Internat. J. Math. & Math. Sci.
VOL. 21 NO. 3 (1998) 571-580

A SAMPLING THEOREM ASSOCIATED WITH BOUNDARY-VALUE PROBLEMS WITH NOT NECESSARILY SIMPLE EIGENVALUES

MAHMOUD H. ANNABY

Lehrstuhl A für Mathematik, RWTH Aachen, D-52056 Aachen, Germany.

HASSAN A. HASSAN

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.

(Received July 10, 1996 and in revised form March 13, 1997)

ABSTRACT. We use a new version of Kramer's theorem to derive a sampling theorem associated with second order boundary-value problems whose eigenvalues are not necessarily simple.

KEY WORDS AND PHRASES: Kramer's theorem, Lagrange interpolations, eigenvalue problems.

1991 AMS SUBJECT CLASSIFICATION CODES: 34A05, 94A24.

1. INTRODUCTION.

In [3], Kramer derived a sampling theorem which generalizes the Whittaker-Shannon-Kotel'nikov sampling theorem [6, 8, pp. 16–17]. It states that

THEOREM 1.1. Let I be a finite closed interval. Let $K(x,t) : I \times \mathbb{C} \to \mathbb{C}$ be a function such that $K(x,t) \in L^2(I)$, $\forall t \in \mathbb{C}$. Let $\{t_k\}_{k\in\mathbf{Z}}$ be a sequence of real numbers such that $\{K(x,t_k)\}_{k\in\mathbf{Z}}$ is a complete orthogonal set in $L^2(I)$. Let $g \in L^2(I)$ and suppose that

$$f(t) = \int_I K(x,t) g(x)\, dx.$$

Then

$$f(t) = \sum_{k\in\mathbf{Z}} f(t_k) S_k(t),$$

where

$$S_k(t) = \frac{\int_I K(x,t)\overline{K(x,t_k)}\, dx}{\|K(x,t_k)\|^2}.$$

DEFINITION 1.1. A function $K(x,t) : I \times \mathbb{C} \longrightarrow \mathbb{C}$ is called a Kramer-type kernel if $K(x,t) \in L^2(I)$, $\forall t \in \mathbb{C}$ and there exists a sequence $\{t_k\} \subset \mathbb{C}$ such that, $\{K(x,t_k)\}$ is a complete orthogonal set in $L^2(I)$.

The point now is that, where can one find Kramer-type kernels? An answer to this question is given by Kramer [3] as follows:

Consider the self-adjoint boundary value problem

$$Ly = \sum_{i=1}^{n} p_i(x) y^{(n-i)}(x) = ty, \quad x \in I = [a,b], \tag{1.1}$$

$$B_j(y) = \sum_{i=1}^{n} \left[\alpha_{ij} y^{(i-1)}(a) + \beta_{ij} y^{(i-1)}(b)\right] = 0, \quad j = 1, 2, \dots, n \tag{1.2}$$

Assume that $u(x,t)$ is a solution of (1.1) such that the zeros, $\{t_k\}$, of $B_j(u(x,t))$ are the same $\forall j$. Thus, [3], the zeros of $B_j(u(x,t))$ are the eigenvalues of the problem (1.1)-(1.2), and $\{u(x,t_k)\}$ is a complete orthogonal set of eigenfunctions. Then

THEOREM 1.2. Let $L(y) = ty$, $B_j(y) = 0$, $j = 1, \ldots, n$, be a self-adjoint boundary value problem on I. Suppose that there exists a solution $u(x,t)$ of (1.1) such that the set of zeros $E_i = \{t_k\}$ of $tB_i(u(x,t))$ is independent of i. Let $g \in L^2(I)$. If

$$f(t) = \int_I u(x,t)g(x)\,dx,$$

then, f has the representation

$$f(t) = \sum_{k\in\mathbf{Z}} f(t_k)S_k(t),$$

where

$$S_k(t) = \frac{\int_I u(x,t)\overline{u(x,t_k)}}{\|u(x,t_k)\|^2}.$$

Kramer's theorem stated above is not always true, since one can find a boundary-value problem of the type (1.1)-(1.2) and a solution $u(x,t)$ such that $B_j(u(x,t))$ has the same zeros $\{t_k\}$, $\forall j$, but neither $\{t_k\}$ is the set of eigenvalues, nor $\{u(x,t_k)\}$ is the complete set of eigenfunctions. For example, consider the boundary value problem

$$-y'' = ty, \quad x \in [0,\pi], \tag{1.3}$$

$$B_1(y) = y(0) - y(\pi) = 0, \quad B_2(y) = y'(0) - y'(\pi) = 0. \tag{1.4}$$

We have, $u(x,t) = \cos\sqrt{t}\,\frac{\pi}{2}\cos\sqrt{t}\,x$ is a solution of (1.3) with

$$B_1(u) = \cos\sqrt{t}\,\frac{\pi}{2}(1-\cos\sqrt{t}\,\pi),\ B_2(u) = -\sqrt{t}\cos\sqrt{t}\,\frac{\pi}{2}\sin\sqrt{t}\,\pi.$$

