

α -DERIVATIONS AND THEIR NORM IN PROJECTIVE TENSOR PRODUCTS OF Γ -BANACH ALGEBRAS

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ABSTRACT. Let (V, Γ) and (V', Γ') be Gamma-Banach algebras over the fields F_1 and F_2 isomorphic to a field F which possesses a real valued valuation, and $(V, \Gamma) \otimes_p (V', \Gamma')$, their projective tensor product. It is shown that if D_1 and D_2 are α -derivation and α' -derivation on (V, Γ) and (V', Γ') respectively and $u = \sum_i x_i \otimes y_i$ is an arbitrary element of $(V, \Gamma) \otimes_p (V', \Gamma')$, then there exists an $\alpha \otimes \alpha'$ -derivation D on $(V, \Gamma) \otimes_p (V', \Gamma')$ satisfying the relation

$$D(u) = \sum_i \left[(D_1 x_i) \otimes y_i + x_i \otimes (D_2 y_i) \right]$$

and possessing many enlightening properties. The converse is also true under a certain restriction. Furthermore, the validity of the results $\|D\| = \|D_1\| + \|D_2\|$ and $\text{sp}(D) = \text{sp}(D_1) + \text{sp}(D_2)$ are fruitfully investigated.

KEY WORDS AND PHRASES : Γ -Banach algebras, projective tensor products, α -derivations.
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1. INTRODUCTION

Γ -Banach algebras and α -derivations are generalisation of ordinary Banach algebras and derivations respectively. The set of all $m \times n$ rectangular matrices and the set of all bounded linear transformations from an infinite dimensional normed linear space X into a Banach space Y are nice examples of Γ -Banach algebras which are not general Banach algebras. Similarly a derivation can't be defined on these spaces as there appears to be no natural way of introducing an algebraic multiplication into these. So, a new concept of derivation known as α -derivation is introduced on a Γ -Banach algebra. Bhattacharya and Maity have defined a Γ -Banach algebra in their paper [1] and have discussed in their another paper [2] various tensor products of Γ -Banach algebras over fields which are isomorphic to another field with a real valued valuation by using semilinear transformations, [3]. Derivations and tensor products of general Banach algebras are discussed in many papers, [4,5,6,7,8]. Now there are some natural questions: Does every pair of derivations D_1 and D_2 on Gamma Banach algebras (V, Γ) and (V', Γ') respectively give rise to a derivation D on their projective tensor product? If yes, then can one estimate the norm of D with the help of norms of D_1 and D_2 ? Can one evaluate the spectrum of D with the help

of those of D_1 and D_2 ? Are the converses of the above problems true? We give affirmative answers to some of these questions. The useful terminologies are forwarded below :

DEFINITION 1.1. Let $X (F_1)$ and $Y (F_2)$ be given normed linear spaces over fields F_1 and F_2 , which are isomorphic to a field F with a real valued valuation, (refer to Backman's book [9]). If $u = \sum_i (x_i \otimes y_i)$ is an element of the algebraic tensor product $X \otimes Y$, then the projective norm p is defined by

$$p(u) = \inf \left\{ \sum_i \|x_i\| \|y_i\| : x_i \in X, y_i \in Y \right\},$$

where the infimum is taken over all finite representations of u . Further the weak norm w on u is defined by

$$w(u) = \sup \left\{ \left| \sum_i \zeta_1 (f(x_i)) \cdot \zeta_2 (g(y_i)) \right| : f \in X^*, g \in Y^*, \|f\| \leq 1, \|g\| \leq 1 \right\}.$$

[Here X^* and Y^* are respective dual spaces of X and Y ; and F_1, F_2 are isomorphic to F under isomorphisms ζ_1 and ζ_2]. The projective tensor product $X \otimes_p Y$ and the weak tensor product $X \otimes_w Y$ are the completions of $X \otimes Y$ with their respective norms. For details, see Bonsall and Duncan's book [10].

DEFINITION 1.2. Let (V, Γ) be a Γ - Banach algebra and α , a fixed element of Γ . Then α -identity, 1_α , is an element of V satisfying the conditions $x\alpha 1_\alpha = x$ and $1_\alpha \alpha x = x$ for every x in V .

