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ADAMS AND STEENROD OPERATORS IN DIHEDRAL HOMOLOGY

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ABSTRACT. In this article, we define the Adam's and Steenrod's operators in the dihedral homology

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INTRODUCTION

The dihedral (co)homology of unital algebra with an involution, symmetry, bisymmetry and Weile has been studied by Looder [1], Krasauskas, Lapin and Solovev [2], Kolosov [3] and others 1987-89 In the present work we are concerned with Adam's and Steenrod's operators in the dihedral homology.

1. THE ADAM'S OPERATOR IN THE DIHEDRAL HOMOLOGY

We recall the Adam's operator ψ^k in the cyclic homology from [4] and [5]. Let A be a commutative, associative, and unital K-algebra with an involution $*$ ($* : A \longrightarrow A$ is an automorphism of degree zero, $*^2 = id$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*, a, b \in A$), and K is a field with characteristic zero. Let $\lambda^k = \wedge^k(1_n - n)$ be the k^{th} exterior dimension representation of the Lie algebra $\mathfrak{gl}_n(k)$ and n is the direct sum of the one dimensional representation (n-argument). Following [6], the ring $R(\mathfrak{gl}_n(k))$ is isometric to the ring of polynomial $K[\lambda^1, \dots, \lambda^n]$. Let $R(\mathfrak{gl}(k)) = \varprojlim R(\mathfrak{gl}_n(k))$. Consider, for an arbitrary representation ρ of an algebra $\mathfrak{gl}_n(k)$, the following sequence:

$$\begin{aligned} &CC_\infty(A) \xrightarrow{S} \wedge^n(\mathfrak{gl}(k))_{\mathfrak{gl}(k)} \xrightarrow{\widehat{\rho}} \wedge^n(\mathfrak{gl}(k))_{\mathfrak{gl}(k)} \xrightarrow{\varphi} \\ &\xrightarrow{\varphi} CC_n(M_\infty(A)) \xrightarrow{Tr} CC_\infty(A), \end{aligned} \tag{1 1}$$

where $\wedge^\bullet(\mathfrak{gl}(k))_{\mathfrak{gl}(k)}$ is the coinvariant complex of Cherilley-Eilenberg Complex $\wedge(\mathfrak{gl}(k))$ (see [4]), $M_\infty(A) = \varinjlim M_n(A)$, $M_n(A)$ is the $(n \times n)$ matrix with coefficients in A. The composition maps in (1.1) are denoted by $\propto_n$ where $\propto = \varprojlim \propto_n$. The morphism S is given by:

$$S(a_1 \otimes a_2 \otimes \dots \otimes a_n) = E_{12}a_1 \wedge E_{23}a_2 \wedge \dots \wedge E_{n-1,n}a_{n-1} \wedge E_{n,1} \cdot a_n,$$

where E_{ij} is the matrix, whose only non zero elements are the identity element $1 \in k$. The map $\widehat{\rho}$ is given by:

$$\widehat{\rho}(X_1 a_1 \wedge \dots \wedge X_n a_n) = \rho(x_1)a_1 \wedge \dots \wedge \rho(x_n)a_n, \ x_i \in \mathfrak{gl}_n(k),$$
$$\varphi(Z_0 \wedge \dots \wedge Z_n) = \sum_\sigma \operatorname{sgn}(\sigma)(-1)^n Z_0 \otimes Z_{\sigma(1)} \otimes \dots \otimes Z_{\sigma(n)}, \ Z_i \in \mathfrak{gl}_N(k),$$

$\rho : \mathfrak{gl}_n(k) \longrightarrow \mathfrak{gl}_N(k)$, and Tr is the trace map defined by:

$$Tr(X_1 a_1 \otimes \dots \otimes X_n a_n) = tr(X_1 \dots X_n) a_1 \otimes \dots \otimes a_n.$$

We can easily check ([4]) that, $\propto(\rho + \tau) = \propto(\otimes)$, where ρ and τ are representations of $\mathfrak{gl}(k)$

