\|_m^{p^*}.$$

Therefore there are $\rho > 0, r > 0$ such that

$$I_m(u) \geq r, \quad \|u\|_m = \rho.$$

On the other hand using (1.7) it follows that

$$F(x,t) \geq C|t|^\theta, \quad |t| \geq M \text{ a.e. } x \in \Omega.$$

Now take $\phi \in C_0^\infty$, $\phi \geq 0$, $\phi \not\equiv 0$ and $\lambda > 0$. Then

$$I_m(\lambda\phi) = \frac{\lambda^2}{2}\|\phi\|_m^2 - \int_{\lambda\phi \leq M} F(x,\lambda\phi) - \int_{\lambda\phi > M} F(x,\lambda\phi)$$

Since

$$F(x,\lambda\phi) \geq -C_1\lambda\phi - C_2 A(\lambda\phi)$$

we get

$$\begin{aligned} I_m(\lambda\phi) &\leq \tfrac{\lambda^2}{2}\|\phi\|_m^2 + \textstyle\int_{\lambda\phi \leq M} (C_1\lambda\phi + C_2A(\lambda\phi)) - \int_{\lambda\phi > M} F(x,\lambda\phi) \\ &\leq \tfrac{\lambda^2}{2}\|\phi\|_m^2 + \textstyle\int_{\lambda\phi \leq M} (C_1 M + C_2 A(M)) - \int_\Omega \phi^\theta \chi_{\phi > \frac{M}{\lambda}} \\ &\leq \tfrac{\lambda^2}{2}\|\phi\|_m^2 + C_M - \lambda^\theta \textstyle\int_\Omega \phi^\theta \chi_{\phi > \frac{M}{\lambda}}. \end{aligned}$$

Now, by Lebesgue Theorem

$$\int_\Omega \phi^\theta \chi_{\phi > \frac{M}{\lambda}} \to \int_\Omega \phi^\theta.$$

Thus

$$I_m(\lambda\phi) \to -\infty \text{ as } \lambda \to \infty.$$

VERIFICATION OF CLAIM 1. If $m = 1$ CLAIM 1 holds by the Sobolev inequality. So let us assume $m = 2$. Letting

$$\|u\|_{2,2} \equiv \max_{|\alpha| \leq 2} |D^\alpha u|_{L^2},$$

it is an easy matter to check that the space H_2 endowed with $\|.\|_{2,2}$ is complete. We claim that

$$\|u\|_2 \leq C\|u\|_{2,2}.$$

Indeed,

$$\|u\|_2^2 = \int_\Omega |\Delta u|^2 \leq C\left(\max_i \int_\Omega |\frac{\partial^2 u}{\partial x_i^2}|^2\right) \leq C\left(\max_{|\alpha|\leq 2} |D^\alpha u|_{L^2}^2\right) = C\|u\|_{2,2}^2$$

Hence we also have

$$\|u\|_{2,2} \leq C\|u\|_2 \tag{3.2}$$

and by Sobolev embedding we get $|u|_{L^{p^*}} \leq C\|u\|_2$, showing CLAIM 1 and thus proving lemma 3.

The proof of lemma 4 is a consequence of a general result due to Donaldson & Trudinger [9] (see also Adams [1, Theorem 8.40]). For the sake of completeness we recall that result in an Appendix. (see THEOREM A.1)

PROOF OF LEMMA 4.

Case $m = 2$. Applying the notations of theorem A.1 let

$$B_0^{-1}(t) = \sqrt{2}t^{\frac{1}{2}},\ t \geq 0$$

and

$$(B_k)^{-1}(t) \equiv \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau \quad t \geq 0,\ \ k = 1, 2.$$

We claim that

$$\textstyle\int_1^\infty \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau = \infty \ \text{ for } \ k = 0, 1. \tag{3.3}$$

and

$$\textstyle\int_1^\infty \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty \ \text{ for some } \ k \geq 2. \tag{3.4}$$

By (3.3) and (3.4) J is defined and $2 \leq J \leq N$.