DEFINITION 1.3. A linear operator D of (V, Γ) into itself is called an α - derivation if

$$D(x \alpha y) = (Dx) \alpha y + x \alpha (Dy), \quad x, y \in V.$$

Every $x \in V$ gives rise to an α - derivation D_x defined by $D_x(y) = x\alpha y - y\alpha x$. Such a derivation is called an α -inner derivation. Further, if (V, Γ) is an involutive Γ - Banach algebra with an involution $*$, then an α - derivation D is called an α -star-derivation if $Dx^* = -(Dx)^*$, x^* being the adjoint of x . Again, we define an operation \circ by $x\circ y = x\alpha y + y\alpha x$, $x, y \in V$. A linear map D on (V, Γ) is called an α -Jordan derivation if $D(x\circ y) = (Dx) \circ y + x \circ (Dy)$ for all x and y in V .

2. THE MAIN RESULTS

Throughout our discussion, unless stated otherwise, (V, Γ) and (V', Γ') are Γ -Banach algebras over F_1 and F_2 , isomorphic to F which possesses a real valued valuation ; α and α' are fixed elements of Γ and Γ' ; and $1_\alpha, 1_{\alpha'}$ are α - identity and α' -identity of V and V' respectively. Moreover, suppose that $\|1_\alpha\| = k_1 \neq 0$ and $\|1_{\alpha'}\| = k_2 \neq 0$.

The following proposition is fundamental for our purpose, and we refer to Bhattacharya and Maity [2] for its proof.

PROPOSITION 2.1. The projective tensor product $(V, \Gamma) \otimes_p (V', \Gamma')$ with the projective norm is a $\Gamma \otimes \Gamma'$ - Banach algebra over the field F , where multiplication is defined by the formula

$$(x \otimes y) (\gamma \otimes \delta) (x' \otimes y') = (x \gamma x') \otimes (y \delta y'), \text{ where } x, y \in V; x', y' \in V'; \gamma \in \Gamma; \delta \in \Gamma'.$$

THEOREM 2.1. Let D_1 and D_2 be bounded α - derivation and α' - derivation on (V, Γ) and (V', Γ') respectively. Then

(i) there exists a bounded $\alpha \otimes \alpha'$ - derivation D on the projective tensor product $(V, \Gamma) \otimes_p (V', \Gamma')$ defined

by the relation

$$D(u) = \sum_1^n \left[(D_1 x_i) \otimes y_i + x_i \otimes (D_2 y_i) \right], \text{ for each vector } u = \sum_1^n x_i \otimes y_i \in (V, \Gamma) \otimes_p (V', \Gamma').$$

(ii) If D_1 and D_2 are α - and α' - inner derivations implemented by the elements $r_\alpha \in V$ and $s_{\alpha'} \in V'$ respectively then D is an $\alpha \otimes \alpha'$ - inner derivation implemented by $r_\alpha \otimes 1_{\alpha'} + 1_\alpha \otimes s_{\alpha'}$.

(iii) If D_1 and D_2 are α - and α' - Jordan derivations, then D is an $\alpha \otimes \alpha'$ - Jordan derivation.

(iv) If (V, Γ) and (V', Γ') are involutive Gamma -Banach algebras, and if D_1 and D_2 are α - and α' - star derivations, then D is $\alpha \otimes \alpha'$ - star derivation.

PROOF. (i) We define a map $D : (V, \Gamma) \otimes_p (V', \Gamma') \rightarrow (V, \Gamma) \otimes_p (V', \Gamma')$ by the rule

$$D(u) = \sum_1^n \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right], \text{ for each vector } u = \sum_1^n x_i \otimes y_i.$$

Clearly, D is well - defined. Before establishing the linearity of D , we first aim at proving the boundedness of D . For any arbitrary element $u \in (V, \Gamma) \otimes_p (V', \Gamma')$ and $\varepsilon > 0$, the definition of the projective norm provides a finite representation $\sum_{i=1}^n x'_i \otimes y'_i$ such that $\|u\|_p + \varepsilon \geq \sum_{i=1}^n \|x'_i\| \|y'_i\|$. Therefore, for this representation of u , we obtain

$$\begin{aligned} \|Du\|_p &= \left\| \sum_1^n \left[D_1 x'_i \otimes y'_i + x'_i \otimes D_2 y'_i \right] \right\|_p \\ &\leq \sum_1^n \left[\|D_1 x'_i \otimes y'_i\|_p + \|x'_i \otimes D_2 y'_i\|_p \right] \\ &= \sum_1^n \left[\|D_1 x'_i\| \|y'_i\| + \|x'_i\| \|D_2 y'_i\| \right], \text{ (because a projective norm is a cross norm)} \\ &\leq (\|D_1\| + \|D_2\|) \sum_1^n \|x'_i\| \|y'_i\|, \text{ (because } D_1 \text{ and } D_2 \text{ are bounded)} \\ &\leq K (\|u\|_p + \varepsilon), \text{ where } K = \|D_1\| + \|D_2\|. \end{aligned}$$

