

LUCAS PARTITIONS

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ABSTRACT. The Lucas sequence is defined by: $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Let $V(n), r(n)$ denote respectively the number of partitions of n into parts, distinct parts from $\{L_n\}$. We develop formulas that facilitate the computation of $V(n)$ and $r(n)$.

KEY WORDS AND PHRASES:

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1. INTRODUCTION

Let S denote a non-empty subset of N , the set of all natural numbers. Let $V(n), r(n), r_E(n), r_0(n)$ denote respectively the number of partitions of n into parts, distinct parts, evenly many distinct parts, oddly many distinct parts from S . Define $V(0) = r(0) = r_E(0) = 1, r_0(0) = 0$. Let $V(n)$ have the generating function:

$$F(z) = \sum_{n=0}^{\infty} V(n)z^n. \quad (1.1)$$

Let

$$1/F(z) = \sum_{n=0}^{\infty} a(n)z^n. \quad (1.2)$$

It follows from (1.1) and (1.2) that

$$\sum_{k=0}^n a(n-k)V(k) = 0 \quad \text{for } n \geq 1. \quad (1.3)$$

Furthermore,

$$a(n) = r_E(n) - r_0(n). \quad (1.4)$$

REMARK. Apostol [1], p.311 and Hardy [3], p.285 prove that (1.4) holds when $S = N$, but the same reasoning applies to the more general case. Since also

$$r(n) = r_E(n) + r_0(n) \quad (1.5)$$

it follows that

$$a(n) = 2r_E(n) - r(n) = r(n) - 2r_0(n). \quad (1.6)$$

In this note, we consider the case where S is the set of all Lucas numbers, L_n , where $n \geq 0$. (The Lucas numbers are defined by: $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$ if $n \geq 2$.) We will show how to compute the $r(n)$ and the $a(n)$: via explicit formulas if $n = L_k$ or $n = 1 + L_{2k+1}$ for some k , recursively otherwise. The $V(n)$ can then be computed recursively via (1.3).

2. PRELIMINARIES

Notation and Definitions

$F_k = k^{\text{th}}$ Fibonacci number ($F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2}$ if $k \geq 2$)

$L_k = k^{\text{th}}$ Lucas number ($L_0 = 2, L_1 = 1, L_k = L_{k-1} + L_{k-2}$ if $k \geq 2$)

$$E(n) = \sum_{k=1}^n |a(k)|$$

$[x]$ denotes the integer part of the real number x

$[a, b]$ denotes the set of all integers, t , such that $a \leq t \leq b$.

In particular, if $k \geq 3$, then

$$I_k = [L_{k+1}, L_{k+2} - 1]$$

$$I_{k,1} = [L_{k+1}, 2L_k - 1]$$

$$I_{k,2} = [2L_k, 5F_k - 1]$$

$$I_{k,3} = [5F_k, L_{k+2} - 1]$$

Lucas Identities

(1) $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $L_0 = 2, L_1 = 1$

(2) $L_j < L_k$ iff $j < k$, unless $j = k - 1 = 0$

(3) $\{L_n\}$ is strictly increasing if $n \geq 1$

(4) $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1}{2}(1 + 5^{\frac{1}{2}}), \beta = \frac{1}{2}(1 - 5^{\frac{1}{2}})$

(5) $L_{2n} = L_n^2 - 2(-1)^n$

(6) $L_n > 1.6^n$ if $n \geq 4$

(7) $\sum_{i=0}^{k+1} L_i = L_{k+3} - 1$

(8) $\sum_{i=0}^{[\frac{1}{2}j]} L_{j-2i} = L_{j+1} + t$, where $t = \begin{cases} 1 & \text{if } 2|j \\ -2 & \text{if } 2 \nmid j \end{cases}$

(9) $L_{n+1} + L_{n-1} = 5F_n$

(10) $L_{k+2} - 5F_k = 2L_k - L_{k+1} = L_{k-2}$

REMARKS. (1) is the definition of the Lucas sequence (2) and (3) follow from (1). (4) follows from (1), using induction and the fact that α, β are the roots of $u^2 - u - 1 = 0$. (5) and (6) follow from (4). (7) through (9) may be proved using induction on n . (10) follows from (1) and (9).

3. THE MAIN THEOREMS

Let n be a natural number. We first address the issue of the representability of n as a sum of distinct Lucas numbers. Such a representation will be called a Lucas representation of n . If in addition, the summands are non-consecutive Lucas numbers, we say that the Lucas representation of n is special. We will show that every natural number has a special Lucas representation. If the special Lucas representation of n is unique, which is usually the case, we call it the minimal Lucas representation of n . Otherwise, n has two special Lucas representations. In this case, we define the minimal Lucas representation of n as the special Lucas representation that does not include $L_0 = 2$ as a summand.

For example, 13 has the unique special (and hence minimal) Lucas representation $13 = 11 + 2 = L_5 + L_0$; 12 has two special Lucas representations: $12 = 11 + 1 = L_5 + L_1$, and $12 = 7 + 3 + 2 = L_4 + L_2 + L_0$. The former is the minimal Lucas representation of 12.

THEOREM 1. Every natural number, n , has a special Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \quad (3.1)$$

where $k_i - k_{i+1} \geq 2$ for all i such that $1 \leq i \leq r - 1$, if $r \geq 2$.

PROOF. (Induction on n) It suffices to consider the case where $n \neq L_k$. Therefore there exists unique $k_1 \geq 3$ such that $L_{k_1} < n < L_{k_1+1}$. Let $n_1 = n - L_{k_1}$. Now (1) implies $0 < n_1 < L_{k_1-1}$. By induction hypothesis, we have $n_1 = L_{k_2} + L_{k_3} + \text{etc.} + L_{k_r}$, with $r \geq 2$ and $k_i - k_{i+1} \geq 2$ for all i such

that $2 \leq i \leq r$, if $r \geq 3$. Thus $L_{k_2} \leq n_1$, hence $L_{k_2} < L_{k_1-1}$. Since $k_1 \geq 3$, (2) implies $k_2 < k_1 - 1$, that is, $k_1 - k_2 \geq 2$. Since $n = L_{k_1} + n_1$, the conclusion now follows.