Obviously $B_1(u), B_2(u)$ have the same set of zeros, $\{t_k = k^2\}_{k=0}^{\infty}$, but neither $\{t_k\}_{k=0}^{\infty}$ is the sequence of eigenvalues, nor $\{\cos\frac{k\pi}{2}\cos kx\}_{k=0}^{\infty}$ is the complete set of eigenfunctions. So it is not practical to discuss the existence of Kramer-type kernels associated with problems of type (1.3)-(1.4), i. e., when the eigenvalues are not necessarily simple. When the eigenvalues of the problem are simple, many Kramer-type expansions associated with the boundary-value problems were derived [1, 2, 9].

There are two ways introduced by Zayed [7, 8] to obtain sampling series associated with problem (1.3)-(1.4). The first one [8, pp. 50–52] is given by taking the kernel of the sampled integral transform to be

$$\phi(x,t) = A\cos\sqrt{t}\,x + B\sin\sqrt{t}\,x.$$

Therefore, if

$$f(t) = \int_0^\pi F(x)\phi(x,t)\,dx, \tag{1.5}$$

for some $F \in L^2(0,\pi)$, then

$$f(t) = \frac{f(0)}{\pi}\left\{\frac{\sin(\pi\sqrt{t})}{\sqrt{t}} + \frac{B}{A}\frac{2\sin^2(\frac{\pi}{2}\sqrt{t})}{\sqrt{t}}\right\} + \sum_{k=1}^{\infty} a_{2k}\left\{A\frac{\sqrt{t}\sin(\pi\sqrt{t})}{(t-4k^2)} + \right.$$
$$\left. B\frac{2\sqrt{t}\sin^2(\frac{\pi}{2}\sqrt{t})}{(t-4k^2)}\right\} + \sum_{k=1}^{\infty} b_{2k}\left\{A\frac{(-4k)\sin^2(\frac{\pi}{2}\sqrt{t})}{(t-4k^2)} + B\frac{(2k)\sin(\pi\sqrt{t})}{(t-4k^2)}\right\},$$

where

$$a_k = \frac{2}{\pi}\int_0^\pi F(x)\cos kx\,dx, \quad b_k = \frac{2}{\pi}\int_0^\pi F(x)\sin kx\,dx.$$

The last series is not a sampling expansion of $f(t)$ since the coefficients a_k and b_k can not be uniquely expressed in terms of the sampled values of f at the eigenvalues. If we denote the Hilbert transform of f by $\tilde{f}$, where

$$\tilde{f}(t) = \int_0^\pi F(x)(A\sin\sqrt{t}x - B\cos\sqrt{t}x)\,dx,$$

we obtain

$$\begin{aligned} f(t) = {} & \frac{f(0)}{\pi\sqrt{t}}\left\{\sin(\pi\sqrt{t}) + 2r\sin^2(\frac{\pi}{2}\sqrt{t})\right\} \\ & + \frac{2}{\pi(1+r^2)}\sum_{k=1}^{\infty}\frac{1}{(t-4k^2)}\left\{f(4k^2)\left[(\sqrt{t}+2kr^2)\sin(\pi\sqrt{t})\right.\right. \\ & \left. + 2r(\sqrt{t}-2k)\sin^2(\frac{\pi}{2}\sqrt{t})\right] \\ & \left.+ \tilde{f}(4k^2)\left[r(2k-\sqrt{t})\sin(\pi\sqrt{t}) - 2(r^2\sqrt{t}+2k)\sin^2(\frac{\pi}{2}\sqrt{t})\right]\right\}, \end{aligned}$$

where $r = B/A$.