From the above discussion we have the homomorphism $\alpha : R(\mathfrak{gl}(k)) \longrightarrow End(CC.(A))$. Clearly, for any $f \in K[\lambda^1, ..., \lambda^n, ...]$, the homomorphism $\alpha(f)$ coincides with the homomorphism α [5]. Suppose that Q_k, $k \geq 1$ is the Newton Polynomial, which is given by the symmetric function $\sum_{\imath=1}^{k} (u_\imath)^k$, such that $\sigma_r = \sum_{\imath_1 < \imath_2 < ... < \imath_r}^{k} u_{\imath_1} ... u_{i_r}$, $1 \leq r \leq k$. By acting with the morphism α on the Newton Polynomial, we get the Adams operators $\psi^k = \alpha(Q_k) = \alpha\big((- 1)^k . k\, \lambda^k\big)$, since $(- 1)^k . k\, \lambda^k$ is the linear part of K-Newton Polynomial. Consider the chain complex $(C\mathcal{H}.(A), b.')$ and the Connes-Tsygan bicomplex (see [1])

$$\begin{tikzcd}
\vdots \arrow[d, "b."'] & \vdots \arrow[d, "b'."'] & \vdots \arrow[d, "b."'] & \vdots \arrow[d, "b'."'] & \vdots \arrow[d, "b."'] & \\
C\mathcal{H}_2(A) \arrow[d, "b."'] & C\mathcal{H}_2(A) \arrow[l, "1-t."'] \arrow[d, "b'."'] & C\mathcal{H}_2(A) \arrow[l, "N"'] \arrow[d, "b."'] & C\mathcal{H}_2(A) \arrow[l, "1-t."'] \arrow[d, "b'."'] & C\mathcal{H}_2(A) \arrow[l, "N"'] \arrow[d, "b."'] & \cdots \arrow[l] \\
C\mathcal{H}_1(A) \arrow[d, "b."'] & C\mathcal{H}_1(A) \arrow[l, "1-t."'] \arrow[d, "b'."'] & C\mathcal{H}_1(A) \arrow[l, "N"'] \arrow[d, "b."'] & C\mathcal{H}_1(A) \arrow[l, "1-t."'] \arrow[d, "b'."'] & C\mathcal{H}_1(A) \arrow[l, "N"'] \arrow[d, "b."'] & \cdots \arrow[l] \\
C\mathcal{H}_0(A) & C\mathcal{H}_0(A) \arrow[l, "1-t."'] & C\mathcal{H}_0(A) \arrow[l, "N"'] & C\mathcal{H}_0(A) \arrow[l, "1-t."'] & C\mathcal{H}_0(A) \arrow[l, "N"'] & \cdots, \arrow[l]
\end{tikzcd} \tag{1.1}$$

then, the subcomplex $(ker(1 - t.), b.') \subset (C\mathcal{H}.(A), b.')$ has the same homology as the complex $(CC.(A), b.)$, that is,

$$\begin{aligned}
\mathcal{H}.(CC.(A)) &= \mathcal{H}.((C\mathcal{H}.(A), b.)/Im(1 - t.)) = \mathcal{H}.((C\mathcal{H}.(A), b.)/Ker\, N.) \\
&= \mathcal{H}.(Im\, N, b.') = \mathcal{H}.(Ker(1 - t.), b.'),
\end{aligned}$$

where $C\mathcal{H}_n(A) = A^{\otimes n+1} = A \otimes ... \otimes A (n + 1 \text{ times})$, b_n, $b'_n : C\mathcal{H}_n(A) \longrightarrow C\mathcal{H}_{n-1}(A)$, such that $b'_n(a_0 \otimes ... \otimes a_n) = \sum_{\imath=0}^{n-1} (- 1)^\imath (a_0 \otimes ... \otimes a_i a_{\imath+1} \otimes ... \otimes a_n), b_n(a_0 \otimes ... \otimes a_n) = b'_n + (- 1)^n (a_n a \otimes ... \otimes a_{n-1})$, $t_n : C\mathcal{H}_n(A) \longrightarrow C\mathcal{H}_n(A)$, such that $t_n(a_0 \otimes ... \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes ... \otimes a_{n-1})$ and $N_n = 1 + t_n^1 + ... + t_n^n$. Therefore, the complex $(Ker(1 - t.), b.')$ is isomorphic to the complex $(CC.(A), b.)$. The isomorphism between them is given by the operator $N. : CC.(A) \longrightarrow (ker(1 - t.), b.')$. Consequently, the action of the group $\mathbb{Z}/2$ on the complex $CC.(A)$, by means of the operator ${}^\epsilon r$, is equal to the action of $\mathbb{Z}/2$ on the complex $\big(\mathrm{Ker}(1 - t.), b'.\big)$, by means of the operator

$${}^\epsilon h : a_0 \otimes a_1 \otimes ... \otimes a_n \longrightarrow (- 1)^{\frac{n(n+1)}{2}} \epsilon\, a_n^* \otimes a_{n-1}^* \otimes ... \otimes_{,} a_0^*,$$