Thus, $\|Du\|_p \leq K (\|u\|_p + \varepsilon)$. Since the left hand side is independent of ε , and ε was arbitrary, it follows that $\|Du\|_p \leq K \|u\|_p$ for every $u \in (V, \Gamma) \otimes_p (V', \Gamma')$. Consequently, D is bounded.

Next to establish the linearity, let $u = \sum_{i=1}^n x_i \otimes y_i$ and $v = \sum_{j=1}^m r_j \otimes s_j$ be any two elements of

$(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u + v = \sum_{i=1}^{n+m} x_i \otimes y_i$, where $x_{n+j} = r_j$ and $y_{n+j} = s_j, j = 1, 2, \dots, m$.

$$\begin{aligned} \text{Now, } D(u + v) &= D\left(\sum_{i=1}^{n+m} x_i \otimes y_i\right) \\ &= \sum_{i=1}^{n+m} \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] \\ &= \sum_{i=1}^n \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] + \sum_{i=n+1}^{n+m} \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] \\ &= \sum_{i=1}^n \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] + \sum_{j=1}^m \left[D_1 r_j \otimes s_j + r_j \otimes D_2 s_j \right] = D(u) + D(v). \end{aligned}$$

The boundedness of D implies that the result, $D(u + v) = D(u) + D(v)$, is also true for any infinite

representations of u and v . Similarly it can be shown easily that $D(au) = aD(u)$ for any scalar a . Consequently D is a bounded linear map.

To show that D is an $\alpha \otimes \alpha'$ -derivation, we suppose that $u = x \otimes y$ and $v = r \otimes s$ are any two elementary tensors of $(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u \alpha \otimes \alpha' v = x \alpha r \otimes y \alpha' s$. Now

$$\begin{aligned} D(u \alpha \otimes \alpha' v) &= (D_1 x \alpha r) \otimes y \alpha' s + x \alpha r \otimes (D_2 y \alpha' s) \\ &= \left[(D_1 x) \alpha r + x \alpha (D_1 r) \right] \otimes y \alpha' s + x \alpha r \otimes \left[(D_2 y) \alpha' s + y \alpha' (D_2 s) \right] \\ &= \left[(D_1 x) \alpha r \otimes y \alpha' s + x \alpha r \otimes (D_2 y) \alpha' s \right] + \left[x \alpha (D_1 r) \otimes y \alpha' s + x \alpha r \otimes y \alpha' (D_2 s) \right] \\ &= (Du) \alpha \otimes \alpha' v + u \alpha \otimes \alpha' (Dv). \end{aligned}$$

Similarly, if $u = \sum_i x_i \otimes y_i$ and $v = \sum_j r_j \otimes s_j$ be any two elements of $(V, \Gamma) \otimes_p (V', \Gamma')$, then summing over i and j we can prove easily that $D(u \alpha \otimes \alpha' v) = (Du) \alpha \otimes \alpha' v + u \alpha \otimes \alpha' (Dv)$. so D is an $\alpha \otimes \alpha'$ -derivation.

(ii) Let D_1 and D_2 be α - and α' -inner derivations implemented by the vectors r_0 and s_0 respectively.

So, $D_1(x) = r_0 \alpha x - x \alpha r_0, \forall x \in V$ and $D_2(y) = s_0 \alpha' y - y \alpha' s_0, \forall y \in V'$.