The second way is given by taking the kernel of the sampled integral transform to be

$$\phi(x,t) = P(t)G(x,\xi_0,t),$$

where $G(x,\xi,t)$ is the Green's function of (1.3)-(1.4), ξ_0 is chosen in $[0,\pi]$ as in [7], and $P(t)$ is the canonical product

$$P(t) = t\prod_{k=1}^{\infty}\left(1-\frac{t}{t_k}\right), \qquad t_k = 4k^2,\ k = 1,2,\ldots.$$

Then, for

$$f(t) = \int_0^\pi \overline{F(x)}\phi(x,t)\,dx,$$

$F \in L^2(0,\pi)$, we have

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} f(t_k)\frac{P(t)}{(t-t_k)P'(t_k)} \\ &= f(0)\frac{2\sin(\frac{\pi}{2}\sqrt{t})}{\pi\sqrt{t}} + \sum_{k=1}^{\infty} f(t_k)\frac{4(-1)^k\sqrt{t}\sin(\frac{\pi}{2}\sqrt{t})}{\pi(t-4k^2)}. \end{aligned}$$

As we have seen there is no Kramer-type representations associated with problem (1.3)-(1.4).

In this article we use another version of Kramer's theorem, Lemma 3.1, so that we can obtain a new Kramer-type sampling representation associated with second order boundary-value problems which may have multiple eigenvalues.

2. PRELIMINARIES.

Consider the second-order eigenvalue problem

$$Ly = y'' - q(x)y = -\lambda y, \quad x \in I = [a,b],\ \lambda \in C, \tag{2.1}$$

$$U_i(y) = \alpha_{i1}y(a) + \alpha_{i2}y'(a) + \beta_{i1}y(b) + \beta_{i2}y'(b) = 0, \quad i = 1,2, \tag{2.2}$$

where

$$\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \beta_{11}\beta_{22} - \beta_{12}\beta_{21}, \tag{2.3}$$

α_{ij}, β_{ij} are real constants, and $q(x)$ is a continuous real-valued function on $[a,b]$.

Let $u, v \in C^2(a,b)$. Then the Green's formula for this eigenvalue problem is

$$\int_I [\bar{v}L(u) - u\overline{L(v)}]\,dx = U_1V_4 + U_2V_3 + U_3V_2 + U_4V_1, \tag{2.4}$$

where U_j, $1 \le j \le 4$, are linearly independent linear forms of $u(a), u'(a), u(b), u'(b)$, and V_j, $1 \le j \le 4$, are linearly independent linear forms of $v(a), v'(a), v(b), v'(b)$. Here $V_j = 0$, $j = 1,2$, are the adjoint boundary conditions of (2.2), cf. [5]. Moreover problem (2.1)-(2.2) with (2.3) is self-adjoint, [5], and has at most countable set of real eigenvalues with no finite limit points.

Let $\{\phi_1(x,\lambda), \phi_2(x,\lambda)\}$ be the fundamental set of solutions of (2.1) defined by

$$\begin{aligned} \phi_1(a,\lambda) &= 1, \quad \phi_1'(a,\lambda) = 0, \\ \phi_2(a,\lambda) &= 0, \quad \phi_2'(a,\lambda) = 1. \end{aligned} \tag{2.5}$$

Any solution of (2.1) can be written as

$$\phi(x,\lambda) = c_1\phi_1(x,\lambda) + c_2\phi_2(x,\lambda),$$

where c_1, c_2 are arbitrary constants. The function $\phi(x,\lambda)$ is an eigenfunction of the self-adjoint eigenvalue problem (2.1)-(2.3) if it satisfies (2.2), i. e., when the system

$$\begin{pmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.6}$$

has a nontrivial solution. This happens when

$$\Delta(\lambda) := \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix} = 0.$$

That is the roots of $\Delta(\lambda)$ are the eigenvalues of the problem. The eigenvalues of problem (2.1)-(2.3) are not necessarily simple. Assume $\{\lambda_{1,k} = \lambda_{2,k}\}$, $\{\lambda_{3,k}\}$ are the sequences of double and simple eigenvalues respectively. Let $\chi_i(x,\lambda)$, $i = 1,2,3$, be the functions

$$\chi_1(x,\lambda) = \phi_1(x,\lambda), \qquad \chi_2(x,\lambda) = \phi_2(x,\lambda) + c(\lambda)\phi_1(x,\lambda),$$

and

$$\chi_3(x,\lambda) = \begin{vmatrix} U_1(\phi_1(x,\lambda)) & U_1(\phi_2(x,\lambda)) \\ \phi_1(x,\lambda) & \phi_2(x,\lambda) \end{vmatrix} + \alpha \begin{vmatrix} \phi_1(x,\lambda) & \phi_2(x,\lambda) \\ U_2(\phi_1(x,\lambda)) & U_2(\phi_2(x,\lambda)) \end{vmatrix},$$

where

$$c(\lambda) = -\frac{\int_a^b \phi_2(x,\lambda)\phi_1(x,\lambda)\,dx}{\|\phi_1(x,\lambda)\|^2},$$

and α is a constant chosen such that $\chi_3(x,\lambda_{3,k}) \not\equiv 0$, $\forall k$, $k = 1,2,\ldots$