Indeed by computing we find that

$$B_1^{-1}(t) = \tfrac{\sqrt{2}(N-2)}{2N} t^{\frac{N-2}{2N}} \tag{3.5}$$

and

$$B_2^{-1}(t) = \tfrac{\sqrt{2}(N-2)}{2N} t^{\frac{N-4}{2N}}. \tag{3.6}$$

Now using (3.5) and (3.6) and computing again we get (3.3). Thus $J \geq 2$.
In order to show (3.4) it suffices to evaluate

$$\int_0^t \frac{(B_{N-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau.$$

But

$$B_k^{-1}(t) = C_{N,k} t^{\frac{N-2k}{2N}} \quad t \geq 0,\ \ C_{N,k} > 0,\ \ k \geq 1$$

and from this

$$\int_1^\infty \frac{B_N^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau < \infty.$$

By computing again we find that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty, \quad k = 1, 2.$$

Therefore by theorem A.1 we have

$$W^2 L_{B_0}(\Omega) \hookrightarrow L_A(\Omega),$$

since as we have shown above $J \geq 2$ and yet by (1.4)

$$\frac{B_2(\lambda t)}{A(t)} = \frac{C_{N,\lambda}|t|^{2^*}}{A(t)} \to \infty \quad \text{as} \quad t \to \infty, \quad \lambda > 0.$$

The case $m = 1$ that is

$$W^1 L_{B_0}(\Omega) \hookrightarrow L_A(\Omega)$$

is similar and even more direct.

Hence

$$W^m L_{B_0}(\Omega) \hookrightarrow L_A(\Omega) \quad m = 1, 2.$$

Using (3.2) we finally get

$$H_m \hookrightarrow L_A(\Omega) \quad m = 1, 2.$$

This completes the proof of lemma 4.

Before proceding to the proof of lemma 5 we consider the function $a^*(t) \equiv 2C_2 a(t)$. We remark that $a^*(t)$ has the same properties of $a(t)$ and in addition its potential $A^*(t) \equiv \int_0^t a^*(\tau)d\tau$ is an N-function having the same properties as $A(t)$. In particular A^* satisfies (1.4) and moreover

$$|f(x,t)| \leq C_1 + \frac{1}{2}a^*(t).$$

PROOF OF LEMMA 5.

Using (1.7) we have

$$C \geq \tfrac{1}{2}\|u\|_m^2 - \textstyle\int_\Omega F(x,u_n) \geq \tfrac{1}{2}\|u\|_m^2 - C - \tfrac{1}{\theta}\textstyle\int_\Omega u_n f(x,u_n). \tag{3.7}$$

Now since $I'_m(u_n) \to 0$ we have

$$|\langle I'_m(u_n), u_n\rangle| \leq \epsilon\|u\|_m \quad \text{for large} n$$

that is

$$\|u\|_m^2 - \int_\Omega u_n f(x,u_n)| \leq \epsilon\|u\|_m \quad \text{for large} n.$$

Hence

$$\begin{aligned} C &\geq \tfrac{1}{2}\|u_n\|_m^2 - C - \tfrac{1}{\theta}\|u_n\|_m^2 - \tfrac{1}{\theta}\epsilon\|u_n\|_m \\ &= \left(\tfrac{1}{2} - \tfrac{1}{\theta}\right)\|u_n\|_m^2 - \tfrac{1}{\theta}\epsilon\|u_n\|_m - C \end{aligned}$$

showing that u_n is bounded in H_m. Hence by lemma 4 there is some $u \in H_m$ such that

$$u_n \rightharpoonup u \ in\ H_m \quad \text{and} \quad u_n \to u \ \ in\ \ L_{A^*}(\Omega).$$

On the other hand, since $I'_m(u_n) \to 0$ we have

$$\langle u_n, \phi\rangle_m - \int_\Omega f(x,u_n)\phi = o(1), \quad \phi \in H_m.$$

We claim that

$$|f(x,u_n)|_{L_{A^*}} \leq C, \quad \text{for some } \ C > 0. \tag{3.8}$$

Assume (3.8) for a while. Using Hölder inequality in Orlicz spaces for L_{A^*} and $L_{\bar{A}^*}$ where $\bar{A}^*$ is the conjugate function of A^* (see e.g. Adams [1, pg 234]) we get

$$|\langle u_n, \phi \rangle_m| \leq o(1) + |f(x,u_n)|_{L_{\bar{A}^*}} |\phi|_{L_{A^*}} \tag{3.9}$$

Now replacing ϕ by $u_n - u$ in (3.9) and using (3.8) we have

$$0 = \lim \langle u_n, u_n - u \rangle_m = \lim \langle u_n, u_n \rangle_m = \lim \langle u_n, u_n \rangle_m - \langle u, u \rangle_m$$

showing that $u_n \to u \ \ in \ \ H_m$.

