

STABILITY PROBLEM OF SOME NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we investigate the asymptotic stability of the recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \dots$$

and the existence of certain monotonic solutions of the equation

$$x_{n+1} = x_n^p f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots$$

which includes as a special case the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^p}{1 + \sum_{i=1}^k \gamma_i x_{n-i}^{p-r}},$$

where $\alpha \geq 0$, $\beta > 0$, $\gamma > 0$, $\gamma_i \geq 0$, $i = 1, 2, \dots, k$, $\sum_{i=1}^k \gamma_i > 0$, $p \in \{2, 3, \dots\}$ and $r \in \{1, 2, \dots, p-1\}$. The case when $r = 0$ has been investigated by Camouzis et. al. [1], and for $r = 0$ and $p = 2$ by Camouzis et. al. [2].

KEY WORDS AND PHRASES: Difference Equations, Monotonic solutions, stability.

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1. INTRODUCTION

Many authors studied the asymptotic behaviour of the recursive sequence

$$x_{n+1} = x_n f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \quad (1.1)$$

which includes as a special case the rational recursive sequence

$$x_{n+1} = \frac{a + bx_n}{1 + \sum_{i=1}^k \gamma_i x_{n-i}}, \quad n = 0, 1, \dots \quad (1.2)$$

See Jaroma et. al. [3]. Also, there are many results about permanence, global attractivity and asymptotic stability of equation (1.2), see Camouzis et. al. [2], Kocic and Ladas [4-5] and Kocic et. al. [6]. The investigation of the behaviour of solutions of the equation

$$x_{n+1} = x_n^p f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \quad (1.3)$$

was suggested by Kocic and Ladas [5]. This equation includes as a special case the equation

$$x_{n+1} = \frac{\alpha + \beta x_n^p}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \dots, \quad (1.4)$$

Our aim in this paper is to study the asymptotic stability of the rational recursive sequence (1.4) when $p = 2$, see section 2. On the basis of the results of section 2 we also investigate the behaviour of solutions of equation (1.3), in section 3. We show that under certain conditions on f , there exists two solutions of (1.3) such that one tends to zero and the other tends to infinity. See theorem 3.1. We apply this theorem to equation (1.4) when $\alpha = 0$.

2. THE RECURSIVE SEQUENCE $x_{n+1} = (\alpha + \beta x_n^2) / (1 + \gamma x_{n-1})$

In this section we study the asymptotic stability for the rational recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \dots \quad (2.1)$$

where $\alpha \geq 0$ and $\beta, \gamma > 0$.

The linearized equation associated with (2.1) about \bar{x} is

$$y_{n+1} - \frac{2\beta\bar{x}}{1 + \gamma\bar{x}}y_n + \frac{\gamma\bar{x}}{1 + \gamma\bar{x}}y_{n-1} = 0, \quad n = 0, 1, \dots \quad (2.2)$$

The characteristic equation of (2.2) about \bar{x} is

$$\lambda^2 - \frac{2\beta\bar{x}}{1 + \gamma\bar{x}}\lambda + \frac{\gamma\bar{x}}{1 + \gamma\bar{x}} = 0. \quad (2.3)$$

Equation (2.3) can be rewritten in the form

$$(\lambda - l\theta)^2 = l^2\theta^2 - \theta, \quad (2.4)$$

where

$$l = \beta/\gamma \text{ and } \theta = \frac{\gamma\bar{x}}{1 + \gamma\bar{x}}. \quad (2.5)$$

We summarize the results of this section in the following

THEOREM 2.1

(1) If $\beta > \gamma$ and $\alpha = 0$, then equation (2.1) has two equilibria:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = \frac{1}{\beta - \gamma}$$

and \bar{x}_1 is asymptotically stable while \bar{x}_2 is unstable. Neither of them is a global attractor.

(2) If $\beta < \gamma$ and $\alpha = 0$, then the unique equilibrium point $\bar{x} = 0$ of equation (2.1) is globally asymptotically stable.

(3) If $\beta < \gamma$ and $\alpha > 0$, then the unique positive equilibrium point

$$\bar{x} = \frac{\sqrt{1 + 4\alpha(\gamma - \beta)} - 1}{2(\gamma - \beta)}$$

of equation (2.1) is asymptotically stable.

(4) If $\beta = \gamma$, equation (2.1) has the unique equilibrium point $\bar{x} = \alpha$ which is asymptotically stable.

(5) If $\beta > \gamma$, $\alpha > 0$ and $1 > 4\alpha(\beta - \gamma)$, then equation (2.1) has two positive equilibria

$$\bar{x}_1 = \frac{1 - \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

which is asymptotically stable, and

$$\bar{x}_2 = \frac{1 + \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

which is unstable. None of them is a global attractor.

(6) If $\beta > \gamma$, $\alpha > 0$ and $1 = 4\alpha(\beta - \gamma)$, then equation (2.1) has the unique equilibrium $\bar{x} = 1/2(\beta - \gamma)$ which is neither stable nor a global attractor.

(7) Assume that $\beta > \gamma$, $\alpha > 0$ and $1 < 4\alpha(\beta - \gamma)$. If the initial conditions $\{x_{-1}, x_0\}$ are such that

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \frac{1}{\beta - \gamma},$$

then $\{x_n\}$ tends to infinity monotonically.

PROOF.

(1) Assume that $\beta > \gamma$ and $\alpha = 0$. The characteristic equation of equation (2.2) about $\bar{x}_1 = 0$ is $\lambda^2 = 0$. Hence \bar{x}_1 is asymptotically stable, by Corollary 1.3.2 Kocic and Ladas [5] page 14. The characteristic equation of equation (2.2) about $\bar{x}_2 = 1/(\beta - \gamma)$ is

$$\lambda^2 - 2\lambda + \frac{\gamma}{\beta} = 0,$$

which has two solutions $\lambda = 1 \pm \sqrt{1 - \gamma/\beta}$. Therefore, \bar{x}_2 is unstable. The nonattractivity of equilibria \bar{x}_1 and \bar{x}_2 follows directly from theorems 3.3 and 5.1 of Camouzis et. al. [1].

(2) Assume that $\beta < \gamma$ and $\alpha = 0$. Let $\{x_n\}$ be a positive solution of equation (2.1). We have

$$\frac{x_{n+1}}{x_n} = \frac{\beta x_n}{1 + \gamma x_{n-1}} < \frac{\beta}{\gamma} \frac{x_n}{x_{n-1}}$$

Hence $x_{n+1}/x_n < (\beta/\gamma)^{n+1}(x_0/x_{-1}) \forall n \in \mathbf{N}$. Since $\beta/\gamma < 1$, then $(\beta/\gamma)^{n+1}(x_0/x_{-1}) < 1 \forall n \geq n_0$ for some $n_0 \in \mathbf{N}$. Therefore, $x_{n+1} < x_n \forall n \geq n_0$. This implies that $\lim_{n \rightarrow \infty} x_n = 0$, i.e., $\bar{x} = 0$ is globally asymptotically stable.

(3) Suppose that $\beta < \gamma$ and $\alpha > 0$. We can see that $|\lambda| < 1$ for every solution λ of the characteristic equation (2.4), about

$$\bar{x} = \frac{\sqrt{1 + 4\alpha(\gamma - \beta)} - 1}{2(\gamma - \beta)}.$$

Indeed, we have the following two cases

First case: $l^2\theta^2 - \theta < 0$. In this case $\lambda = l\theta \pm ir$, where $r^2 = \theta - l^2\theta^2$. Hence $|\lambda|^2 = l^2\theta^2 + r^2 = \theta < 1$.

Second case: $l^2\theta^2 - \theta \geq 0$. In this case $\lambda = l\theta \pm \sqrt{l^2\theta^2 - \theta}$. Hence, $|\lambda| \leq l\theta + \sqrt{l^2\theta^2 - \theta}$. Since $l < 1$, then $(2l - 1)\theta < 1$. Hence $(1 - l\theta)^2 > l^2\theta^2 - \theta$ whence $l\theta + \sqrt{l^2\theta^2 - \theta} < 1$. Therefore $|\lambda| < 1$. In both cases $|\lambda| < 1$ and thus \bar{x} is asymptotically stable.

(4) Assume that $\beta = \gamma$. For every solution λ of the characteristic equation (2.4) about $\bar{x} = \alpha$, we have $|\lambda|^2 = \gamma\alpha/(1 + \gamma\alpha) < 1$. Therefore, \bar{x} is asymptotically stable.

(5) Suppose that $\beta > \gamma$, $\alpha > 0$ and $1 > 4\alpha(\beta - \gamma)$. The characteristic equation of (2.2) about

$$\bar{x}_1 = \frac{1 - \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

is obtained by setting $\bar{x} = \bar{x}_1$ in equation (2.5). Since $\bar{x}_1 < 1/2(\beta - \gamma) < 1/(\beta - \gamma)$, then $l\theta < 1$. We can see that $|\lambda| < 1$ for every solution λ of equation (2.4). Indeed, we have the following two cases

First case: $l^2\theta^2 - \theta < 0$. In this case $\lambda = l\theta \pm ir$, where $r^2 = \theta - l^2\theta^2$. Hence $|\lambda|^2 = l^2\theta^2 + r^2 = \theta < 1$.

Second case: $l^2\theta^2 - \theta \geq 0$. In this case $\lambda = l\theta \pm \sqrt{l^2\theta^2 - \theta}$. Hence, $|\lambda| \leq l\theta + \sqrt{l^2\theta^2 - \theta}$. Since $\bar{x}_1 < 1/2(\beta - \gamma)$, then $\gamma\bar{x}_1/(1 + \gamma\bar{x}_1) < \gamma/(2\beta - \gamma)$, i.e., $\theta < 1/(2l - 1)$. Hence $(1 - l\theta)^2 > l^2\theta^2 - \theta$ whence $l\theta + \sqrt{l^2\theta^2 - \theta} < 1$. Therefore $|\lambda| < 1$. In both cases $|\lambda| < 1$ and thus \bar{x}_1 is asymptotically stable. In a similar manner, it can be shown that

$$\bar{x}_2 = \frac{1 + \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

is unstable. To show the nonattractivity of \bar{x}_1 and \bar{x}_2 , one chooses the initial conditions $\{x_{-1}, x_0\}$ such that

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \max\left\{\frac{1}{\beta - \gamma}, \bar{x}_2\right\}.$$

We show by induction that $\{x_n\}$ is increasing. Indeed, we have

$$x_{n+1} > \frac{\beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \dots$$

Then

$$x_1 > x_0 \frac{\beta x_0}{1 + \gamma x_0} \geq x_0.$$

Assume that there exists $m_0 \geq 0$ such that

$$x_{n+1} > x_n \quad \forall n \leq m_0.$$

Hence

$$x_{m_0+1} > x_{m_0} \frac{\beta x_{m_0}}{1 + \gamma x_{m_0-1}} > x_{m_0} \frac{\beta x_{m_0}}{1 + \gamma x_{m_0}} > x_{m_0} \frac{\beta x_0}{1 + \gamma x_0} > x_{m_0},$$

i.e., $\{x_n\}$ is increasing. The condition $x_0 \geq \bar{x}_2$ implies that x_n tends to infinity.

(6) Suppose that $\beta > \gamma$, $\alpha > 0$ and $1 = 4\alpha(\beta - \gamma)$. Substituting by $\bar{x} = 1/2(\beta - \gamma)$ in equation (2.3) one can easily see that $\bar{x} = 1/2(\beta - \gamma)$ is unstable. The nonattractivity of \bar{x} follows directly by considering a solution $\{x_n\}$ with the initial conditions $\{x_{-1}, x_0\}$ satisfying

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \frac{1}{\beta - \gamma}.$$

As in the proof of (5), it is easy to show that $\{x_n\}$ tends to infinity.

(7) Assume that $\beta > \gamma$, $\alpha > 0$ and $1 < 4\alpha(\beta - \gamma)$. Then in a similar way as in (5), one can easily show that the solution $\{x_n\}$ with the initial conditions $\{x_{-1}, x_0\}$ are such that

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \frac{1}{\beta - \gamma}$$

is increasing. Since equation (2.1) has no real equilibria, then x_n tends to infinity.

3. THE EQUATION $x_{n+1} = x_n^p f(x_n, x_{n-1}, \dots, x_{n-k})$

Let $f \in C([0, \infty)^{k+1}, (0, \infty))$ such that f satisfies the following conditions

(C1) $f(x, u_1, \dots, u_k)$ is nonincreasing in u_1, u_2, \dots, u_k .

(C2) $x^{p-1}f(x, x, \dots, x)$ is increasing.

(C3) The equation $x^{p-1}f(x, x, \dots, x) = 1$ has a unique positive equilibrium \bar{x} .

We show that the asymptotic behaviour of the positive solutions of the difference equation

$$x_{n+1} = x_n^p f(x_n, x_{n-1}, \dots, x_{n-k}) \tag{3.1}$$

depends on the initial conditions, see theorem 3.1. More precisely, we can choose the initial conditions such that the corresponding solution $\{x_n\}$ may tend to zero or infinity.

LEMMA 3.1. Assume that $\{x_n\}$ is a solution of equation (3.1). Under conditions (C1-C3) the following statements are true

(a) If for some $n_0 \geq -k$,

$$x_{n_0+k} \leq x_{n_0+j}, \quad j = 0, 1, \dots, k-1 \text{ and } x_{n_0+k} < \bar{x},$$

then

$$x_{n+k+1} < x_{n+k} < \bar{x} \quad \forall n \geq n_0.$$

(b) If for some $n_0 \geq -k$,

$$x_{n_0+k} \geq x_{n_0+j}, \quad j = 0, 1, \dots, k-1 \text{ and } \bar{x} < x_{n_0+k},$$

then

$$x_{n+k} < x_{n+k+1} \quad \forall n \geq n_0.$$

PROOF.

(a) Assume that for some $n_0 \geq -k$,

$$x_{n_0+k} \leq x_{n_0+j}, \quad j = 0, 1, \dots, k-1 \text{ and } x_{n_0+k} < \bar{x},$$

Then

$$\begin{aligned} x_{n_0+k+1} &= x_{n_0+k}^p f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) = x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) \\ &\leq x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k}, \dots, x_{n_0+k}) < x_{n_0+k}. \end{aligned}$$

We can see by induction that

$$x_{n+k+1} < x_{n+k} < \bar{x} \quad \forall n \geq n_0.$$

(b) Assume that for some $n_0 \geq -k$,

$$x_{n_0+k} \geq x_{n_0+j}, \quad j = 0, 1, \dots, k-1 \text{ and } \bar{x} < x_{n_0+k},$$

Then

$$\begin{aligned} x_{n_0+k+1} &= x_{n_0+k}^p f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) = x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) \\ &\geq x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k}, \dots, x_{n_0+k}) > x_{n_0+k}. \end{aligned}$$

By induction we see that

$$x_{n+k} < x_{n+k+1} \quad \forall n \geq n_0.$$

THEOREM 3.1. Under conditions (C1-C3) the following statements are true

If $\{x_n\}$ is a solution of equation (3.1) with initial conditions $\{x_{-k}, \dots, x_0\}$ that satisfy

$$x_{-j} \geq x_0 > 0, \quad j = 1, \dots, k \text{ and } \bar{x} > x_0,$$

then x_n tends to zero monotonically.

If the initial conditions $\{x_{-k}, \dots, x_0\}$ are such that

$$x_{-j} \leq x_0, \quad j = 1, \dots, k \text{ and } \bar{x} < x_0,$$

then x_n tends to infinity monotonically.

PROOF.

(1) From Lemma 3.1 we see that the solution $\{x_n\}$ is decreasing whence it converges to a nonnegative number, say l . Since $l < \bar{x}$, then $l = 0$, because of condition (C3).

(2) We can see in a similar manner that $\{x_n\}$ is increasing and $x_n > \bar{x} \quad \forall n \in \mathbb{N}$. Therefore, $x_n \rightarrow \infty$ as $n \rightarrow \infty$ by condition (C3).

As a direct consequence we obtain the following result

COROLLARY 3.1. Under conditions (C1-C3), equation (3.1) is not permanent.

4. MONOTONE SOLUTIONS OF $x_{n+1} = \beta x_n^p / (1 + \sum_{i=1}^k \gamma_i x_{n-i}^{p-r})$

We apply theorem (3.1) to the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^p}{1 + \sum_{i=1}^k \gamma_i x_{n-i}^{p-r}}, \tag{4.1}$$

where $\beta > 0, \gamma_i \geq 0 \quad \forall i = 1, \dots, k, p \in \{2, 3, \dots\}, r \in \{1, 2, \dots, p-1\}$ and $\gamma = \sum_{i=1}^k \gamma_i > 0$.

We verify that the function $f(x, u_1, \dots, u_k) = \beta / (1 + \sum_{i=1}^k \gamma_i u_i^{p-r})$ satisfies conditions (C1-C3). We can see easily that conditions (C1-C2) are satisfied. The equation

$$x = \frac{\beta x^p}{1 + \gamma x^{p-r}} \tag{4.2}$$

has a unique positive solution if and only if the function

$$h(x) = \beta x^{p-1} - \gamma x^{p-r} - 1$$

has a unique positive zero. Since

$$h'(x) = x^{p-r-1}[\beta(p-1)x^{r-1} - \gamma(p-r)],$$

then we have the following two cases

If $r \in \{2, \dots, p-1\}$, then h has a unique positive zero $\bar{x} > [\gamma(p-r)/\beta(p-1)]^{1/r-1} = x_0$ which is the unique equilibrium point of (4.1). Indeed, the function h is decreasing for $0 < x < x_0$ and increasing for $x > x_0$. Moreover, $\lim_{x \rightarrow \infty} h(x) = \infty$ and $h(0) = -1 < 0$. Then equation (4.1) has a unique positive equilibrium \bar{x} .

If $r = 1$ and $\beta > \gamma$, then equation (4.1) has the unique positive equilibrium

$$\bar{x} = \left[\frac{1}{\beta - \gamma} \right]^{\frac{1}{p-1}}.$$

Now we can apply theorem (3.1) to equation (4.1) to obtain the following result.

COROLLARY 4.1. Assume that either

$$r \in \{2, \dots, p-1\}$$

or

$$r = 1 \text{ and } \beta > \gamma.$$

Let \bar{x} be the unique positive equilibrium point of equation (4.1) and let $\{x_n\}$ be a solution of equation (4.1).

If for some $n_0 \geq -k$

$$x_{n_0+k} \leq x_{n_0+j}, \quad j = 0, 1, \dots, k-1 \text{ and } x_{n_0+k} < \bar{x},$$

then

$$x_{n+k+1} < x_{n+k} \quad \forall n \geq n_0.$$

If for some $n_0 \geq -k$,

$$x_{n_0+k} \geq x_{n_0+j}, \quad j = 0, 1, \dots, k-1 \text{ and } \bar{x} < x_{n_0+k},$$

then

$$x_{n+k} < x_{n+k+1} \quad \forall n \geq n_0.$$

If the initial conditions $\{x_{-k}, \dots, x_0\}$ are such that

$$x_{-j} \geq x_0 > 0, \quad j = 1, \dots, k \text{ and } \bar{x} > x_0,$$

then x_n tends to zero monotonically.

If the initial conditions $\{x_{-k}, \dots, x_0\}$ are such that

$$x_{-j} \leq x_0, \quad j = 1, \dots, k \text{ and } \bar{x} < x_0,$$

then $x_n \rightarrow \infty$ monotonically.

Now, we consider the equation

$$x_{n+1} = \frac{\beta x_n^p}{1 + \gamma x_{n-1}^{p-1}}, \tag{4.2}$$

where $\beta > 0, \gamma > 0, p \in \{2, 3, \dots\}$. We prove that there exists a solution $\{x_n\}$ which tends monotonically to \bar{x} . We follow the proof by Camouzis et. al. [2].

THEOREM 4.1. If $\beta > \max\{\gamma, 2\sqrt{\gamma}\}$, then equation (4.2) has two solutions $\{x_n\}$ and $\{y_n\}$ such that $\{x_n\}$ increases to \bar{x} and $\{y_n\}$ decreases to \bar{x}

PROOF. First, define the functions f_{-1} and f_0 on $[0, \infty)$ by

$$f_{-1}(x) = x^2, \quad f_0(x) = x$$

and

$$f_{n+1} = \frac{\beta f_n^p}{1 + \gamma f_n^{p-1}}, \quad n = 0, 1, \dots$$

Let

$$A = \{x \in [0, \infty) : \sup_{n \geq 0} f_n(x) < \bar{x}\}.$$

We show that $A \neq \emptyset$. Indeed, let θ be a positive number such that

$$\theta < \min \left\{ \bar{x}, \left(\frac{\beta}{2\gamma} - \frac{1}{2\gamma} \sqrt{\beta^2 - 4\gamma} \right)^{\frac{1}{p-1}} \right\}.$$

We have

$$f_1(\theta) = \frac{\beta \theta^p}{1 + \gamma \theta^{2p-2}}.$$

One can easily show that $f_1(\theta) < f_0(\theta) = \theta < \bar{x}$. By Corollary 4.1 (3), $f_{n+1}(\theta) < f_n(\theta) \forall n \geq 0$. This implies that $\sup_{n \geq 0} f_n(\theta) = f_0(\theta) < \bar{x}$.

We define the function S by

$$S(x) = \sup_{n \geq 0} f_n(x).$$

We claim that S is continuous on A and A is open. Fix $x \in A$. There exists $N \geq 0$ such that

$$f_0(x) \leq f_1(x) \leq \dots \leq f_N(x) < \bar{x} \quad \text{and} \quad f_{N+1}(x) < f_N(x).$$

If this were not true, then

$$f_0(x) \leq f_1(x) \leq \dots \leq S(x) < \bar{x},$$

whence $f_n(x) \rightarrow S(x) = \bar{x}$ which is a contradiction. This implies that

$$S(x) = f_N(x) \quad \text{and} \quad f_{N+1}(x) < f_N(x).$$

Let $\epsilon > 0$ be such that $\epsilon < \min\{\bar{x} - f_N(x), (f_N(x) - f_{N+1}(x))/2\}$. From the continuity of f_0, \dots, f_{N+1} , there exists $\delta > 0$ such that for $x' \in A$ we have

$$|x - x'| < \delta \Rightarrow \sup_{0 \leq n \leq N+1} |f_n(x) - f_n(x')| < \epsilon.$$

Since $f_{N+1}(x') < f_{N+1}(x) + \epsilon < f_N(x) - \epsilon < f_N(x') < f_N(x) + \epsilon < \bar{x}$, then

$$S(x) - \epsilon = f_N(x) - \epsilon < f_N(x') \leq S(x'),$$

and

$$\begin{aligned} S(x') &= \sup_{0 \leq n \leq N} f_n(x') < \sup_{0 \leq n \leq N} (f_n(x) + \epsilon) \\ &= f_N(x) + \epsilon < f_N(x) + \bar{x} - f_N(x) = \bar{x}. \end{aligned}$$

Therefore, S is continuous and A is open. Set $\lambda = \sup A$. Then $\lambda \notin A$ whence $S(\lambda) \geq \bar{x}$. The continuity of f_m for every $m \geq 0$ implies that $S(\lambda) \leq \bar{x}$. Hence $S(\lambda) = \bar{x}$. Now, we claim that $f_0(\lambda) < f_1(\lambda) < \dots < \bar{x}$. Indeed, we can see that $f_1(\lambda) > f_0(\lambda)$. If not, then $f_0(\lambda) \geq f_1(\lambda) \geq f_2(\lambda) \dots$, because of corollary 4.1. Hence $S(\lambda) = f_0(\lambda) = \lambda = \bar{x}$ whence

$$f_1(\lambda) = f_1(\bar{x}) = \frac{\beta \bar{x}^p}{1 + \gamma \bar{x}^{2p-2}} = \frac{\beta(\bar{x})^p}{1 + \gamma(\bar{x})^{p-1}} \frac{1 + \gamma(\bar{x})^{p-1}}{1 + \gamma \bar{x}^{2p-2}} > \bar{x}.$$

Note that $\bar{x} < 1$. Now assume that $f_0(\lambda) < f_1(\lambda) < \dots < f_N(\lambda)$ and $f_N(\lambda) \geq f_{N+1}(\lambda)$ for some $N \geq 1$. Then $S(\lambda) = f_N(\lambda) = \bar{x}$ whence

$$\begin{aligned} f_{N+1}(\lambda) &= \frac{\beta f_N^p(\lambda)}{1 + \gamma f_N^{p-1}(\lambda)} > \frac{\beta f_N^p(\lambda)}{1 + \gamma f_N^{p-1}(\lambda)} \\ &= \frac{\beta \bar{x}^p}{1 + \gamma \bar{x}^{p-1}(\lambda)} = \bar{x}, \end{aligned}$$

which is a contradiction. Therefore, $f_n(\lambda)$ is increasing to \bar{x} .

Next, we define the functions f_{-1} and f_0 on $[0, \infty)$ by

$$f_{-1}(x) = x, \quad f_0(x) = x^2$$

and

$$f_{n+1} = \frac{\beta f_n^p}{1 + \gamma f_n^{p-1}}, \quad n = 0, 1, \dots$$

We denote by

$$A = \{x \in [0, \infty) : \inf_{n \geq 0} f_n(x) > \bar{x}\}.$$

We can see that $A \neq \emptyset$. Indeed, let θ be such that

$$\theta > \max \left\{ \sqrt{\bar{x}}, \left(\frac{\gamma}{2\beta} + \frac{1}{2\beta} \sqrt{\gamma^2 + 4\beta} \right)^{\frac{1}{p-1}} \right\}.$$

We have

$$f_1(\theta) = \frac{\beta f_0^p(\theta)}{1 + \gamma f_0^{p-1}(\theta)} = \frac{\beta \theta^{2p}}{1 + \gamma \theta^{p-1}}.$$

Set $a = \theta^{p-1}$. Then $a > (\gamma + \sqrt{\gamma^2 + 4\beta})/2\beta$ whence $\beta a^2 > 1 + a\gamma$ i. e. $(\beta \theta^{2p-2})/(1 + \gamma \theta^{p-1}) > 1$. Hence $f_1(\theta) > f_0(\theta) = \theta^2 > \bar{x}$. This implies that $\inf_{n \geq 0} f_n(\theta) = f_0(\theta) > \bar{x}$.

Define the function S by

$$S(x) = \inf_{n \geq 0} f_n(x).$$

We show that S is continuous on A and A is open. In fact, fix $x \in A$, there exists a natural number N such that

$$f_0(x) \geq f_1(x) \geq \dots \geq f_N(x) > \bar{x} \text{ and } f_{N+1}(x) > f_N(x)$$