where a^* is the image of element $a \in A$ under involution $* : A \longrightarrow A$, $\epsilon = \pm 1$. Since ${}^\epsilon h.t. = t.^{-1}\, {}^\epsilon h.$. Hence, $N.({}^\epsilon h.) = ({}^\epsilon h.)N.$. On the other hand, since ${}^\epsilon r. = t.{}^\epsilon h.$, then ${}^\epsilon h.N. = N.\, {}^\epsilon h. = (N.t.)\, {}^\epsilon h. = N.(t.\, {}^\epsilon h.) = N.\, {}^\epsilon r.$. So, the dihedral homology of A is given by the formula

$${}^\epsilon\mathcal{HD}.(A) = \mathcal{H}.(ker(1 - t.)/(Im(1 - {}^\epsilon h.) \cap ker(1 - t.))).$$

Assume that the complex $CC.(A)$ is a subcomplex of $(C\mathcal{H}.(A), b.')$, then the direct calculation of homomorphism $\alpha\big((- 1)^k k\, \lambda^k\big)$ gives the Adam's operator Ψ^k in additive algebraic K-theory (see [4]), that is, $\Psi(a_0 \otimes ... \otimes a_n) = \sum_I sgn(\sigma_I) a_{\sigma_I(0)} \otimes ... \otimes a_{\sigma_I(n)}$, where I is the division of the set $\{0, 1, 2, ..., n\}$ into non-empty intersected subsets, that is, $I = I_0 \cup ... \cup I_{k-1}$, and $\sigma_I \in \sum_{n+1}$ is the permutation of the set $\{0, 1, ..., n\}$, such that:

(i) If $i_1 \in I_{p_1}$, $i_2 \in I_{p_2}$, $P_1 < P_2$, then $\sigma_I(i_1) > \sigma_I(i_2)$,

(ii) For any P, $I_P = \{i_0, ..., i_q\}$, $(i_1 < i_2 < ... < i_q)$.

The permutation σ_I satisfies the following condition:

$$\sigma_I(i_q) = \sigma_I(i_{q-1}) + 1 = ... = \sigma_I(i_0) + q.$$

LEMMA 1.1. The following diagram is commutative:

$$\begin{tikzcd} CC.(A) \arrow[r, "\psi^k"] \arrow[d, "\epsilon_r"'] & CC.(A) \arrow[d, "\epsilon_r"] \\ CC.(A) \arrow[r, "\psi^k"] & CC.(A) \end{tikzcd} \tag{1.2}$$

PROOF. Assume that the complex $CC.(A)$ is a subcomplex of the complex $(CH.(A), b.')$ and the element $a_0 \otimes ... \otimes a_n \in ker(1 - t_n)$, then

$$\begin{aligned} \epsilon_h \psi^k(a_0 \otimes a_1 \otimes ... \otimes a_n) &= {}^{\epsilon}h \sum_I sgn(\sigma_I) a_{\sigma_I(0)} \otimes ... \otimes a_{\sigma_I(n)} \\ &= (-1)^{\frac{n(n+1)}{2}} \epsilon \sum_I sgn(\sigma_I) a^*_{\sigma_I(n)} \otimes ... \otimes a^*_{\sigma_I(0)}. \end{aligned} \tag{1.2}$$

On the other hand

$$\begin{aligned} \psi^k({}^{\epsilon}h)(a_0 \otimes ... \otimes a_n) &= (-1)^{\frac{n(n+1)}{2}} \epsilon \psi^k(a^*_n \otimes ... \otimes a^*_0) \\ &= (-1)^{\frac{n(n+1)}{2}} \epsilon \sum_J sgn(g_J) a^*_{g_J(n)} \otimes ... \otimes a^*_{g_J(0)}, \end{aligned} \tag{1.3}$$

where g_j is the permutation of the ordered set $\{n, n-1, ..., 0\}$ satisfies the conditions (i), (ii) and J is the division of the ordered set $\{n, n-1, ..., 0\}$. Note that, in general, the permutation g_J of the ordered set $\{0, 1, ..., n\}$, satisfies the following conditions:

i)' If $i_1 \in J_{p_1}$, $i_2 \in J_{p_2}$, $p_1 < p_2$, then $g_J(i_1) > g_J(i_2)$,

ii)" For any p, $g_J = \{i_1, ..., i_0\}$, $i_q > ... > i_0$, we have

$$g_J(i_0) = g_J(i_1) - 1 = ... = g_J(i_q) - q.$$

Note that the decreasing (by one) of the elements in the set $\{0, 1, ..., n\}$ met the increasing of elements (also by one) in the set $\{n, n-1, ..., 0\}$. Suppose that the arguments of the summation in (1 2) correspond to the permutation σ_I. The permutation g_J of the set $\{n, n-1, ..., 0\}$, where $g_J(i) = \sigma_I(i)$ will correspond to the division $J = I^*_{k-1} \cup ... \cup I^*_0$, where

$$I^*_i = \{P^i_{q_i}, ..., P^i_0\} \, (I = \{P^i_0, ..., P^i_{q_i}\}, P^i_0 < ... < P^i_{q_i}).$$

We can easily check, for any P and $I^*_p = \{i^p_{q_p}, ..., i^p_0\}$, $i^p_{q_p} < ... < i^p_0$, that $g_J(i^p_0) = g_J(i^p_1) - 1 = ... = g_J(i^p_{q_p}) - q_p$. If $i_1 \in I^*_{p_1}$, $i_2 \in I^*_{p_2}$, $p_1 < p_2$, then $g_J(i_1) > g_J(i_2)$. From the definition of σ_I and g_J we have ${}^{\epsilon}h\, \psi^k = \psi^k({}^{\epsilon}h)$ in $ker((1-t), b')$ and, hence ${}^{\epsilon}r\, \psi^k = \psi^k({}^{\epsilon}r)$ in $(CC.(A), b.)$. Clearly the inverse of the isomorphism $(CC.(A)) \longrightarrow ker(1 - t.)$ is $\frac{1}{n} id : (ker(1 - t.), b.') \longrightarrow (CC.(A)b)$. The operator ψ^k in $CC.(A)$ is given by $\frac{1}{n} \psi^k N$, where ψ^k is an operator in $(ker(1-t), b.')$. Since the operator ψ^k, on $CC.(A)$ commutes with the operator ${}^{\epsilon}r$, then we have the Adam's operator ${}^{\epsilon}\psi^k$ in the dihedral homology. Following [6] the multiplication in the cyclic homology of the algebra A is given as follows

$$\cup : \mathcal{H}C_p(A) \otimes \mathcal{H}C_q(A) \longrightarrow \mathcal{H}C_{p+q+1}(A),$$

such that

$$\cup : TotB(A) \otimes TotB(A) \longrightarrow TotB(A),$$

$$xuy = \left[\begin{array}{ll} \rightarrow (x)T(\beta y), & r = 0 \\ \rightarrow 0 & , r \neq 0 \end{array} \right] \in B(A)_{\ell+r, m+s+1}, x \in B(A)_{\ell, m} = A \otimes \overline{A}^{\otimes(m-\ell)},$$

$y \in B(A)_{r,s} = A \otimes \overline{A}^{\otimes(s-r)}$, where T is a product map [7], $Tot\,B(A)$ is the total complex of the bicomplex $\mathcal{B}(A)$, β is the Connes's operator. The group $\mathbb{Z}/2$ acts on the column of the bicomplex $\mathcal{B}(A)$ with the numbers $2\ell \, (n > 0)$ by means of the operator ${}^{\epsilon}r$, on the column with the numbers $(2\ell + 1)$ by means of the operator $(-1)^{\epsilon}r$, and on the complex $Tot^{\epsilon}B(A) \otimes Tot^{\delta}B(A)$ by means of ${}^{\epsilon}\hat{r} \otimes {}^{\delta}\hat{r}$, where ${}^{\epsilon}\hat{r}$ is the action of $\mathbb{Z}/2$ on $Tot^{\epsilon}B(A)$ induced by the action $\mathbb{Z}/2$ on ${}^{\epsilon}\mathcal{B}(A)$. Since the action of the group $\mathbb{Z}/2$ on the complex $Tot^{\epsilon}B(A) \otimes Tot^{\delta}B(A)$ commutes with the multiplication in the cyclic homology, then

$${}^{\epsilon}\hat{r} \otimes {}^{\delta}\hat{r}(a \otimes b) = {}^{E}\hat{r}(a) \otimes {}^{\delta}\hat{r}(b) \xrightarrow{\cup} {}^{\epsilon}\hat{r}(a)T\beta\big({}^{\delta}\hat{r}(b)\big),$$

$a \in Tot^{\epsilon}B(A)$, $b \in Tot^{\delta}B(A)$. On the other hand

$$\big(-\big({}^{\epsilon}\hat{r}(a)T\beta\big({}^{\delta}\hat{r}(b)\big)\big) = {}^{\epsilon}\hat{r}(a)T\beta\big(-{}^{\delta}\hat{r}(b)\big) = -{}^{\epsilon}\hat{r}(a)T\big({}^{\delta}\hat{r}(\beta(b)\big) = {}^{(\epsilon\delta)}\hat{r}(a \cup b).$$

Therefore ${}^{\epsilon}r(a) \cup {}^{\delta}r(b) = {}^{-(\epsilon\delta)}r(a \cup b)$. From the above we have the multiplication in the dihedral homology

$$\cup : {}^{\epsilon}\mathcal{HD}_p(A) \otimes {}^{\delta}\mathcal{HD}_q(A) \longrightarrow {}^{-(\epsilon\delta)}\mathcal{HD}_{p+q+1}(A).$$

It is well known that (see [1], [2]), the dihedral homology can be considered as the hyperhomology of the group $\mathbb{Z}/2$ with the coefficient in $Tot^{\epsilon}B(A)$, then

$$\begin{aligned}\mathbb{H}.(\mathbb{Z}/2, Tot^{\epsilon}B(A)) \otimes \mathbb{H}.\big(\mathbb{Z}/2, Tot^{\delta}B(A)\big) &\longrightarrow \mathbb{H}.\big(\mathbb{Z}/2, Tot^{\epsilon}B(A) \otimes Tot^{\delta}B(A)\big)\\ &\longrightarrow \mathbb{H}.\big(\mathbb{Z}/2, Tot^{-(\epsilon\delta)}B(A)\big).\end{aligned}$$

Consider the Adam's operator properties in the cyclic homology [4]. Since the Adam's operator ψ^k commutes with the action of the group $\mathbb{Z}/2$ and the multiplication $\cup$ in the cyclic homology anti-commutes with the action of group $\mathbb{Z}/2$, we get the following theorem.

THEOREM 1.2. Assume that A is a commutative K-algebra, where K is a field of characteristic zero. The Adam's operator ψ^k has the following properties:

1) ${}^{\epsilon}\psi^k \circ {}^{\epsilon}\psi^k = {}^{\epsilon}\psi^{k\ell}$,
2) ${}^{\epsilon}\psi^k(\alpha) \cup {}^{\delta}\psi^k(\beta) = {}^{-(\epsilon\delta)}\psi^k(\alpha \cup \beta)$, where $\alpha \in \mathcal{HD}.(A)$, $b \in \mathcal{HD}.(A)$.

2. THE STEENROD'S OPERATOR IN THE DIHEDRAL HOMOLOGY

In this part we define the Steenrod's operator in the dihedral homology. Let A be a commutative K-Hopf algebra, where K is a field with characteristic (not essential) zero. Let Ξ be the dihedral category and $K[\Xi]$ be an algebra associated with $[\Xi]$ over K (see [1], [2]). We can define on the $K[\Xi]$-module ${}^{\epsilon}A^D$, the structure of a co-commutative $K[\Xi]$-co-algebra by the formula

$${}^{\epsilon}A^D \xrightarrow{\nabla} {}^{\epsilon}(A \otimes A) \xrightarrow{f} {}^{\epsilon}A^D \otimes {}^{\epsilon}A^D,$$

where ∇ is the $K[\Xi]$ homomorphism, and f is given by

$$f((a_0 \otimes b_0) \otimes (a_1 \otimes b_1) \otimes \ldots \otimes (a_n \otimes b_n)) = (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \otimes (b_0 \otimes b_1 \otimes \ldots \otimes b_n).$$

Suppose that $f \circ \nabla = {}^{\epsilon}\nabla^D$ gives the co-commutative co-multiplication in ${}^{\epsilon}A^D$. We show that ${}^{\epsilon}\nabla^D$ is a $K[\Xi]$-homomorphism Define on the algebra $K[\Xi]$ the co-multiplication

$$K[\Xi] \longrightarrow K[\Xi] \underset{k}{\otimes} K[\Xi]; \quad \text{such that} \quad x \longrightarrow x \otimes x, \quad x \in K[\Xi].$$

Since ${}^{\epsilon}A^D \underset{k}{\otimes} {}^{\epsilon}A^D$ is $K[\Xi] \underset{k}{\otimes} K[\Xi]$ module, then by using the multiplication on ${}^{\epsilon}A^D \underset{k}{\otimes} {}^{\epsilon}A^D$, one can define the $K[\Xi]$-module structure and the $K[\Xi]$-module homomorphism f, since

$$\begin{aligned}f(x((a_0 \otimes b_0) \otimes (a_1 \otimes b_1) \otimes \ldots \otimes (a_n \otimes b_n))) &= x(a_0 \otimes a_1 \otimes \ldots \otimes a_n) \otimes x(b_0 \otimes b_1 \otimes \ldots \otimes b_n)\\ &= x((a_0 \otimes a_1 \otimes \ldots \otimes a_n) \otimes (b_0 \otimes b_1 \otimes \ldots \otimes b_n))\\ &= xf((a_0 \otimes b_0) \otimes (a_1 \otimes b_1) \otimes \ldots \otimes (a_n \otimes b_n)),\end{aligned}$$

$x \in K[\Xi]$. Hence the morphism ${}^{\epsilon}\nabla^D$ is a $K[\Xi]$-module homorphism.

The dihedral cohomology $Ext^n_{K[\Xi]}\big({}^{\epsilon}A^D; (K^D)^*\big)$ can be calculated by using the normalized bar construction $\beta(\mathfrak{L})$ (see [6]). Assume that $\mathfrak{L}$ and $\mathfrak{F}$ be the triples $({}^{\epsilon}A^D, K[\Xi], K^D)$, $(K[\Xi], K[\Xi], K^D)$, and $JK[\Xi]$ be the cokernel identity: $k \longrightarrow K[\Xi]$. The normalized bar construction $\beta(\mathfrak{L})$ is defined to be a k module $\beta(\mathfrak{L}) = {}^{\epsilon}A^D \otimes_{K[\Xi]} T(JK[\Theta]) \otimes_{K[\Xi]} K^D$, where $T(JK[\Xi])$ is the tensor algebra of $JK[\Xi]$. Clearly the K module $\beta(\mathfrak{L})$ is graded. The elements of the K-module $\beta(\mathfrak{L})$ can be written as follows: $a[g_1, g_2, \ldots, g_s]k \in \beta(\mathfrak{L})_s$, $a \in {}^{\epsilon}A$, $g_i \in k[\Xi]$ and $k \in K^D$. The differential $d : \beta(\mathfrak{L})_s \longrightarrow \beta(\mathfrak{L})_{s-1}$ and the argument $f : \beta(\mathfrak{L}) \longrightarrow {}^{\epsilon}A^D \otimes_{K[\Xi]} K^D$ can be written as follows

$$d[a[g_1|g_2\cdots|g_s]k] = ag_1[g_2|g_3|\cdots|g_s]k + \sum_{i=1}^{s-1}(-1)^i a[g_1|\cdots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\cdots|g_s]k$$
$$+ (-1)^s a[g_1|\cdots|g_{s-1}g_s]k, \quad \text{and} \quad f[g_1|\cdots|g_s]k = 0, \quad f(a[\ ]k) = 0.$$

We can define also, for $\mathfrak{F}$,the maps d and f in the same manner. Note that for $\mathfrak{L}$, the differential d is a left $K[\Xi]$-module homomorphism, and $dS + Sd = 1 - \sigma f$, where the homomorphism σ

$$\sigma : K^D \longrightarrow \beta(\mathfrak{F}), \quad \text{and} \quad S : \beta(\mathfrak{F})_s \longrightarrow \beta(\mathfrak{F})_{s+1}$$

is given by the formulas

$$\sigma(k) = [\ ]k \otimes [\ ],\ S(g[g_1|\cdots|g_s]k) = [g|g_1|\cdots|g_s]k.$$

Clearly, that the differential d in the complex $\beta(\mathfrak{l}) = {}^{\epsilon}A^D \otimes_{K[\Xi]} \beta(\mathfrak{F})$ is equal to $1 \otimes_{K[\Xi]} d$. From [6], we have the following

$$Hom_{K[\Xi]}\left(\beta(\mathfrak{F}); ({}^{\epsilon}A^D)^*\right) = (\beta(\mathfrak{F}))^* = Hom_{K[\Xi]}\left(\beta({}^{\epsilon}A^D), K[\Xi], K[\Xi], (k^D)^*\right).$$

Then

$${}_{\epsilon}\mathcal{HD}^n(A) = Ext^n_{K[\Xi]}\left({}^{\epsilon}A^D; (K^D)^*\right) = \mathcal{H}^n(\beta(\mathfrak{L})^*).$$