LEMMA 1. Let $n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s}$, with $s \geq 2$ and $j_i - j_{i+1} \geq 2$ for all i such that $1 \leq i \leq s - 1$. Let $j = j_1$. Then $n \leq L_{j+1} + (-1)^j$. Furthermore, $n = L_{j+1} + (-1)^j$ iff $s = 1 + [\frac{1}{2}j]$, $j_s = 0$, and $j_i = j - 2(i - 1)$ for all i such that $1 \leq i \leq s - 1$.

PROOF. Using (2) and our hypothesis, we have $L_{j_i} \leq r + L_{j-2(i-1)}$ where $r = \begin{cases} 1 & \text{if } j_i = 0 \text{ and } j = 2i - 1 \\ 0 & \text{otherwise} \end{cases}$. Thus we have:

$$n = \sum_{i=1}^s L_{j_i} \leq r + \sum_{i=1}^s L_{j-2(i-1)} = r + \sum_{i=0}^{s-1} L_{j-2i} \leq r + \sum_{i=0}^{[\frac{1}{2}j]} L_{j-2i}. \tag{3.2}$$

Now (8) implies $n \leq L_{j+1} + (-1)^j$. If $s = 1 + [\frac{1}{2}j]$, $j_s = 0$, and $j_i = j - 2(i - 1)$ for all i such that $1 \leq i \leq s - 1$, then the weak inequalities in (3.2) may be replaced with equalities, which yields $n = L_{j+1} + (-1)^j$. Conversely, if $n = L_{j+1} + (-1)^j$, then the weak inequalities in (3.2) become equalities. This implies $s = 1 + [\frac{1}{2}j]$, $L_{j_i} = L_{j-2(i-1)}$ (and hence $j_i = j - 2(i - 1)$) for all i such that $1 \leq i \leq s - 1$, and $j_s = 0$.

LEMMA 2. $L_i = L_j + 1$ iff $(i, j) = (0, 1), (2, 0)$, or $(3, 2)$.

PROOF. Suppose $L_i = L_j + 1$. If $j = 0$, then $L_i = 3$, so $i = 2$. If $j = 1$ and $i < j$, then $i = 0$. Now suppose $i > j \geq 1$. Then (3) and (1) imply $1 = L_i - L_j \geq L_i - L_{i-1} = L_{i-2}$. Therefore $L_{i-2} = 1$, so $i = 3$ and $j = 2$. The converse follows by direct substitution.

LEMMA 3. If

$$n = L_k \tag{3.3}$$

then this special Lucas representation of n is unique.

PROOF. Let k be the least index such that the special representation (3.3) is not unique. By inspection, $k \geq 4$. Thus n has a second special Lucas representation:

$$n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s} \tag{3.4}$$

with $j_i - j_{i+1} \geq 2$ for all i such that $1 \leq i \leq s - 1$. In fact, (2) implies $s \geq 2$. Let $j = j_1$. Now (3.4) implies $L_j < n$, so $L_j < L_k$. If $2 \nmid j$, then Lemma 1 implies $L_k \leq L_{j+1} - 1$, so $L_k < L_{j+1}$. But then $L_j < L_k < L_{j+1}$, an impossibility. If $2 \mid j$, then Lemma 1 implies $L_k \leq L_{j+1} + 1$. Since $k \geq 4$, Lemma 2 implies $L_k \neq L_{j+1} + 1$. Therefore $L_k < L_{j+1} + 1$, so that $L_k \leq L_{j+1}$. Since $L_j < L_k \leq L_{j+1}$, we must have $L_k = L_{j+1}$, hence $k = j + 1$. Now (3.4) yields $L_k = L_{k-\tau} + L_{j_2} + \text{etc.} + L_{j_s}$, hence $L_{k-2} = L_{j_2} + \text{etc.} + L_{j_s}$. By definition of k , we must have $s = 2$, $j_2 = k - 2$. But then $j_1 - j_2 = 1$, an impossibility.

THEOREM 2. Let n have two distinct special Lucas representations:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \quad \text{with } k_i - k_{i+1} \geq 2 \text{ for all } i \\ \text{such that } 1 \leq i \leq r - 1; \tag{3.5}$$

$$n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s} \quad \text{with } j_i - j_{i+1} \geq 2 \text{ for all } i \\ \text{such that } 1 \leq i \leq s - 1. \tag{3.6}$$

Assume also that $j = j_1 < k_1$. Then $k_1 = 2s - 1$, $k_2 = k_r = 1$, and $j_i = 2(s - i)$ for all i with $1 \leq i \leq s$.

PROOF. Let $k_1 = k$. Note that Lemma 3 implies $\text{Min}\{r, s\} \geq 2$. Thus $n \geq 5$ and $j \geq 2$. Let $j = k - m$, where $m \geq 1$. Lemma 1 implies $n \leq L_{j+1} + (-1)^j = L_{k-m+1} + (-1)^{k-m}$. By hypothesis, $L_k < n$, so that $L_k < L_{k-m+1} + (-1)^{k-m}$. If $m \geq 2$, then $L_{k-m+1} \leq L_{k-1}$ by (2), since if $k = 2$, then $j = 1 = s$. Thus $L_k < L_{k-1} + (-1)^{k-m}$, which implies $L_{k-2} < 1$, an impossibility, since $L_n \geq 1$ for all n . Therefore $m = 1$, so $0 < (-1)^{k-1}$ implies k is odd. Since $L_k < n \leq L_k + 1$, we

must have $n = L_k + 1 = L_k + L_1$, so $k_2 = k_r = 1$. Now (3.6) and Lemma 1 imply $j_i = 2(s - i)$ for all i such that $1 \leq i \leq s$.

THEOREM 3. Let n have the special Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r}. \tag{3.7}$$

This special Lucas representation is unique unless $k_r = 1$ and $k_{r-1} = 2h + 1$ for some $h \geq 1$, in which case n has a second special Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_{r-2}} + L_{2h} + L_{2h-2} + \text{etc.} + L_2 + L_0. \tag{3.8}$$

PROOF. If $r = 1$, then the special Lucas representation (3.7) is unique by Lemma 3. If $r \geq 2$, suppose that n has a second special Lucas representation:

$$n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s}. \tag{3.9}$$

Again, by Lemma 3, $s \geq 2$. If $j_1 < k_1$, then the conclusion follows from Theorem 2, with $r = 2$ and $h = s$. Now suppose that $j_i = k_i$ for all i such that $1 \leq i \leq u - 1$ (for some $u \geq 2$), but $j_u < k_u$. Let

$$m = n - \sum_{i=1}^{u-1} L_{k_i} = L_{k_u} + L_{k_{u+1}} + \text{etc.} + L_{k_r};$$

also

$$m = n - \sum_{i=1}^{u-1} L_{j_i} = L_{j_u} + L_{j_{u+1}} + \text{etc.} + L_{j_s}.$$

Now Theorem 2 implies $j_{u+1} = 2(s - u - i)$ for all i such that $0 \leq i \leq s - u$, $u = r - 1$, $k_u = k_{r-1} = 2(s - u) - 1 = 2(s - r) + 1$, $k_r = 1$. The conclusion now follows from Theorem 2, with $h = s - r$.

Combining the results of Theorems 1, 2, and 3, we have:

THEOREM 4. Every natural number, n , has a unique minimal Lucas representation:

$$n = L_{k_1} + L_{k_2} + \dots + L_{k_{r-1}} + L_{k_r} \tag{3.10}$$

where (i) $k_i - k_{i+1} \geq 2$ for all i such that $1 \leq i \leq r - 1$, if $r \geq 2$; (ii) if $r \geq 2$ and $k_r = 0$, then $k_{r-1} \geq 3$.

LEMMA 4. Let n have the minimal Lucas representation given by (3.10) in Theorem 4 above. Then $L_{k_1} < n < L_{k_1+1}$, if $r \geq 2$.

PROOF. (Induction on r) Clearly, $L_{k_1} < n$, so it suffices to show that $n < L_{k_1+1}$. Let $r = 2$, so $n = L_{k_1} + L_{k_2}$. If $k_2 \geq 1$, then by hypothesis, $k_2 \leq k_1 - 2$, so (2) implies $L_{k_2} \leq L_{k_1-2} < L_{k_1-1}$. Thus $n \leq L_{k_1} + L_{k_1-2} < L_{k_1} + L_{k_1-1} = L_{k_1+1}$. If $k_2 = 0$, then by (1) and (2), we have $n = L_{k_1} + L_{k_2} = L_{k_1} + L_0 = L_{k_1} + 2 < L_{k_1} + 3 = L_{k_1} + L_2 \leq L_{k_1} + L_{k_1-1} = L_{k_1+1}$, so $n < L_{k_1+1}$. If $r \geq 3$, let $n_1 = n - L_{k_1} = L_{k_2} + \text{etc.} + L_{k_r}$. Clearly, this is a minimal Lucas representation of n_1 , so by induction hypothesis, we have $n_1 < L_{k_2+1}$, hence $n < L_{k_1} + L_{k_2+1}$. Since $1 \leq k_2 + 1 \leq k_1 - 1$ by hypothesis, (2) implies $L_{k_2+1} \leq L_{k_1-1}$. Therefore $n < L_{k_1} + L_{k_1-1} = L_{k_1+1}$.

The three following theorems permit the computation of $r(L_n)$ and $a(L_n)$.

THEOREM 5. $r(L_n) = \lfloor \frac{1}{2}(n + 2) \rfloor$ if $n \geq 0$.

PROOF. (Induction on n) The statement is true by inspection if $n = 0$ or 1 . If $n \geq 2$, and if L_n is partitioned into several distinct parts, then (7) implies that the largest part must be L_{n-1} . Therefore, by (1), we have $r(L_n) = 1 + r(L_{n-2}) = 1 + \lfloor \frac{1}{2}n \rfloor$ (by induction hypothesis) $= \lfloor \frac{1}{2}(n + 2) \rfloor$. (The "1" in the last equation arises from the trivial partition: $L_n = L_n$.)

THEOREM 6. $r_E(L_n) = \lfloor \frac{1}{4}(n + 2) \rfloor$ if $n \geq 0$.

PROOF. (Induction on n) The statement is true by inspection if $n = 0$ or 1 . If $n \geq 2$, then reasoning as in the proof of Theorem 5, we have $r_E(L_n) = r_0(L_{n-2}) = r(L_{n-2}) - r_E(L_{n-2}) = [\frac{1}{2}n] - [\frac{1}{4}n] = [\frac{1}{4}(n+2)]$ by Theorem 5 and induction hypothesis.

THEOREM 7. $a(L_n) = \begin{cases} 0 & \text{if } n \equiv 2, 3 \pmod{4} \\ -1 & \text{if } n \equiv 0, 1 \pmod{4} \end{cases}$

PROOF. This follows from (1.6) and from Theorems 5 and 6.

Having settled the case where n is a Lucas number, we now consider the case where n is a sum of two or more distinct, non-consecutive Lucas numbers. Then, by Theorem 4, n has a unique minimal Lucas representation:

$$n = \sum_{k=1}^r L_{k_i} \tag{3.11}$$

where $r \geq 2$, $k_i - k_{i+1} \geq 2$ for all i such that $1 \leq i \leq r - 1$, and if $k_r = 0$, then $k_{r-1} \geq 3$.

Alternatively, we could write:

$$n = \sum_{j=0}^s c_j L_j \tag{3.12}$$

where (i) $c_s = 1$; (ii) $c_j = 0$ or 1 for all j such that $0 \leq j \leq s - 1$; (iii) $c_{j-1}c_j = 0$ for all j such that $1 \leq j \leq s$; (iv) if $c_0 = 1$, then $c_2 = 0$.

If we omit the conditions (iii) and (iv), then (3.12) corresponds to a Lucas representation of n . The c_j will be called the digits of the representation.

Referring again to (3.11), let $n_1 = n - L_{k_1} > 0$, $n_2 = n_1 - L_{k_2} \geq 0$. Given any Lucas representation of n , define the initial segment as the first $k_1 - k_2$ digits; define the terminal segment as the remaining digits. In the minimal Lucas representation of n , the initial segment consists of a 1 followed by $k_1 - k_2 - 1$ 0's, and corresponds to the minimal Lucas representation of $L_{k_1 - k_2 - 1}$, while the terminal segment corresponds to the minimal Lucas representation of n_1 . Lucas representations of n may be obtained as follows:

Type I. Arbitrary combinations of Lucas representations of the integers corresponding to the initial and terminal segments in the minimal Lucas representation of n , namely $L_{k_1 - k_2 - 1}$ and n_1 . Clearly, the number of Type I Lucas representations of n is $r(L_{k_1 - k_2 - 1})r(n_1) = [\frac{1}{2}(k_1 - k_2 + 1)]r(n_1)$.