$$\begin{aligned} \text{Now, } D(u) &= \sum_i \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] \\ &= \sum_i \left[(r_0 \alpha x_i - x_i \alpha r_0) \otimes y_i + x_i \otimes (s_0 \alpha' y_i - y_i \alpha' s_0) \right] \\ &= \sum_i \left[r_0 \alpha x_i \otimes y_i - x_i \alpha r_0 \otimes y_i + x_i \otimes s_0 \alpha' y_i - x_i \otimes y_i \alpha' s_0 \right] \\ &= \sum_i \left[(r_0 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (x_i \otimes y_i) - (x_i \otimes y_i) (\alpha \otimes \alpha') (r_0 \otimes 1_{\alpha'}) \right. \\ &\quad \left. + (1_{\alpha} \otimes s_0) (\alpha \otimes \alpha') (x_i \otimes y_i) - (x_i \otimes y_i) (\alpha \otimes \alpha') (1_{\alpha} \otimes s_0) \right] \\ &= \sum_i \left[(r_0 \otimes 1_{\alpha'} + 1_{\alpha} \otimes s_0) (\alpha \otimes \alpha') (x_i \otimes y_i) - (x_i \otimes y_i) (\alpha \otimes \alpha') (r_0 \otimes 1_{\alpha'} + 1_{\alpha} \otimes s_0) \right] \\ &= D_{t_0}(u), \text{ where } t_0 = r_0 \otimes 1_{\alpha'} + 1_{\alpha} \otimes s_0. \end{aligned}$$

Consequently, D is an $\alpha \otimes \alpha'$ -inner derivation implemented by t_0 .

(iii) The proof is routine.

(iv) Let D_1 and D_2 be star derivations. If $u = \sum_i x_i \otimes y_i$ is an element of $(V, \Gamma) \otimes_p (V', \Gamma')$, then the adjoint of u is given by $u^* = \sum_i x_i^* \otimes y_i^*$. Now,

$$\begin{aligned} Du^* &= D \left(\sum_i x_i^* \otimes y_i^* \right) \\ &= \sum_i \left[D_1 x_i^* \otimes y_i^* + x_i^* \otimes D_2 y_i^* \right] \\ &= \sum_i \left[- (D_1 x_i)^* \otimes y_i^* + x_i^* \otimes \{ -(D_2 y_i)^* \} \right], \text{ because } D_1 \text{ and } D_2 \text{ are star derivation.} \end{aligned}$$

$$= -\sum_i \left[(D_1 x_i)^* \otimes y_i^* + x_i^* \otimes (D_2 y_i)^* \right] = -(Du)^*. \text{ So, } D \text{ is a star-derivation. Q.E.D.}$$

REMARK 2.1. (i) The above theorem can be extended to the projective tensor product of n number of Γ - Banach algebras.

(ii) If $u = x \otimes 1_{\alpha'} \in (V, \Gamma) \otimes_p (V', \Gamma')$, then from the definition of D, we get

$$Du = D_1 x \otimes 1_{\alpha'}, \text{ because } D_2 1_{\alpha'} = 0 \quad \dots \quad (2.1)$$

From this result, we can ascertain that for each derivation D on $(V, \Gamma) \otimes_p (V', \Gamma')$, there may **not** exist derivations D_1 and D_2 on (V, Γ) and (V', Γ') respectively such that D, D_1 and D_2 are connected by the relation given in Theorem 2.1. For example, let D' be an $\alpha \otimes \alpha'$ - inner derivation implemented by an element $r_0 \otimes s_0$, where s_0 is not a scalar multiple of the identity element $1_{\alpha'}$. Then

$D' u = (r_0 \otimes s_0) (\alpha \otimes \alpha') u - u (\alpha \otimes \alpha') (r_0 \otimes s_0)$, for every $u \in (V, \Gamma) \otimes_p (V', \Gamma')$. Now if $u = x \otimes 1_{\alpha'}$, then

$$\begin{aligned} D' u &= (r_0 \otimes s_0) (\alpha \otimes \alpha') (x \otimes 1_{\alpha'}) - (x \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (r_0 \otimes s_0) \\ &= r_0 \alpha x \otimes s_0 \alpha' 1_{\alpha'} - x \alpha r_0 \otimes 1_{\alpha'} \alpha' s_0 = (r_0 \alpha x - x \alpha r_0) \otimes s_0 \\ &= (D_{1, r_0} x) \otimes s_0, \text{ where } D_1 \text{ is a derivation on } (V, \Gamma) \text{ implemented by } r_0 \quad \dots \quad (2.2) \end{aligned}$$

From the results (2.1) and (2.2) we can conclude that unless s_0 is a scalar multiple of the identity element $1_{\alpha'}$, $D' (x \otimes 1_{\alpha'})$ may not be of the form $x_1 \otimes 1_{\alpha'}$, where $x_1 \in V$, [x_1 may be different from x]. This implies that D' may not equal D in general. However, we have a converse of Theorem 2.1 as follows. Recall that an element $x \in V$ is called an α - idempotent element if $x \alpha x = x$.

THEOREM 2.2. The following results are true :

(i) If D is a derivation on $(V, \Gamma) \otimes_p (V', \Gamma')$ such that $D(\sum_i x_i \otimes y_i) = \sum_i z_i \otimes y_i$, $z_i \in V$ and y_i 's are α' - idempotent elements of V' , then there exists an α' -derivation D_1 on V defined by the rule $D_1 x \otimes y = D(x \otimes y)$ for all $x \in V$ and for every α' - idempotent element $y \in V'$;

(ii) If D is bounded, so is D_1 ;

(iii) If D is an $\alpha \otimes \alpha'$ -inner derivation implemented by an element w of the form $w = \sum_i x_i \otimes y_i$, where y_i 's are α' - idempotent elements, then D_1 is also an α - inner derivation implemented by the element $\sum_i x_i$;

(iv) If (V, Γ) and (V', Γ') are involutive Gamma-Banach algebras, and D is a star derivation, then so is D_1 ;

(v) If D is an $\alpha \otimes \alpha'$ - Jordan derivation then D_1 is an α - Jordan derivation;

(vi) If D is an $\alpha \otimes \alpha'$ - derivation on $(V, \Gamma) \otimes_p (V', \Gamma')$ such that $D(\sum_i x_i \otimes y_i) = \sum_i x_i \otimes s_i$ for α -idempotent elements x_i 's in V, and $s_i \in V'$, then there exists an α' - derivation D_2 on (V', Γ') given by the relation $x \otimes D_2 y = D(x \otimes y)$ for every α - idempotent element $x \in V$ and for all elements $y \in V'$. The above results (ii), (iii), (iv) and (v) are also true for D_2 .

PROOF. (i) We define a map $D_1 : V \rightarrow V$ by

$$D_1 x \otimes y = D(x \otimes y), \text{ for all } x \in V \text{ and for every } \alpha'\text{-idempotent element } y \in V'.$$

Clearly, D_1 is well-defined. In particular, we have $D_1 x \otimes 1_{\alpha'} = D(x \otimes 1_{\alpha'})$, $\forall x \in V$. We first establish the linearity of D_1 . Let $x_1, x_2 \in V$.

Then

$$\begin{aligned}
 D_1(x_1 + x_2) \otimes 1_{\alpha'} &= D((x_1 + x_2) \otimes 1_{\alpha'}) \\
 &= D(x_1 \otimes 1_{\alpha'} + x_2 \otimes 1_{\alpha'}) \\
 &= D(x_1 \otimes 1_{\alpha'}) + D(x_2 \otimes 1_{\alpha'}) \\
 &= (D_1 x_1 \otimes 1_{\alpha'} + D_1 x_2 \otimes 1_{\alpha'}) \\
 &= (D_1 x_1 + D_1 x_2) \otimes 1_{\alpha'}
 \end{aligned}$$

So, $(D_1(x_1+x_2) \otimes 1_{\alpha'})(f,g) = ((D_1x_1 + D_1x_2) \otimes 1_{\alpha'})(f,g), \quad \forall f \in V^*, \forall g \in V''.$

This gives, $f(D_1(x_1+x_2))g(1_{\alpha'}) = f(D_1x_1 + D_1x_2)g(1_{\alpha'}), \quad \forall f \in V^*, \forall g \in V''.$

The Hahn-Banach theorem provides a functional $g_0 \in V''$ in such a way that $g_0(1_{\alpha'}) = \|1_{\alpha'}\| = k_2$.

Then, $f(D_1(x_1 + x_2)) = f(D_1x_1 + D_1x_2), \forall f \in V^*.$ This yields, $D_1(x_1+x_2) = D_1x_1 + D_1x_2.$

By appealing to the same mechanism, we can show that $D_1(ax) = aD_1(x)$ for any scalar a . So D_1 is linear.

Next, to show that D_1 is an α - derivation.

$$\begin{aligned}
 D_1(x_1 \alpha x_2) \otimes 1_{\alpha'} &= D(x_1 \alpha x_2 \otimes 1_{\alpha'}) \quad (x_1, x_2 \in V) \\
 &= D \left[(x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (x_2 \otimes 1_{\alpha'}) \right] \\
 &= (D(x_1 \otimes 1_{\alpha'})) (\alpha \otimes \alpha') (x_2 \otimes 1_{\alpha'}) + (x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') D(x_2 \otimes 1_{\alpha'}) \\
 &\quad \text{(because } D \text{ is an } \alpha \otimes \alpha' \text{-derivation)} \\
 &= (D_1 x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (x_2 \otimes 1_{\alpha'}) + (x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (D_1 x_2 \otimes 1_{\alpha'}) \\
 &= (D_1 x_1) \alpha x_2 \otimes 1_{\alpha'} + (x_1 \alpha (D_1 x_2)) \otimes 1_{\alpha'} = \left[(D_1 x_1) \alpha x_2 + x_1 \alpha (D_1 x_2) \right] \otimes 1_{\alpha'}
 \end{aligned}$$

So, $D_1(x_1 \alpha x_2) = (D_1 x_1) \alpha x_2 + x_1 \alpha (D_1 x_2).$ Therefore, D_1 is an α - derivation. The rest of the results are routine.

3. THE NORM OF D

We now shift our attention to study the possibility of the result $\|D\| = \|D_1\| + \|D_2\|$, when D_1 and D_2 are related as in Theorem 2.1.

THEOREM 3.1. If D, D_1 and D_2 are related as in Theorem 2.1, then

$$\|D\| \leq \|D_1\| + \|D_2\| \leq 2\|D\|.$$

PROOF. For each $u \in (V, \Gamma) \otimes_p (V', \Gamma')$ with $\|u\|_p = 1$ and for each $\epsilon > 0, \exists a$ (finite) representation

$$u = \sum_i x_i \otimes y_i \text{ such that } \|u\|_p + \epsilon \geq \sum_i \|x_i\| \|y_i\|.$$

Now, $\|D\| = \sup_u \{ \|Du\|_p : \|u\|_p = 1 \}$

$$\begin{aligned}
 &= \sup_{\mathbf{u}} \left\{ \left\| \sum_i [D_1 x_i \otimes y_i + x_i \otimes D_2 y_i] \right\|_p : \|\mathbf{u}\|_p = 1 \right\} \\
 &\leq \sup_{\mathbf{u}} \left\{ \sum_i [\|D_1 x_i \otimes y_i\|_p + \|x_i \otimes D_2 y_i\|_p] : \|\mathbf{u}\|_p = 1 \right\} \\
 &= \sup_{\mathbf{u}} \left\{ \sum_i [\|D_1 x_i\| \|y_i\| + \|x_i\| \|D_2 y_i\|] : \|\mathbf{u}\|_p = 1 \right\} \\
 &\leq \sup_{\mathbf{u}} \left\{ \sum_i [\|D_1\| \|x_i\| \|y_i\| + \|x_i\| \|D_2\| \|y_i\|] : \|\mathbf{u}\|_p = 1 \right\} \\
 &\leq (\|D_1\| + \|D_2\|) \sup_{\mathbf{u}} \{1 + \varepsilon : \|\mathbf{u}\|_p = 1\} \\
 &= (\|D_1\| + \|D_2\|)(1 + \varepsilon)
 \end{aligned}$$

Since ε was arbitrary, it follows that $\|D\| \leq \|D_1\| + \|D_2\|$ (3.1)

Next, let $x \in V$ be such that $\|x\| = 1$. Then $\|x/k_2 \otimes 1_\alpha\| = \|x/k_2\| \|1_\alpha\| = 1$

Now, $\|D\| = \sup_{\mathbf{u}} \{ \|D\mathbf{u}\|_p : \|\mathbf{u}\|_p = 1 \}$

$$\geq \|D(x/k_2 \otimes 1_\alpha)\|_p = \|D_1(x/k_2) \otimes 1_\alpha\|_p, (\text{Since } D_2(1_\alpha) = 0) = \|D_1 x\|$$

Thus, $\|D_1 x\| \leq \|D\|$ for every $x \in V$ with $\|x\| = 1$. This gives $\|D_1\| \leq \|D\|$. Similarly, we can prove that $\|D_2\| \leq \|D\|$. Hence, we have $\|D_1\| + \|D_2\| \leq 2\|D\|$ (3.2)

The inequalities (3.1) and (3.2) together imply $\|D\| \leq \|D_1\| + \|D_2\| \leq 2\|D\|$. Q.E.D.

Our next question is - can one improve the above result - ? We illustrate the possibility with the help of examples :

Let V be the set of 2×3 rectangular matrices and Γ be the set of all 3×2 rectangular matrices with real (or complex) entries. Then V and Γ are Banach spaces under usual matrix addition, scalar multiplication, and the norm defined by $\|A\|_\infty = \max_{i,j} |a_{ij}|$, where $A = (a_{ij})$. Then (V, Γ) is a Γ -Banach algebra. Now the following result is true :

THEOREM 3.2. For a fixed $\alpha \in \Gamma$, each α - derivation on V is inner.

Since α -derivations on a finite dimensional Γ -Banach algebra are all inner, the result follows immediately, see [10].

We show below with an example in the Γ -Banach algebra of 2×3 rectangular matrices that the equality $\|D\| = \|D_1\| + \|D_2\|$ holds.

AN EXAMPLE 3.1.

Let $\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}$ be a fixed element in Γ , and let $D_{1\alpha}$ and $D_{2\alpha}$ be two α -derivations on V

implemented by A_0 and B_0 respectively, where $A_0 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ and $B_0 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}$

Now $\|A_0\| = 2$ and $\|B_0\| = 3$. and $D_{1\alpha}(A) = A_0 \alpha A - A \alpha A_0, \forall A \in V$.

Then $\|D_{1\alpha} A\| \leq 2 \|A_0\| \|\alpha\| \|A\| = 2 \|A_0\| \|A\|$, because $\|\alpha\| = 1$.

Hence, $\|D_{1\alpha}\| \leq 2 \|A_0\| = 2.2 = 4$. Next, suppose that $X_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\|X_0\| = 1$.

Also $\|A_0 \alpha X_0 - X_0 \alpha A_0\| = \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \right\| = 4$. Hence $\|D_{1\alpha}\| = 4$

Similarly we can show that $\|D_{2\alpha}\| = 6$. So $\|D_{1\alpha}\| + \|D_{2\alpha}\| = 4 + 6 = 10$.

If D is the derivation defined by the relation as in Theorem 3.1, then we always have

$$\|D\| \leq \|D_{1\alpha}\| + \|D_{2\alpha}\| = 10 \quad (3.1)$$

Next, consider the element $u_0 = e_1 \otimes e_1$, where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\|u_0\|_p = 1$.

Now, $\|D\| \geq \|Du_0\|_p$

$$\begin{aligned} &= \|D_{1\alpha} e_1 \otimes e_1 + e_1 \otimes D_{2\alpha} e_1\|_p \\ &\geq \|D_{1\alpha} e_1 \otimes e_1 + e_1 \otimes D_{2\alpha} e_1\|_w \\ &\text{(because the projective norm is always greater than or equal to the weak norm)} \\ &= \sup \left\{ |f(D_{1\alpha} e_1)g(e_1) + f(e_1)g(D_{2\alpha} e_1)| : f, g \in V^*, \|f\| = \|g\| = 1 \right\} \quad (3.2) \end{aligned}$$

Again $D_{1\alpha} e_1 = A_0 \alpha e_1 - e_1 \alpha A_0$

$$\begin{aligned} &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \end{aligned}$$

$D_{2\alpha} e_1 = B_0 \alpha e_1 - e_1 \alpha B_0$

$$\begin{aligned} &= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \end{aligned}$$

We know that if we define

$f_i(e_j) = 1$ if $i = j$ and $= 0$ if $i \neq j$, then $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a basis for V^*

In (3.2) put $f = g = f_1$. Then we find that $\|D\| \geq 10$ (3.3)

The inequalities (3.1) and (3.3) combinedly give $\|D\| = 10$. Hence $\|D\| = \|D_{1\alpha}\| + \|D_{2\alpha}\|$

ANOTHER EXAMPLE 3.2.

Next we wish to illustrate that the result in Theorem 3.1 cannot be improved in general. If we assume V and Γ represent the same set of all 2×2 real matrices, then (V, Γ) is a particular Γ -Banach

algebra with the usual operations. The ordinary identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of (V, Γ) under multiplication.

If $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\beta = \{e_1, e_2, e_3, e_4\}$ is the standard basis for (V, Γ) . For a simple example, let D_1 and D_2 be derivations on (V, Γ) implemented by the

matrices $A_o = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $B_o = \begin{pmatrix} 4 & -7 \\ 0 & 2 \end{pmatrix}$ respectively. Then the matrix representations of D_1 and D_2

with respect to the basis β are respectively

$$[D_1]_\beta = \begin{pmatrix} 0 & 0 & 3 & 0 \\ -3 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \text{ and } [D_2]_\beta = \begin{pmatrix} 0 & 0 & -7 & 0 \\ 7 & 2 & 0 & -7 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 7 & 0 \end{pmatrix}$$

So, $\|D_1\| = 3$ and $\|D_2\| = 7$. Again, $\gamma = \{e_i \otimes e_j \mid i, j = 1, 2, 3, 4\}$ is a basis for $(V, \Gamma) \otimes_p (V, \Gamma)$ and the matrix representation of D with respect to the basis γ is

$$[D]_\gamma = \begin{pmatrix} 0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 2 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 7 & 3 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 7 & 2 & 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 7 & 0 & 0 \end{pmatrix}$$

Hence $\|D\| = 7$. Thus the strict inequality $\|D\| < \|D_1\| + \|D_2\| < 2\|D\|$ holds.

4. THE SPECTRUM OF D

We next devote to studying the validity of the result $sp(D) = sp(D_1) + sp(D_2)$. Recall that $sp(D_1)$ consists of all scalars λ_1 such that $D_1 - \lambda_1 I_1$ is singular. Analogous definitions apply to $sp(D_2)$ and $sp(D)$. Further, for the singularity and invertibility of a rectangular matrix, see . Joshi [11].

THEOREM 4.1. The derivations D, D_1 and D_2 are defined as in Theorem 2.1. Then

$$sp(D_1) + sp(D_2) \subseteq sp(D)$$

PROOF. Let $\lambda_1 \in sp(D_1)$ and $\lambda_2 \in sp(D_2)$.

$$\Rightarrow D_1 - \lambda_1 I_1 \text{ and } D_2 - \lambda_2 I_2 \text{ are singular}$$

$$\Rightarrow \exists \text{ nonzero vectors } x_o \in V \text{ and } y_o \in V' \text{ such that } (D_1 - \lambda_1 I_1)x_o = 0 \text{ and } (D_2 - \lambda_2 I_2)y_o = 0$$

Now, $x_o \otimes y_o$ is a non-zero element in $(V, \Gamma) \otimes_p (V, \Gamma')$.

$$\begin{aligned}
 \text{Again, } [D - (\lambda_1 + \lambda_2) I] (x_0 \otimes y_0) &= D(x_0 \otimes y_0) - (\lambda_1 + \lambda_2)(x_0 \otimes y_0) \\
 &= D_1 x_0 \otimes y_0 + x_0 \otimes D_2 y_0 - (\lambda_1 + \lambda_2) x_0 \otimes y_0 \\
 &= (D_1 - \lambda_1 I_1) x_0 \otimes y_0 + x_0 \otimes (D_2 - \lambda_2 I_2) y_0 = 0
 \end{aligned}$$

So, $D - (\lambda_1 + \lambda_2) I$ is singular and hence $\lambda_1 + \lambda_2 \in \text{sp}(D)$. Thus, we obtain $\text{sp}(D_1) + \text{sp}(D_2) \subseteq \text{sp}(D)$. Q.E.D

REMARK 4.1. (i) We conjecture that the above result cannot be improved in general.

(ii) However, the equality holds in finite dimensional Γ - Banach algebras. For, if $\dim(V, \Gamma) = m$, $\dim(V', \Gamma') = n$, then $\dim((V, \Gamma) \otimes_p (V', \Gamma')) = mn$. So, $\text{sp}(D_1)$, $\text{sp}(D_2)$ and $\text{sp}(D)$ have m, n and mn eigenvalues respectively. Again, $\text{sp}(D_1) + \text{sp}(D_2)$ gives mn values which are precisely the eigenvalues of D .

Further, we have the following illuminating result.

THEOREM 4.2. As usual, let D_1, D_2 and D be derivations connected by the relation as in Theorem 2.1(i). If (V, Γ) and (V', Γ') are finite dimensional Gamma-Banach algebras, D_1 and D_2 are implemented by $r \in V$ and $s \in V'$ respectively, then

$$\begin{aligned}
 \text{sp}(D_1) &= \{ a = \lambda - \mu \mid \lambda, \mu \in \text{sp}(r) \}, \\
 \text{sp}(D_2) &= \{ b = \lambda' - \mu' \mid \lambda', \mu' \in \text{sp}(s) \}
 \end{aligned}$$

$$\text{and } \text{sp}(D) = \{ a + b \mid a \in \text{sp}(D_1), b \in \text{sp}(D_2) \}.$$

PROOF. The first two results will follow from Proposition 9, §18, Ch2 in [10], and the last result will follow from Remark 4.1 (ii). Q.E.D.

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