Suppose the triples $\mathfrak{L}(({}^{\epsilon}A^D), k[\Xi], K^D)$ and $\hat{\mathfrak{F}} = \left(\left({}^{\epsilon}\widehat{A}^D\right), \hat{k}[\Xi], \hat{K}^D\right)$ and consider the product $\perp : (\beta(\mathfrak{L} \otimes \hat{\mathfrak{F}}) \longrightarrow \beta(\mathfrak{L}) \otimes \beta(\hat{\mathfrak{F}})$. Define on $\beta(\mathfrak{L})$ the structure of co-associative co-algebra by means of co-multiplication $\tilde{\nabla} = \perp \beta({}^{\epsilon}\nabla^D, \nabla_{k[\Xi]}, \nabla_k D) : \beta(\mathfrak{L}) \longrightarrow \beta(\mathfrak{L}) \otimes \beta(\mathfrak{L})$and on the complex $\beta(\mathfrak{L})^*$ the following multiplication

$$\beta(\mathfrak{L})^* \otimes \beta(\mathfrak{L})^* \longrightarrow (\beta(\mathfrak{L}) \otimes \beta(\mathfrak{L}))^* \xrightarrow{(\tilde{\nabla})^*} \beta(\mathfrak{L})^*.$$

The following lemma can easily be proved by using the ordinary techniques of homological algebra (see [8]).

LEMMA 2.1. Let μ be an arbitrary subgroup of the symmetry group Σ_r, W is the $K[\mu]$-free resolution $K[\mu]$-module K that $W_0 = K[\mu]$ with the $K[n]$ generator e_0 and the module $W \otimes \beta(\mathfrak{L})$ is a graded module, since: $[W \otimes \beta(\mathfrak{L})]_s = \sum_{i+j=s} W_i \otimes \beta_j(\mathfrak{L})$, then there exist graded $K[n]$ complexes, with the following conditions of the homorphism $\triangle : W \otimes \beta(\mathfrak{L}) \longrightarrow \beta(\mathfrak{L})^{\otimes r}$:

1) $\triangle(W \otimes b) = 0$, $b \in \beta(\mathfrak{L})_0$ and $w \in W_i$, $i > 0$.

2) $\triangle(e_0 \otimes b) = \tilde{\nabla}^{\otimes r}(b)$, if $b \in \beta(\mathfrak{L})$, $\tilde{\nabla}^{\otimes r} : \beta(\mathfrak{L}) \longrightarrow \beta(\mathfrak{L})^{\otimes r}$.

3) For $\beta(\mathfrak{L})$ the map $\triangle$ is a left $K[\Xi]$-module homorphism, where $K[\Xi]$ acts on $W \otimes \beta(\mathfrak{L})$ by the relation $K(w \otimes b) = w \otimes kb$.

4) $\triangle(w_i \otimes \beta(\mathfrak{L})_s) = 0$, when $i > (r-1)_s$. Furthermore, there exists a $k[\mu]$-homotopy between any two homomorphisms $\triangle$ with the same properties. Now, define the $K[\mu]$-homomorphism Θ as follows: Θ:W $\otimes (\beta(\mathfrak{L})^*)^{\otimes r} \longrightarrow \beta(\mathfrak{L})^*$, since $\Theta(w \otimes x)(m) = \mathcal{B}(x)\triangle(w \otimes m)$, $w \in W$, $x \in (\beta(\mathfrak{L})^*)^{\otimes r}$, and $m \in \beta(\mathfrak{L})$, $\mathcal{B} : (\beta(\mathfrak{L})^*)^{\otimes r} \longrightarrow (\beta(\mathfrak{L})^{\otimes r})^*$ is a trivial homomorphism. Now we shall define the operator in $H(\beta(\mathfrak{L})^*)$. In the above lemma, let $\mu = Z/p$, $K = Z/p$. Consider the standard $K[Z/p]$-free resolution W. In this case W_i, $i \geq 0$, is a free $K[Z/p]$-module with the generator e_i. By considering the graded $W_i = W^{-i}$, which is a free $K[Z/p]$-module with the generator e^{-i}, let $x \in H^q(\beta(\mathfrak{L})^*)$, and define the following homomorphism: $R_i : H^q(\beta(\mathfrak{L})^*) \longrightarrow H^{pq-i}(\beta(\mathfrak{L})^*)$, since $R_i(x) = \Theta^*(e^{-i} \otimes x^p)$, $i \geq 0$ Now we can define the Steenrod operator P^i, by using the operator R_i, as follows:

1. If $p = 2$ then, $p^s(x) = R_{q-s}(x) \in H^{q+s}(\beta(\mathfrak{L})^*)$, where $R_i = 0$ if $i < 0$;

2. If $P > 2$, then

$$p^s(x) = (-1)^s \gamma(-q) R_{(q-2s)(p-1)}(x) \in H^{q+2s(p-1)}(\beta(\mathfrak{L})^*),$$
$$\mathcal{B}P^s(x) = (-1)^s \gamma(-q) R_{(q-2s)(p-1)-1}(x) \in H^{q+2s(p-1)+1}(\beta(\mathfrak{L})^*),$$

where $R_i = 0$ if $i < 0$, and if $q = 2j - \ell$, where $\ell = 0$ or 1, then $\gamma(-q) = (-1)^j (mI)^{\mathcal{L}}$, $m = \frac{p-1}{2}$.