Type II. Suppose that in a non-minimal Lucas representation of n , the initial segment ends in 10, while the terminal segment starts with 0. If this block of digits, consisting of 100, is replaced by 011, then a new Lucas representation of n is obtained. A necessary condition for the existence of Type II Lucas representations is that $2|(k_1 - k_2)$.

Type III. In the minimal Lucas representation of n , if $k_r = 1$ and $k_{r-1} = 2h + 1$ for some $h \geq 1$, then by Theorem 3, a new Lucas representation of n is obtained by replacing $L_{2h+1} + L_1$ by $L_{2h} + L_{2h-2} + \dots + L_2 + L_0$.

The three following theorems enable us to compute $r(1 + L_{2k+1})$ and $a(1 + L_{2k+1})$.

THEOREM 8. If $k \geq 1$, then $r(L_{2k+1} + 1) = k + 1$.

PROOF. Let $n = L_{2k+1} + 1 = L_{2k+1} + L_1$. Here $n_1 = L_1$, so the number of Type I Lucas representations of n is $r(L_{2k+1})r(L_1) = [\frac{2k+1}{2}][\frac{3}{2}] = k$. Since L_1 has no Lucas representation but the minimal one, there are no Type II Lucas representations of n . By hypothesis and Theorem 3, there is a unique Type III Lucas representation of n . Therefore $r(L_{2k+1} + 1) = k + 1$.

THEOREM 9. If $k \geq 1$, then $r_E(L_{4k+1} + 1) = k$; $r_E(L_{4k-1} + 1) = k + 1$.

PROOF. Let $n = L_{2j+1} + 1 = L_{2j+1} + L_1$. As in the proof of Theorem 8, n has no Type II Lucas representations. A Type I Lucas representation has an even number of terms iff its initial segment has an odd number of terms. Therefore the number of such Type I Lucas representations of n is $r_0(L_{2j-1}) = r(L_{2j-1}) - r_E(L_{2j-1}) = [\frac{1}{2}(2j+1)] - [\frac{1}{4}(2j+1)] = j - [\frac{1}{4}(2j+1)]$ by (1.5) and Theorems 5 and 6. Whether $j = 2k$ or $2k - 1$, the number of Type I Lucas representations of n with evenly many terms is k , since $2k - [\frac{1}{4}(4k+1)] = k = (2k-1) - [\frac{1}{4}(4k-1)]$. The unique Type III

Lucas representation of n has $j+1$ terms, and thus contributes to $r_E(L_{2j+1}+1)$ iff j is odd. The conclusion now follows.

THEOREM 10. If $k \geq 1$, then $a(L_{4k+1}+1) = -1$; $a(L_{4k-1}+1) = 2$.

PROOF. This follows from (1.6) and from Theorems 8 and 9.

In Theorems 11, 12, and 13 below, we develop formulas for $r(n)$, $r_E(n)$, and $a(n)$ in the case where $n \neq L_k$, $n \neq L_{2k+1}+1$. In order to do so, we must be able to count the number of Type II Lucas representations of n . We therefore need to determine the number of Lucas representations of n that do not include the largest possible Lucas number as a part. This question is addressed by Lemma 5.

LEMMA 5. Let n have the minimal Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r}.$$

Let $n_1 = n - L_{k_1} \geq 0$. Let $\bar{r}(n)$ denote the number of Lucas representations of n that do not include L_{k_1} as a part; let $r_E(n)$, $r_0(n)$ denote respectively the number of such representations consisting of evenly, oddly many parts. Then

$$\bar{r}(n) = r(n) - r(n_1); \quad (3.13)$$

$$\bar{r}_E(n) = r_E(n) - r_0(n_1); \quad (3.14)$$

$$\bar{r}_0(n) = r_0(n) - r_0(n) - r_E(n_1). \quad (3.15)$$

PROOF. It follows from the definitions of $\bar{r}(n)$, $\bar{r}_E(n)$, $\bar{r}_0(n)$ that $r(n) - \bar{r}(n)$ is the number of Lucas representations of n that do include L_{k_1} as a part; $r_E(n) - \bar{r}_E(n)$, $r_0(n) - \bar{r}_0(n)$ are respectively the number of such representations consisting of evenly, oddly many parts. If $n \neq L_k$ let

$$n = L_{k_1} + L_{j_2} + \text{etc.} + L_{j_s} \quad (\text{with } s \geq 2) \quad (3.16)$$

be a Lucas representation of n that includes L_{k_1} as a part. (It follows from Lemma 4 that L_{k_1} is the largest part.) Corresponding to (3.16), there is a Lucas representation of n_1 :

$$n_1 = L_{j_2} + \text{etc.} + L_{j_s}. \quad (3.17)$$

This correspondence is clearly a bijection, so that $r(n) - \bar{r}(n) = r(n_1)$. Furthermore, the number of parts in (3.16) and (3.17) differ in parity. Therefore $r_E(n) - \bar{r}_E(n) = r_0(n_1)$ and $r_0(n) - \bar{r}_0(n) = r_E(n_1)$. The conclusions (3.13), (3.14), (3.15) now follow if $n \neq L_k$. If $n = L_k$, so that $n_1 = 0$, then clearly no other Lucas representation of n includes L_k as a part. Therefore $\bar{r}(n) = r(n) - 1 = r(n) - r(0) = r(n) - r(n_1)$. Furthermore, $\bar{r}_E(n) = r_E(n) - r_E(n) - 0 = r_E(n) - r_0(n_1)$; $\bar{r}_0(n) = r_0(n) - 1 = r_0(n) - r_E(n_1)$.