VERIFICATION OF (3.8). We have

$$\begin{array}{rcl} \int_\Omega \overline{A^*}(|f(x,u_n)|) & \leq & \int_\Omega \overline{A^*}\left(C_1 + \frac{1}{2} a^*(|u_n|)\right) \\ & \leq & \frac{1}{2}\int_\Omega \overline{A^*}(2C_1) + \frac{1}{2}\int_\Omega \overline{A^*}(a^*(|u_n|)) \\ & \leq & C + \frac{1}{2}\int_\Omega A^*(|u_n|) + \int_\Omega |u_n| a^*(|u_n|) \\ & \leq & C + C_1 \left[\int_\Omega |u_n|^{p^*} + \int_\Omega |u_n|^{p^*}\right] \leq C \end{array}$$

showing (3.8) and consequently lemma 5.

PROOF OF THEOREM 1.

We have already shown using the lemmata above that I_m has a critical point $u \in H_m$ so that

$$\langle u, v \rangle_m = \int_\Omega f(x,u)v, \quad v \in H_m.$$

In the case $m = 1$, we have $H_1 = H_o^1$ and so u is a weak solution of $(*)_1$.

In the case $m = 2$ it remains to show that $\Delta u = 0$ *on* $\partial\Omega$. We use here an argument of [4].

By (1.3) and (1.5), we have

$$f(x,u) \in L^{p^{*'}}(\Omega) \ \text{with} \ \frac{1}{p^*} + \frac{1}{p^{*'}} = 1.$$

Letting $g(x) = f(x,u)$ using the fact that $p^* > 2$ it follows that $W \equiv W^{2,p^*}(\Omega) \cap W_0^{1,p^*}(\Omega) \subset H_2$ and we have

$$\int_\Omega \Delta u \Delta z = \int_\Omega g(x)z, \quad z \in W.$$

Since $g(x) \in L^{p^{*'}}(\Omega)$ there is a unique $w \in W^{2,p^{*'}}(\Omega) \cap W_0^{1,p^{*'}}(\Omega)$ such that

$$\Delta w = g(x), \quad x \in \Omega.$$

Hence

$$\int_\Omega \Delta u \Delta z = \int_\Omega \Delta w z = \int_\Omega w \Delta z, \quad z \in W.$$

On the other hand given $h \in L^{p^*}(\Omega)$, there is a unique $z \in W$, such that

$$\Delta z = h(x), \quad x \in \Omega.$$

Thus

$$\int_{\Omega}(\Delta u - w)h = 0, \quad h \in L^{p^*}(\Omega)$$

showing that

$$\Delta u = w \quad in \quad \Omega$$

and so

$$\Delta u = 0, \quad \text{on} \quad \partial\Omega.$$

This proves theorem 1.

4. APPENDIX

At first we recall a general result due to Donaldson & Trudinger [9] (see also Adams [1, theorem 8.40]).

Let C be an N-function and consider the sequence of N-functions

$$B_0(t) \equiv C(t), \quad t \geq 0$$

$$(B_k)^{-1}(t) \equiv \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau, \quad k = 1, 2, \cdots, \quad t \geq 0.$$

It follows that

$$\int_1^{\infty} \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty \quad \text{for some} \quad k \geq 1.$$

Let us label $J \equiv J(C)$ the least such k.

THEOREM A.1. Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain with the cone property. Assume also that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty, \quad k = 1, 2, \ldots.$$

Then

$$W^m L_{B_0}(\Omega) \rightarrow L_{B_m}(\Omega) \tag{3.10}$$

provided $J \geq m$,

$$W^m L_{B_0}(\Omega) \hookrightarrow L_A(\Omega) \tag{3.11}$$

provided both $J \geq m$ and A is an N-function such that

$$\frac{B_m(\lambda t)}{A(t)} \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty, \quad \lambda > 0.$$

Next we present an example to illustrate our assumptions $(1.2) - (1.5)$.

EXAMPLE A.2. Let $a : [0, \infty) \rightarrow \mathbb{R}$ be given by $a(t) = t^{p^*-1}$ if $0 \leq t < 1$, $a(t) = t^{(p^*-1) - \frac{1}{log(log(2))}}$ if $1 \leq t < 3$ and $a(t) = t^{(p^*-1) - \frac{1}{log(log(n))}}$ if $n \leq t < (n+1)$ for $n = 3, 4, \ldots.$

Then a satisfies $(1.2), (1.5)$ and it is a straightfoward calculation to show that A satisfies (1.4).

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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Manuscript Due	March 1, 2009
First Round of Reviews	June 1, 2009
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Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru