Internat. J. Math. & Math. Sci.
VOL. 21 NO. 2 (1998) 369-374

ANGULAR ESTIMATIONS OF CERTAIN INTEGRAL OPERATORS

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(Received April 1, 1996 and in revised form September 30, 1996)

ABSTRACT. The object of the present paper is to derive some argument properties of certain integral operators. Our results contain some interesting corollaries as the special cases.

KEY WORDS AND PHRASES: Argument, integral operators, starlike functions, Bazilević functions.
1991 AMS SUBJECT CLASSIFICATION CODES: 30C45.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$, if there exists a Schwarz function $w(z)$ in U such that $f(z) = g(w(z))$. A function $f \in A$ is said to be in the class $S^*[E,F]$ if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Ez}{1+Fz}\,(z \in U,\ -1 \le F < E \le 1).$$

The class $S^*[E,F]$ was studied in [1,2]. In particular, $S^*[1-2\alpha,\ -1] \equiv S^*(\alpha)(0 \le \alpha < 1)$ is the well known class of starlike functions of order α. We observe [2] that a function f is in $S^*[E,F]$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-EF}{1-F^2}\right| < \frac{E-F}{1-F^2}\,(z \in U, F \neq -1) \tag{1.2}$$

and

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-E}{2}\,(z \in U, F = -1). \tag{1.3}$$

A function $f \in A$ is said to be in the class $B(\mu,\alpha,\beta)$ if it satisfies

$$Re\left\{\frac{zf'(z)f^{\mu-1}}{g^{\mu}(z)}\right\} > \beta(z \in U)$$

for some $\mu(\mu > 0)$, $\beta(0 \le \beta < 1)$ and $g \in S^*(\alpha)$. Furthermore, we denote $B_1(\mu,\alpha,\beta)$ by the subclass of $B(\mu,\alpha,\beta)$ for $g(z) \equiv z \in S^*(\alpha)$. The classes $B(\mu,\alpha,\beta)$ and $B_1(\mu,\alpha,\beta)$ are the subclasses of Bazilević functions in U [3]. We also note that $B(1,\alpha,\beta) \equiv C(\alpha,\beta)$ is an important subclass of close-to-convex functions [4].

For a positive real number $\mu > 0$ and a function $f \in A$, we define the integral operator $J_{c,\mu}$ by

$$J_{c,\mu}(f) = \left(\frac{c+\mu}{z^c}\int_0^z t^{c-1}f^{\mu}(t)dt\right)^{\frac{1}{\mu}} (c > -\mu). \tag{1 4}$$

Kumar and Shukla [5] showed that the integral operator $J_{c,\mu}(f)$ defined by (1.4) belongs to the class $S^*[E,F]$ for $c \geq \frac{\mu(E-1)}{1-F}$, whenever $f \in S^*[E,F]$. The operator $J_{c,1}$, when $c \in N = \{1,2,3,\cdots\}$, was introduced by Bernardi [6]. Further, the operator $J_{1,1}$ was studied earlier by Libera [7] and Livingston [8].

In the present paper, we give some argument properties of the integral operator defined by (1.4). We also generalize the previous results of Libera [7], Owa and Srivastava [9] and Owa and Obradović [10].

2. MAIN RESULTS

In proving our main results, we shall need the following lemmas.

LEMMA 1 ([11]). Let $M(z)$ and $N(z)$ be regular in U with $M(0) = N(0) = 0$, and let β be real. If $N(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin, then

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > \beta(z \in U) \Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\} > \beta(z \in U)$$

and

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} < \beta(z \in U) \Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\} < \beta(z \in U).$$

LEMMA 2 ([12]). Let $p(z)$ be analytic in U, $p(0) = 1$, $p(z) \neq 0$ in U and suppose that there exists a point $z_0 \in U$ such that

$$\left|arg\, p(z)\right| < \frac{\pi\beta}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$\left|arg\, p(z_0)\right| = \frac{\pi\beta}{2},$$

where $\beta > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad arg\, p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad arg\, p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia(a > 0).$$

With the help of Lemma 1 and Lemma 2, we now derive

THEOREM 1. Let c and μ be real numbers with $c \geq 0$, $\mu > 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$ If

$$\left|arg\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}-\beta\right)\right|<\frac{\pi\delta}{2}\,(0\leq\beta<1,0<\delta\leq1)$$

for some $g\in S^*[E,F]$, then

$$\left|arg\left(\frac{z(J_{c,\mu}(f))'J_{c,\mu}^{\mu-1}(f)}{J_{c,\mu}^{\mu}(g)}\right)-\beta\right|<\frac{\pi\eta}{2},$$

where $J_{c,\mu}$ is the integral operator defined by (1.4) and $\eta(0<\eta\leq1)$ is the solution of the equation

$$\delta=\begin{cases}\eta+\dfrac{2}{\pi}Tan^{-1}\left(\dfrac{\eta\,sin\,\frac{\pi}{2}(1-t_c(E,F))}{c+\frac{1+E}{1+F}+\eta\,cos\,\frac{\pi}{2}(1-t_c(E,F))}\right) & \text{for } F\neq-1,\\ \eta & \text{for } F=-1,\end{cases}\tag{2.1}$$

when

$$t_c(E,F)=\frac{2}{\pi}sin^{-1}\left(\frac{E-F}{c(1-F^2)+1-EF}\right).\tag{2.2}$$

PROOF. Let us put

$$p(z)=\frac{M(z)}{N(z)},$$

where

$$M(z)=\frac{1}{1-\beta}\left\{z^cf^{\mu}(z)-c\int_0^z t^{c-1}f^{\mu}(t)dt-\beta\mu\int_0^z t^{c-1}g^{\mu}(t)dt\right\}$$

and

$$N(z)=\mu\int_0^z t^{c-1}g^{\mu}(t)dt.$$

Then $p(z)$ is analytic in U with $p(0)=1$. By a simple calculation, we have

$$\begin{aligned}\frac{M'(z)}{N'(z)}&=p(z)\left(1+\frac{N(z)}{zN'(z)}\frac{zp'(z)}{p(z)}\right)\\&=\frac{1}{1-\beta}\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}-\beta\right).\end{aligned}$$

Since $g\in S^*[E,F]$, $J_{c,\mu}(g)\in S^*[E,F]$ [5] and hence $N(z)$ is (possibly many-sheeted) starlike function with respect to the origin. Therefore, from our assumption and Lemma 1, $p(z)\neq0$ in U.

If there exists a point $z_0\in U$ such that

$$|arg\,p(z)|<\frac{\pi\eta}{2}\quad\text{for}\quad|z|<|z_0|$$

and

$$|arg\,p(z_0)|=\frac{\pi\eta}{2},$$

then, from Lemma 2, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\eta,$$

where

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\quad\text{when}\quad arg\,p(z_0)=\frac{\pi\eta}{2}$$

and

$$k \le -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad arg\, p(z_0) = -\frac{\pi\eta}{2}$$

where

$$p(z_0)^{\frac{1}{\eta}} = \pm ia (a > 0).$$

Since $J_{c,\mu}(g) \in S^*[E,F]$, from (1.2) and (1.3), we have

$$\frac{zN'(z)}{N(z)} = \frac{z(J_{c,\mu}(g))'}{J_{c,\mu}(g)} + c = \rho e^{i\frac{\pi\theta}{2}},$$

where

$$\begin{cases} c+\dfrac{1-E}{1-F} < \rho < c+\dfrac{1+E}{1+F}, \\ -t_c(E,F) < \phi < t_c(E,F) \quad \text{for } F \neq -1, \end{cases}$$

when $t_c(E,F)$ is given by (2.2), and

$$\begin{cases} c+\dfrac{1-E}{2} < \rho < \infty, \\ -1 < \phi < 1 \quad \text{for } F = -1. \end{cases}$$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = ia(a>0)$. For the case $F \neq -1$, we obtain

$$\begin{aligned} arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) &= arg\,\frac{(1-\beta)M'(z_0)}{N'(z_0)} \\ &= arg\, p(z_0) + arg\left(1 + \frac{1}{\frac{z(J_{c,\mu}(g))'}{J_{c,\mu}(g)} + c}\,\frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \frac{\pi\eta}{2} + arg\left(1 + \left(\rho e^{i\frac{\pi\phi}{2}}\right)^{-1} i\eta k\right) \\ &= \frac{\pi\eta}{2} + Tan^{-1}\left(\frac{\eta k \sin\frac{\pi}{2}(1-\phi)}{\rho + \eta k \cos\frac{\pi}{2}(1-\phi)}\right) \\ &\ge \frac{\pi\eta}{2} + Tan^{-1}\left(\frac{\eta \sin\frac{\pi}{2}(1-t_c(E,F))}{c+\frac{1+E}{1+F} + \eta\cos\frac{\pi}{2}(1-t_c(E,F))}\right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where $t_c(E,F)$ and δ are given by (2.2) and (2.1), respectively. Similarly, for the case $F = -1$, we have

$$arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) \ge \frac{\pi\eta}{2}.$$

These are a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\eta}} = -ia(a>0)$. For the case $F \neq -1$, applying the same method as the above, we have

$$arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) = \le -\frac{\pi\eta}{2} - Tan^{-1}\left(\frac{n \sin\frac{\pi}{2}(1-t_c(E,F))}{c+\frac{1+E}{1+F} + \eta\cos\frac{\pi}{2}(1-t_c(E,F))}\right)$$

where $t_c(E,F)$ and δ are given by (2.2) and (2.1), respectively and for the case $F = -1$, we have

$$arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) \leq -\frac{\pi\eta}{2},$$

which are contradictions to the assumption. Therefore we complete the proof of our theorem.

Taking $E = 1 - 2\alpha(0 \leq \alpha < 1)$ and $F = -1$ in Theorem 1, we have

COROLLARY 1. Let $c \geq 0$, $\mu > 0$ and $f \in A$. If

$$\left|arg\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)} - \beta\right)\right| < \frac{\pi\delta}{2}\ (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in S^*(\alpha)$, then

$$\left|arg\left(\frac{z(J_{c,\mu}(f))' J_{c,\mu}^{\mu-1}(f)}{J_{c,\mu}^{\mu}(g)} - \beta\right)\right| < \frac{\pi\delta}{2},$$

where $J_{c,\mu}$ is the integral operator defined by (1.4).

REMARK 1. For $\delta = 1$, Corollary 1 is the result obtained by Owa and Obradović [10].

Setting $E = 1$, $F = -1$, $\mu = 1$, $\delta = 1$ and $g(z) = z$ in Theorem 1, we have

COROLLARY 2. Let $c \geq 0$ and $f \in A$. If

$$Re\, f'(z) > \beta(0 \leq \beta < 1),$$

then

$$Re\,(J_{c,1}(f))' > \beta,$$

where $J_{c,1}$ is the integral operator defined by (1.4).

Letting $\mu = 1$ in Theorem 1, we have

COROLLARY 3. Let $c \geq 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$. If

$$\left|arg\left(\frac{zf'(z)}{g(z)} - \beta\right)\right| < \frac{\pi\delta}{2}\ (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in S^*[E, F]$, then

$$\left|arg\left(\frac{z(J_{c,1}(f))'}{J_{c,1}(g)} - \beta\right)\right| < \frac{\pi\eta}{2},$$

where $J_{c,1}$ is the integral operator defined by (1.4) and $\eta(0 < \eta \leq 1)$ is the solution of the equation (2.1).

Taking $E = 1 - 2\alpha(0 \leq \alpha < 1)$ and $F = -1$ in Corollary 3, we have

COROLLARY 4. Let $c \geq 0$ and $f \in A$. If

$$\left|arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\delta}{2}\ (0 \leq \alpha < 1, 0 < \delta \leq 1),$$

then

$$\left|arg\left(\frac{z(J_{c,1}(f))'}{J_{c,1}(f)} - \alpha\right)\right| < \frac{\pi\delta}{2},$$

where $J_{c,1}$ is the integral operator defined by (1.4).

Putting $E = 1 - 2\alpha(0 \leq \alpha < 1)$, $F = -1$ and $\delta = 1$ in Corollary 3 and Corollary 4, we obtain the following result of Owa and Srivastava [9].

COROLLARY 5. If the function f defined by (1.1) is in the class $C(\alpha, \beta)$, then the integral operator $J_{c,1}(f)(c \geq 0)$ defined by (1.4) is also in the class $c(\alpha, \beta)$.

REMARK 2. Taking $\alpha = \beta = 0$ and $c = 1$ in Corollary 5, we obtain the result given earlier by Libera [7]

By using the same technique as in proving Theorem 1, we have

THEOREM 2. Let c and μ be real numbers with $c \geq 0$, $\mu > 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$. If

$$\left|arg\left(\beta - \frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}\right)\right| < \frac{\pi\delta}{2}\,(\beta > 1, 0 < \delta \leq 1)$$

for some $g \in S^{*}[E,F]$, then

$$\left|arg\left(\beta - \frac{z(J_{c,\mu}(f))'J_{c,\mu}^{\mu-1}(f)}{J_{c,\mu}^{\mu}(g)}\right)\right| < \frac{\pi\eta}{2},$$

where $J_{c,\mu}$ is the integral operator defined by (1.4) and $\eta(0 < \eta \leq 1)$ is the solution of the equation (2.1)

Putting $E = 1 - 2\alpha(0 \leq \alpha < 1)$, $F = -1$, $\mu = 1$ and $\delta = 1$ in Theorem 2, we have the following result by Owa and Srivastava [9].

COROLLARY 6. Let $c \geq 0$ and $f \in A$. If

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} < \beta(\beta > 1)$$

for some $g \in S^{*}(\alpha)$, then

$$Re\left\{\frac{z(J_{c,1}(f))'}{J_{c,1}(g)}\right\} < \beta,$$

where $J_{c,1}$ is the integral operator defined by (1.4).

ACKNOWLEDGEMENT. The authors would like to thank Professor M. Nunokawa for his thoughtful encouragement and much valuable advice in the preparation of this paper. This work was partially supported by Non Directed Research Fund, Korea Research Foundation, 1996 and the Basic Science Research Program, Ministry of Education, Project No. BSRI-96-1440.

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