Otherwise,

$$f_0(x) \geq \dots \geq f_n(x) \geq \dots \geq S(x) > \bar{x}$$

and therefore

$$\lim_{n \rightarrow \infty} f_n(x) \geq S(x) > \bar{x}$$

which is a contradiction. Hence

$$S(x) = f_N(x) \text{ and } f_{N+1}(x) > f_N(x).$$

Choose

$$0 < \epsilon < \min \left\{ f_N(x) - \bar{x}, \frac{f_{N+1}(x) - f_N(x)}{2} \right\}.$$

From the continuity of f_n , there exists $\delta > 0$ such that

$$\forall x' \in [0, \infty) \quad (|x - x'| < \delta \Rightarrow |f_n(x) - f_n(x')| < \epsilon),$$

where $n = 0, 1, \dots, N + 1$. Hence for $x' \in (x - \delta, x + \delta) \cap [0, \infty)$ we have

$$f_{N+1}(x') > f_{N+1}(x) - \epsilon > f_N(x) + \epsilon > f_N(x') > f_N(x) - \epsilon > \bar{x}$$

Therefore

$$S(x) + \epsilon = f_N(x) + \epsilon > f_N(x') \geq \inf_{n \geq 0} f_n(x') = S(x').$$

Also

$$S(x) \leq f_n(x) < f_n(x') + \epsilon \quad 0 \leq n \leq N.$$

Hence

$$S(x') + \epsilon > S(x)$$

and

$$S(x) < S(x') + f_N(x) - \bar{x}.$$

This implies that $S(x') > \bar{x}$ and

$$|S(x) - S(x')| < \epsilon,$$

i.e., S is continuous and A is open.

Let $\lambda = \inf A$. Then $\lambda \notin A$. The continuity of f_n for every n implies that $S(\lambda) = \bar{x}$. Now, we show that $\{f_n(\lambda)\}_{n \geq 0}$ is decreasing to \bar{x} . We can see that $f_1(\lambda) < f_0(\lambda)$. Assume for the sake of contradiction that $f_0(\lambda) \leq f_1(\lambda)$. Then $\bar{x} \leq f_0(\lambda) \leq f_1(\lambda) \leq \dots$ whence $S(\lambda) = f_0(\lambda) = \lambda^2 = \bar{x}$. Hence

$$f_1(\lambda) = \frac{\beta \lambda^{2p}}{1 + \gamma \lambda^{p-1}} = \frac{\beta \bar{x}^p}{1 + \gamma \bar{x}^{\frac{p-1}{2}}} = \bar{x} \frac{1 + \gamma \bar{x}^{p-1}}{1 + \gamma \bar{x}^{\frac{p-1}{2}}} < \bar{x},$$

which is a contradiction. By induction we can show that

$$f_0(\lambda) > f_1(\lambda) > \dots > \bar{x}.$$

REFERENCES

- [1] E. CAMOUZIS, E. A. GROVE and G. LADAS, Monotone unstable solutions of difference equations and conditions for boundedness, *J. Difference Equations and Appl.* **1** (1995), 17–44.
- [2] E. CAMOUZIS, G. LADAS, I. W. RODRIGUES and S. NORTHSHIELD, The rational recursive sequence $x_{n+1} = (\beta x_n^2)/(1 + x_{n-1}^2)$, *Computers Math. Applic.* **28** (1994), 37–43.
- [3] J. H. JAROMA, V. L. KOCIC and G. LADAS, Global asymptotic stability of a second-order difference equation, in J. Wiener and J. Hale, Eds, *Partial Differential Equations, Pitman Research Notes in Mathematics*, No. 273, Longman Scientific and Technical, Essex (1992) 80–84.
- [4] V. L. KOCIC and G. LADAS, Permanence and global attractivity in nonlinear difference equations, *Proceedings of the First World Congress of Nonlinear Analysis, Tampa, Florida, 1992*.
- [5] V. L. KOCIC and G. LADAS, Global behavior of nonlinear difference equations of higher order with applications, *Kluwer Academic Publishers, Dordrecht*, (1993).
- [6] V. L. KOCIC, G. LADAS and I. W. RODRIGUES, On rational recursive sequences, *J. Math. Anal. Appl.* **173** (1993), 127–157.

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