THEOREM 11. Let n have the minimal Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \quad (3.18)$$

where (i) $r \geq 2$; (ii) if $k_r = 1$, then $2|k_{r-1}$. Let $n_1 = n - L_{k_1}$, $n_2 = n_1 - L_{k_2} \geq 0$. Then

$$r(n) = \begin{cases} \frac{1}{2}(k_1 - k_2 + 1)r(n_1) & \text{if } 2 \nmid (k_1 - k_2) \\ \left(1 + \frac{1}{2}(k_1 - k_2)\right)r(n_1) - r(n_2) & \text{if } 2|(k_1 - k_2) \end{cases}$$

PROOF. By hypothesis, there are no Type III Lucas representations of n . As mentioned earlier, the number of Type I Lucas representations of n is $\left[\frac{1}{2}(k_1 - k_2 + 1)\right]r(n_1)$. If $2 \nmid (k_1 - k_2)$, then there are no Type II Lucas representations of n , so that $r(n) = \left[\frac{1}{2}(k_1 - k_2 + 1)\right]r(n_1) = \frac{1}{2}(k_1 - k_2 + 1)r(n_1)$. If $2|(k_1 - k_2)$, then the number of Type II Lucas representations of n is the number of Lucas representations of n_1 that do not include L_{k_2} as a part, namely $\bar{r}(n_1)$. By Lemma 5, we have $\bar{r}(n_1) = r(n_1) - r(n_2)$. Therefore

$$r(n) = \left[\frac{1}{2}(k_1 - k_2 + 1)\right]r(n_1) + r(n_1) - r(n_2) = \left(1 + \frac{1}{2}(k_1 - k_2)\right)r(n_1) - r(n_2).$$

COROLLARY 1. If $n \geq 2$, then $r(L_n - 3) = \lfloor \frac{1}{2} n \rfloor$.

PROOF. (Induction on n) By inspection, the conclusion is true if $2 \leq n \leq 5$. If $n \geq 6$, then $L_n - 3 = L_{n-1} + (L_{n-2} - 3) = L_{n-1} + L_{n-3} + (L_{n-4} - 3)$. Now Theorem 11 implies $r(L_n - 3) = 2r(L_{n-2} - 3) - r(L_{n-4} - 3)$. By induction hypothesis, we have $r(L_{n-2} - 3) = \lfloor \frac{1}{2}(n-2) \rfloor = \lfloor \frac{1}{2} n \rfloor - 1$, and $r(L_{n-4} - 3) = \lfloor \frac{1}{2}(n-4) \rfloor = \lfloor \frac{1}{2} n \rfloor - 2$. Therefore $r(L_n - 3) = 2(\lfloor \frac{1}{2} n \rfloor - 1) - (\lfloor \frac{1}{2} n \rfloor - 2) = \lfloor \frac{1}{2} n \rfloor$.

COROLLARY 2. If $n \geq 1$, then $r(L_n^2 - 1) = n$.

PROOF. If $m \geq 1$, then via (5), we have $r(L_{2m}^2 - 1) = r(L_{4m} + 1) = r(L_{4m} + L_1)$. Now Theorem 11 implies $r(L_{2m}^2 - 1) = (2m)r(L_1) = (2m)1 = 2m$. again, via (5), we have $r(L_{2m-1}^2 - 1) = r(L_{4m-2} - 3)$. Now Corollary 1 implies $r(L_{2m-1}^2 - 1) = \lfloor \frac{1}{2}(4m - 2) \rfloor = 2m - 1$.

REMARK. Corollaries 1 and 2 imply (independently) that the function $r(n)$ is a surjection from N to N .

THEOREM 12. Let $n \neq L_k, n \neq L_{2k+1} + 1$. Then

$$r_E(n) = \begin{cases} \frac{1}{4}(k_1 - k_2 + 1)r(n_1) & \text{if } k_1 - k_2 \equiv 3 \pmod{4} \\ \frac{1}{4}(k_1 - k_2 + 3)r(n_1) - r_E(n_1) & \text{if } k_1 - k_2 \equiv 1 \pmod{4} \\ \frac{1}{4}(k_1 - k_2 + 2)r(n_1) - r_0(n_2) & \text{if } k_1 - k_2 \equiv 2 \pmod{4} \\ \left(1 + \frac{1}{4}(k_1 - k_2)\right)r(n_1) - r_E(n_1) - r_E(n_2) & \text{if } k_1 - k_2 \equiv 0 \pmod{4} \end{cases}$$

PROOF. By hypothesis and Theorem 3, any Type III Lucas representation of n must arise from a corresponding Type III Lucas representation of $n_1 = n - L_{k_1}$. Thus it suffices to count the Type I and II Lucas representations of n consisting of evenly many parts. A Type I Lucas representation of n with evenly many parts will occur whenever the initial and terminal segments agree in parity. Therefore the number of such representations is given by:

$$\begin{aligned} r_E(L_{k_1 - k_2 - 1})r_E(n_1) + r_0(L_{k_1 - k_2 - 1})r_0(n_1) = \\ \left[\frac{1}{2}(k_1 - k_2 + 1) \right] r_E(n_1) + \left(\left[\frac{1}{2}(k_1 - k_2 + 1) \right] - \left[\frac{1}{4}(k_1 - k_2 + 1) \right] \right) (r(n_1) - r_E(n_1)) = \\ \left(\left[\frac{1}{2}(k_1 - k_2 + 1) \right] - \left[\frac{1}{4}(k_1 - k_2 + 1) \right] \right) r(n_1) + \left(2 \left[\frac{1}{4}(k_1 - k_2 + 1) \right] - \left[\frac{1}{2}(k_1 - k_2 + 1) \right] \right) r_E(n_1). \end{aligned}$$

If $2 \nmid (k_1 - k_2)$, then no Type II Lucas representations of n can arise. In particular, if $k_1 - k_2 \equiv 3 \pmod{4}$, by simplifying the last formula, we obtain $r_E(n) = \frac{1}{4}(k_1 - k_2 + 1)r(n_1)$. Similarly, if $k_1 - k_2 \equiv 1 \pmod{4}$, we obtain $r_E(n) = \frac{1}{4}(k_1 - k_2 + 3)r(n_1) - r_E(n_1)$. If $2 \mid (k_1 - k_2)$, we wish to count the number of Type II Lucas representations on n that have evenly many terms. Each such representation originates from a Lucas representation of n whose initial segment has $\frac{1}{2}(k_1 - k_2)$ terms, and whose terminal segment's number of terms therefore differs in parity from $\frac{1}{2}(k_1 - k_2)$. If $k_1 - k_2 \equiv 2 \pmod{4}$, then the number of Type II Lucas representations of n is $\bar{r}_E(n_1) = r_E(n_1) - r_0(n_2)$, by Lemma 5. In this case, the number of Type I Lucas representations of n is $\frac{1}{4}(k_1 - k_2 + 2) - r_E(n_1)$. Thus we obtain: $r_E(n) = \frac{1}{4}(k_1 - k_2 + 2)r(n_1) - r_0(n_2)$. If $k_1 - k_2 \equiv 0 \pmod{4}$, then the number of Type II Lucas representations of n is $\bar{r}_0(n_1) = r_0(n_1) - r_E(n_2)$, by Lemma 5. In this case, the number of Type I Lucas representations of n is $\frac{1}{4}(k_1 - k_2)r(n_1)$. Therefore we obtain:

$$r_E(n) = \frac{1}{4}(k_1 - k_2)r(n_1) + r_0(n_1) - r_E(n_2) = \left(1 + \frac{1}{4}(k_1 - k_2)\right)r(n_1) - r_E(n_1) - r_E(n_2).$$

THEOREM 13. Let $n \neq L_k, n \neq L_{2k+1} + 1$. Then

$$a(n) = \begin{cases} -a(n_1) - a(n_2) & \text{if } k_1 - k_2 \equiv 0 \pmod{4} \\ -a(n_1) & \text{if } k_1 - k_2 \equiv 1 \pmod{4} \\ a(n_2) & \text{if } k_1 - k_2 \equiv 2 \pmod{4} \\ 0 & \text{if } k_1 - k_2 \equiv 3 \pmod{4} \end{cases}$$

PROOF. This follows from (1.6) and from Theorems 11 and 12.

LEMMA 6. If $n \geq 1$, then

$$a(L_n - 1) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{if } n \equiv 0, 3 \pmod{4} \end{cases}$$

PROOF. (Induction on n) The conclusion holds by inspection if $1 \leq n \leq 4$. If $n \geq 5$, then $L_n - 1 = L_{n-1} + (L_{n-2} + 1) = L_{n-1} + L_{n-3} + (L_{n-4} - 1)$. Now Theorem 13 implies $a(L_n - 1) = a(L_{n-4} - 1)$, so the conclusion follows from the induction hypothesis.

LEMMA 7. If $n \geq 1$, then $a(2L_n - 1) = (-1)^n$.

PROOF. The conclusion holds by inspection if $1 \leq n \leq 4$. If $n \geq 5$, then $2L_n - 1 = L_{n+1} + (L_{n-2} - 1) = L_{n+1} + L_{n-3} + (L_{n-4} - 1)$. Now Theorem 13 implies $a(2L_n - 1) = -a(L_{n-2} - 1) - a(L_{n-4} - 1)$. The conclusion now follows from Lemma 6.

LEMMA 8. If $j \leq n \leq L_{k-3} - 1$, then $a(2L_k + n) = 0$.

PROOF. $a(2L_k + n) = a(L_{k+1} + L_{k-2} + n)$ by (1), so the conclusion follows from the hypothesis and Theorem 13.

LEMMA 9. If $0 \leq n \leq L_{k-2} - 1$, then $a(5F_k + n) = a(n)$.

PROOF. By (9), we have $a(5F_k + n) = a(L_{k+1} + L_{k-1} + n)$. The conclusion now follows from the hypothesis and Theorem 13.

THEOREM 14. If n belongs to I_k , where $k \geq 2$, and $m = L_{k+3} - 1 - n$, then (i) $r(m) = r(n)$ and (ii) $a(m) = (-1)^k a(n)$.

PROOF. It is easily seen that m belongs to I_k iff n does. Now (7) implies there is a bijection between the partitions of m, n respectively into distinct Lucas parts. Thus $r(m) = r(n)$. Furthermore, since the left side of (7) has $k + 2$ terms, it follows that under this bijection, corresponding partitions of m and n will have numbers of parts that agree or disagree in parity accordingly as k is even or odd. Therefore $a(m) = (-1)^k a(n)$.

COROLLARY 3. If $k \geq 1$, then $a(L_{4k-2} - 2) = -1$; $a(L_{4k} - 2) = 2$.

PROOF. This follows from Theorems 10 and 14.

THEOREM 15. If $k \geq 3$, then $E(L_{k+2} - 1) = E(L_{k+1} - 1) + 2E(L_{k-2} - 1) + 2$.

PROOF. If $1 \leq i \leq 3 \leq k$, let $x_{k,i} = \Sigma\{|a(n)| : n \in I_{k,i}\}$. Thus $x_{k,1} + x_{k,2} + x_{k,3} = E(L_{k+2} - 1) - E(L_{k+1} - 1)$. Now $x_{k,3} = \Sigma\{|a(n)| : 5F_k \leq n \leq L_{k+2} - 1\} = E(L_{k+2} - 1) - E(5F_k - 1)$. But Lemma 9 implies $a(5F_k) = a(0) = 1$, so $E(5F_k) = 1 + E(5F_k - 1)$. Thus $x_{k,3} = E(L_{k+2} - 1) - E(5F_k) + 1$. But (10) implies $L_{k+2} = 5F_k + L_{k-2}$, so $x_{k,3} = E(5F_k + L_{k-2} - 1) - E(5F_k) + 1$. Now Lemma 9 implies $x_{k,3} = E(L_{k-2} - 1) + 1$. Also, $x_{k,2} = \Sigma\{|a(n)| : 2L_k \leq n \leq 5F_k - 1\} = 0$ by Lemma 8. Now $x_{k,1} = \Sigma\{|a(n)| : L_{k+1} \leq n \leq 2L_k - 1\}$. Theorem 14 implies that $x_{k,1} = \Sigma\{|a(n)| : 5F_k \leq n \leq L_{k+2} - 1\} = x_{k,3}$. Thus we have: $E(L_{k+2} - 1) - E(L_{k+1} - 1) = 2(1 + E(L_{k-2} - 1))$, from which the conclusion follows.

THEOREM 16. If $k \geq 2$ and if $L_{k+1} \leq n \leq L_{k+2} - 1$, then

$$E(n) = \begin{cases} E(2L_k) - E(2L_k - 2 - n) - 1 & \text{if } L_{k+1} \leq n \leq 2L_k - 2 \\ E(2L_k) & \text{if } 2L_k - 1 \leq n \leq 5F_k - 1 \\ E(2L_k) + E(n - 5F_k) + 1 & \text{if } 5F_k \leq n \leq L_{k+2} - 1 \end{cases}$$

PROOF. If $2L_k - 1 \leq n \leq 5F_k - 1$, then Lemma 8 implies $E(n) = E(2L_k)$. If $5F_k \leq n \leq L_{k+2} - 1$, then $E(n) - E(5F_k) = \sum_{j=1}^{n-5F_k} |a(5F_k + j)| = \sum_{j=1}^{n-5F_k} |a(j)| = E(n - 5F_k)$ by Lemma 9. Also, Lemmas 8 and 9 imply $E(5F_k) = 1 + E(2L_k)$, so $E(n) = 1 + E(2L_k) + E(n - 5F_k)$. Finally, if $L_{k+1} \leq n \leq 2L_k - 2$, let $m = 2L_k - n$, so that $2 \leq m \leq L_{k-2}$. We must show that $E(2L_k - m) = E(2L_k) - E(m - 2) - 1$. Now Lemmas 8 and 9 imply $a(2L_k) = 0$ and $|a(2L_k - 1)| = 1$. Therefore $E(2L_k) = E(2L_k - 1) = 1 + E(2L_k - 2)$. Thus it suffices to show that

$E(2L_k - m) = E(2L_k - 2) - E(m - 2)$ when $2 \leq m \leq L_{k-2}$. This is trivially true when $m = 2$. If $3 \leq m \leq L_{k-2}$, then $L_{k+1} + 3 \leq 2L_k - m \leq 2L_k - 3$, so that by (1), (9), Lemma 9 and Theorem 14, we have $|a(2L_k - m)| = |a(L_{k+3} - 1 - 2L_k + m)| = |a(5F_k + m - 1)| = |a(m - 1)|$. Therefore $E(2L_k - 2) - E(2L_k - m) = \sum_{j=2}^{m-1} |a(2L_k - j)| = \sum_{j=2}^{m-1} |a(j-1)| = \sum_{i=1}^{m-2} |a(i)| = E(m-2)$, so we are done.

THEOREM 17. $\lim_{n \rightarrow \infty} \frac{E(n)}{n} = 0$.

PROOF. If $k \geq 2$, let $t_k = \max\{E(n)/n : n \in I_k\}$. It suffices to show that $\lim_{k \rightarrow \infty} t_k = 0$. If $n \in I_k$, then by Theorem 16, we have: $E(n) \leq E(2L_k) + E(n - 5F_k) + 1$. Since $E(n)$ is non-decreasing and $n - 5F_k \leq L_{k-2}$, it follows that $E(n) \leq E(2L_k) + E(L_{k-2}) + 1$. By Theorem 16, we have $E(L_{k+1}) = E(2L_k) - E(L_{k-2} - 2) - 1$, so we obtain $E(n) \leq E(L_{k+1}) + E(L_{k-2}) + E(L_{k-2} - 2) + 2$, hence $E(n) \leq E(L_{k+1}) + 2E(L_{k-2})$. Since $n \leq L_{k+1}$, we get $\frac{E(n)}{n} \leq \frac{E(L_{k+1})}{L_{k+1}} + 2 \frac{E(L_{k-2})}{L_{k+1}}$, so that $t_k \leq E(L_{k+1})/L_{k+1} + 2E(L_{k-2})/L_{k+1}$. Since $E(n)$ is non-decreasing and L_k tends to infinity with k , it suffices to show that $\lim_{k \rightarrow \infty} E(L_k)/L_k = 0$. In fact, since $E(L_k) \leq 1 + E(L_k - 1)$, it suffices to show that $\lim_{k \rightarrow \infty} E(L_k - 1)/L_k = 0$. If $k \geq 1$, let $c_k = E(L_k - 1)$. Thus $c_1 = 0, c_2 = c_3 = 2, c_4 = 4$. By Theorem 15, we have: $c_{k+2} = c_{k+1} + 2c_{k-2} + 2$. Let the $\{c_k\}$ have the generating function: $F(z) = \sum_{k=1}^{\infty} c_k z^k$. Using the method of [2], p. 337-350, we obtain: $F(z) = (2z^2 - 2z^3 + 2z^4)/(1 - z^2)(1 - 2z + 2z^2 - 2z^3)$. Therefore $c_k = b_1 t_1^k + b_2 t_2^k + b_3 t_3^k + b_4 + b_5(-1)^k$, where the b_i are constants, and the t_i are the roots of the equation: $x^3 - 2x^2 + 2x - 2 = 0$. Using Cardan's formula, if $u = \frac{1}{3}(17 + 3\sqrt{33})^{1/3}, v = \frac{1}{3}(17 - 3\sqrt{33})^{1/3}$, then $t_1 = \frac{2}{3} + u + v \doteq 1.544, t_2 = \frac{2}{3} - \frac{1}{2}(u + v) + \frac{i\sqrt{3}}{2}(u - v), t_3 = \bar{t}_2$. Thus $|t_2| = |t_3| \doteq 1.138, b_2 = b_3$, and $|b_2 t_2^k + b_3 t_3^k| = |2b_2 Re(t_2^k)| \leq 2|b_2| |t_2^k|$. Now $0 < |c_k/L_k| \leq (|b_1|1.544^k + 2|b_2|1.138^k + |b_4| + |b_5|)/L_k$. But (6) implies that the right side of the last inequality tends to 0 as k tends to infinity. Therefore $\lim_{k \rightarrow \infty} c_k/L_k = 0$, we are done.

THEOREM 18. $a(n)$ assumes each of the values $0, \pm 1, \pm 2$ infinitely often.

