

ON A PROBLEM OF COMMUTATIVITY OF AUTOMORPHISMS

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ABSTRACT. In this note we provide a partial answer to a problem proposed by M. Brešar. We prove that if α, β are automorphisms of a commutative prime ring of characteristic not equal to 2 satisfying the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, then either $\alpha = \beta$ or $\alpha = \beta^{-1}$. As a consequence α and β commute and in this situation the equation itself ensures the commutativity of α and β .

KEY WORDS AND PHRASES: Prime ring, automorphism.

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The equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1} \quad (*)$$

where α and β are automorphisms of a von Neumann algebra has been extensively studied. This equation (in case α and β commute) has played an important role in the study of Tomita-Takesaki theory [1,2,3]. Several conditions have been considered where the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ itself implies the commutativity of α and β and thus making the additional assumption that α and β commute as redundant. For instance, it has been shown in [4] that if M is a commutative semisimple Banach algebra and α, β are automorphisms of M satisfying equation (*), then an application of Gelfand's theory implies that α and β commute. Also, it has been shown in [5] that if α and β are $*$ -automorphisms of a C^* -algebra A satisfying equation (*) and if either α or β is inner, then α and β commute. Recently Brešar [6,7] has studied this equation on prime and semiprime rings and has remarkably extended most of the decomposition results of [4,8] on von Neumann algebras about this equation to semiprime and prime rings, using purely algebraic techniques. As an application of Posner's result for (α, β) -derivations, Brešar [6, Corollary 3] has shown the following generalization of a result of Thaheem [4,8].

THEOREM A. *Let R be a prime ring of characteristic not 2. Suppose that automorphisms α, β of R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. If α and β commute then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.*

In [6] Brešar has proposed an open question whether or not the assumption that α and β commute can be removed in Theorem A. In this note we are precisely concerned with this question and provide a partial answer to his problem. We prove that in case R is commutative then the assumption of commutativity of α and β can indeed be removed from Theorem A. We prove the following theorem:

THEOREM B. *Let R be a commutative prime ring of characteristic not 2. Suppose that automorphisms α, β of R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.*

PROOF. It follows from the equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1} \quad (1)$$

that for any $x \in R$,

$$(\alpha - \beta)(x^2) = (\beta^{-1} - \alpha^{-1})(x^2). \quad (2)$$

Rewriting (2), we obtain $\alpha(x^2) - \beta(x^2) = \beta^{-1}(x^2) - \alpha^{-1}(x^2)$. That is,

$$(\alpha(x))^2 - (\beta(x))^2 = (\beta^{-1}(x))^2 - (\alpha^{-1}(x))^2. \quad (3)$$

Since R is commutative, then using (1) we may rewrite (3) as

$$(\alpha(x) - \beta(x))(\alpha(x) + \beta(x)) = (\alpha(x) - \beta(x))(\beta^{-1}(x) + \alpha^{-1}(x))$$

or what is same

$$(\alpha(x) - \beta(x))(\alpha(x) + \beta(x) - \beta^{-1}(x) - \alpha^{-1}(x)) = 0. \quad (4)$$

By equation (1), we may rewrite (4) as

$$(\alpha(x) - \beta(x))(\beta(x) + \beta^{-1}(x) - \alpha^{-1}(x) + \beta(x) - \beta^{-1}(x) - \alpha^{-1}(x)) \quad (5)$$

or equivalently

$$2(\alpha(x) - \beta(x))(\beta(x) - \alpha^{-1}(x)) = 0. \quad (6)$$

In view of the commutativity of R , equation (6) implies that for any $y \in R$, $2(\alpha(x) - \beta(x))y(\beta(x) - \alpha^{-1}(x)) = 0$. Since R is prime and characteristic of R is not 2, therefore we have $\alpha(x) - \beta(x) = 0$ or $\beta(x) - \alpha^{-1}(x) = 0$ for any $x \in R$. Thus either $\alpha = \beta$ or $\alpha = \beta^{-1}$. This completes the proof.

It follows from the conclusion of the above theorem that α and β commute. In other words, the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ ensures the commutativity of α and β . It would be interesting to resolve the problem for certain types of noncommutative prime rings.

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