Now we prove the main second theorem in this work.

THEOREM 2.2. Let A be a commutative K-Hopf algebra, where $K = Z/p$, then on the dihedral cohomology group ${}_{\epsilon}\mathcal{HD}^{\cdot}(A)$, we can define the following homomorphisms (Steenrod map):

a) $P^i : {}_{\epsilon}\mathcal{HD}^s(A) \longrightarrow {}_{\epsilon}\mathcal{HD}^{s+i}(A)$, if $p = 2$,

b) $P^i : {}_{\epsilon}\mathcal{HD}^s(A) \longrightarrow {}_{\epsilon}\mathcal{HD}^{s+2i(p-1)}(A)$, and $\mathcal{B}P^i : {}_{\epsilon}\mathcal{HD}^s(A) \longrightarrow {}_{\epsilon}\mathcal{HD}^{s+i+2i(p-1)}(A)$, if $p > 2$.

The operators P^i, βP^i have the following properties:

1) $P^i|_{{}_{\epsilon}\mathcal{HD}^s(A)} = 0$, if $p = 2$, $i > s$,
$P^i|_{{}_{\epsilon}\mathcal{HD}^s(A)} = 0$, if $p > 2$, $2i > s$,
$\mathcal{B}P^i|_{{}_{\epsilon}\mathcal{HD}^s(A)} = 0$, if $p > 2$, $2i \geq s$

2) $P^i(x) = x^p$, if $p = 2$ and $i = s$, or $p > 2$ and $2i = s$

3) $P_j = \Sigma P^i \otimes P^{j-i}$ and $\mathcal{B}P^j = \Sigma \mathcal{B}P^i \otimes P^{j-i} + P^i \otimes \mathcal{B}P^{j-i}$

4) The operators P^i and $\mathcal{B}P^i$ satisfy the following Adam's relations:

i) if $p \geq 2$ and $a < pb$, then

$$\mathcal{B}^{\gamma} P^a P^b \sum_i (-1)^{a+i} (a - pi, (p-1)b - a + i - 1) . \mathcal{B}^{\gamma} P^{a+b-i} P^i,$$

where $\gamma = 0$ or 1 for $p = 2$, $\gamma = 1$ for $p > 2$, and for any two integers i and j let

$$(i, j) = \left[\begin{array}{ll} \dfrac{(i+j)!}{i!j!}, & \text{if} \quad i \geq 0, j \geq 0, \\ & \\ 0 & \text{if} \quad i < 0, j < 0, \end{array} \right.$$

ii) if $p > 2$, $a \leq Pb$, and $\gamma = 0$ or 1, then

$$\mathcal{B}^{\gamma} P^a P^b = (1 - \gamma) \sum_i (-1)^{a+i} (a - pi, (p-1)(b - a + i - 1) . \mathcal{B}P^{a+b-i} p^i - \sum_i (-1)^{a+i}$$
$$.(a - pi - 1, (p-1)b - a + i) \mathcal{B}^{\gamma} P^{a+b-i} \mathcal{B}P^i.$$

Note that the operators $\mathcal{B}^0 P^s$ and $\mathcal{B}^1 P^s$ are P^s and $\mathcal{B}P^s$, respectively.

PROOF. Suppose the triple $C = (E, \mathcal{A}, F)$ where $\mathcal{A}$ is a co-commutative Hopf algebra over $K = Z/p$, E and F are respectively the right and left co-commutative $\mathcal{A}$-co-algebra. From the above discussion and considering the triple $\mathfrak{L} = ({}^{\epsilon}A^D, K[\Xi], k^D)$, then $K[\Xi]$ is a co-commutative Hopf algebra over $K = Z/P$, ${}^{\epsilon}A^D$ and K^D are the left and right co-commutative $K[\Xi]$-co-algebra and hence $\mathcal{H}^{\cdot}(\mathcal{B}(\mathfrak{L})^*) = {}_{\epsilon}\mathcal{HD}^{\cdot}(A)$.

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