PROOF. Theorem 7 implies $a(n) = 0, -1$ infinitely often, while Theorem 10 implies $a(n) = 2$ infinitely often. By Theorems 13 and 7, $a(L_{4k+5} + L_{4k}) = -a(L_{4k}) = 1$. Therefore $a(n) = 1$ infinitely often. Finally, with $k \geq 2$, let $n = L_{4k} + 5 = L_{4k} + L_3 + L_1$. Now Theorem 13 implies $a(n) = -a(5) = -2$. Therefore $a(n) = -2$ infinitely often.

THEOREM 19. $|a(n)| \leq 2$ for all n .

PROOF. If $|a(n)| \geq 3$ for some n , let n be the least such integer. By Theorems 7 and 10, $n \neq L_k, L_{2k+1} + 1$. Let n have the minimal Lucas representation:

$$n = n_0 = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \text{ where } r \geq 2; \text{ let } n_i = n_{i-1} - L_{k_i}$$

for $1 \leq i \leq r$, with $n_r = 0$. By hypothesis and Theorem 13, we must have: $a(n) = -(a(n_1) + a(n_2))$, with $k_1 - k_2 \equiv 0 \pmod{4}$. By Theorem 13 implies

$$a(n_1) + a(n_2) = \begin{cases} -a(n_3) & \text{if } k_2 - k_3 \equiv 0 \pmod{4} \\ 0 & \text{if } k_2 - k_3 \equiv 1 \pmod{4} \\ a(n_2) + a(n_3) & \text{if } k_2 - k_3 \equiv 2 \pmod{4} \\ a(n_2) & \text{if } k_2 - k_3 \equiv 3 \pmod{4} \end{cases}$$

Therefore $a(n) = -(a(n_2) + a(n_3))$, with $k_2 - k_3 \equiv 2 \pmod{4}$. If we apply Theorem 13 repeatedly, we eventually get $a(n) = -(a(n_{r-1}) + a(n_r)) = -(a(L_{k_r}) + a(0))$. Now Theorem 7 implies $|a(n)| \leq 1$, contrary to hypothesis.

ACKNOWLEDGMENT. Theorems 14 through 17 are Lucas analogues of results about Fibonacci partitions announced by Weinstein in [4]. For each integer n , such that $0 \leq n \leq 100$, Table 1 lists $r(n), r_E(n), a(n), E(n)$, and $V(n)$.

Table 1

n	$r(n)$	$r_E(n)$	$a(n)$	$E(n)$	$V(n)$	n	$r(n)$	$r_E(n)$	$a(n)$	$E(n)$	$V(n)$
0	1	1	1	0	1	51	6	3	0	30	6308
1	1	0	-1	1	1	52	6	2	-2	32	6877
2	1	0	-1	2	2	53	6	3	0	32	7491
3	2	1	0	2	3	54	8	4	0	32	8155
4	2	1	0	2	5	55	5	3	1	33	8862
5	2	2	2	4	6	56	5	3	1	34	9622
6	2	1	0	4	9	57	7	3	-1	35	10438
7	3	1	-1	5	12	58	6	3	0	35	11316
8	2	1	0	5	16	59	6	3	0	35	12247
9	2	1	0	5	20	60	6	3	0	35	13249
10	3	2	1	6	26	61	6	3	0	35	14319
11	3	1	-1	7	33	62	6	3	0	35	15464
12	3	1	-1	8	41	63	6	3	0	35	16678
13	3	2	1	9	50	64	6	3	0	35	17981
14	4	2	0	9	62	65	7	4	1	36	19369
15	3	2	1	10	75	66	5	2	-1	37	20845
16	3	1	-1	11	90	67	5	2	-1	38	22413
17	3	1	-1	12	107	68	8	4	0	38	24089
18	4	2	0	12	129	69	6	3	0	38	25868
19	3	2	1	13	151	70	6	4	2	40	27754
20	3	2	1	14	178	71	6	3	0	40	29759
21	5	2	-1	15	208	72	7	3	-1	41	31893
22	4	2	0	15	244	73	4	2	0	41	34149
23	4	2	0	15	281	74	4	2	0	41	36541
24	4	2	0	15	326	75	5	3	1	42	39078
25	5	3	1	16	375	76	7	3	-1	43	41771
26	3	1	-1	17	431	77	5	2	-1	44	44609
27	3	1	-1	18	491	78	5	3	1	45	47619
28	4	2	0	18	561	79	8	4	0	45	50802
29	4	2	0	18	638	80	7	4	1	46	54170
30	4	3	2	20	723	81	7	3	-1	47	57715
31	4	2	0	20	816	82	7	3	-1	48	61471
32	6	3	0	20	922	83	9	5	1	49	65434
33	5	2	-1	21	1037	84	6	3	0	49	69613
34	5	2	-1	22	1163	85	6	3	0	49	74013
35	5	3	1	23	1302	86	7	3	-1	50	78664
36	6	3	0	23	1458	87	8	4	0	50	83561
37	4	2	0	23	1624	88	8	5	2	52	88715
38	4	2	0	23	1808	89	8	4	0	52	94140
39	6	3	0	23	2009	90	6	3	0	52	99862
40	5	3	1	24	2231	91	7	3	-1	53	105871
41	5	2	-1	25	2467	92	7	3	-1	54	112190
42	5	2	-1	26	2729	93	7	4	1	55	118835
43	6	3	0	26	3012	94	8	4	0	55	125830
44	4	2	0	26	3321	95	6	3	0	55	133160
45	4	3	2	28	3651	96	6	3	0	55	140867
46	4	2	0	28	4014	97	10	5	0	55	148958
47	5	2	-1	29	4406	98	8	4	0	55	157456
48	4	2	0	29	4828	99	8	4	0	55	166353
49	4	2	0	29	5282	100	8	4	0	55	175400
50	7	4	1	30	5777						

REFERENCES

- [1] APOSTOL, T.M., *Introduction to Analytic Number Theory*, Springer Verlag, 1976.
- [2] GRAHAM, R.L., KNUTH, D.E. and PATASHNIK, O., *Concrete Mathematics*, Addison-Wesley, 2nd Ed., 1994.
- [3] HARDY, G.H. and WRIGHT, E.M., *An Introduction to the Theory of Numbers*, Oxford University Press, 4th Ed., 1960.
- [4] WEINSTEIN, F.V., private communication.

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