We can see that $\{\chi_1(x,\lambda_{1,k}), \chi_2(x,\lambda_{2,k})\}$ and $\{\chi_3(x,\lambda_{3,k})\}$ are the sequences of orthogonal eigenfunctions corresponding to $\{\lambda_{1,k} = \lambda_{2,k}\}$, $\{\lambda_{3,k}\}$ respectively. This argument can be easily derived using the fact that an eigenvalue λ^* of problem (2.1)-(2.3) is simple if and only if one of the entries of $\Delta(\lambda^*)$ does not vanish.

Now assume that the zeros of $\Delta(\lambda)$, i.e. the eigenvalues $\{\lambda_{i,k}\}$, $i = 1,2,3$, have the asymptotic behaviour $\lambda_{i,k} = O(k^2)$ as $k \longrightarrow \infty$. This, for example, takes place if the boundary conditions are regular [5, p. 64]. Also assume that their multiplicities as zeros of $\Delta(\lambda)$ are at most two.

3. A SAMPLING THEOREM.

In this section, we state and prove the main theorem of the paper. Theorem 3.1 below is a sampling theorem associated with a second-order boundary-value problem whose eigenvalues are not necessarily simple. We start our study by the following Lemma, taken from [1]. It is a new version of Kramer's theorem.

LEMMA 3.1. Let $\{\lambda_{i,k}\}_{i=1}^{n}$ be sequences of numbers. Let $K_i : [a,b] \times \mathbb{C} \to \mathbb{C}$, $i = 1, 2, \ldots, n$ be n functions such that $K_i(x,\lambda) \in L^2(a,b)$, $\forall \lambda \in \mathbb{C}$, and that $\cup_{i=1}^{n} \{K_i(x,\lambda_{i,k})\}$ forms a complete orthogonal set in $L^2(a,b)$. Let H_i be the subspace generated by $\{K_i(x,\lambda_{i,k})\}$, $i = 1, 2, \ldots, n$. Then $L^2(a,b) = \sum_{i=1}^{n} \oplus H_i$. Assume that $f = \sum_{i=1}^{n} \oplus f_i \in L^2(a,b)$, $f_i \in H_i$, and

$$F(\lambda) = \sum_{i=1}^{n} F_i(\lambda) = \sum_{i=1}^{n} \int_a^b f_i(x) K_i(x,\lambda)\, dx. \tag{3.1}$$

Then

$$F(\lambda) = \sum_{i=1}^{n} \sum_{k=1}^{\nu_i} F_i(\lambda_{i,k}) S^*_{i,k}(\lambda), \tag{3.2}$$

where

$$S^*_{i,k}(\lambda) = \frac{\int_a^b K_i(x,\lambda) \overline{K_i(x,\lambda_{i,k})}\, dx}{\int_a^b |K_i(x,\lambda_{i,k})|^2\, dx}, \tag{3.3}$$

and $\nu_i = \dim H_i$.

THEOREM 3.1. Let H_i be the subspace generated by $\{\chi_i(x,\lambda_{i,k})\}$, $i = 1, 2, 3$, and let $L^2(a,b) = \sum_{i=1}^{3} \oplus H_i$. Let $f = \sum_{i=1}^{3} \oplus f_i \in L^2(a,b)$, $f_i \in H_i$. Assume that

$$F(\lambda) = \sum_{i=1}^{3} F_i(\lambda) = \sum_{i=1}^{3} \int_a^b f_i(x) \chi_i(x,\lambda)\, dx. \tag{3.4}$$

Then F admits the following representation

$$F(\lambda) = \sum_{i=1}^{3} \sum_{k=1}^{\infty} F_i(\lambda_{i,k}) \frac{G_{i,k}(\lambda)}{(\lambda - \lambda_{i,k}) G'_{i,k}(\lambda_{i,k})}, \tag{3.5}$$

where

$$G_{i,k}(\lambda) = [\chi_i(x,\lambda), \chi_i(x,\lambda_{i,k})]_a^b,\ i = 1, 2, 3,$$

and $[u,v] = uv' - u'v$. The three series converge uniformly on any compact subset of the complex plane. Moreover

$$G_{3,k}(\lambda) = G(\lambda) = \prod_{i=1}^{3} \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{i,k}}\right), \tag{3.6}$$

if zero is not an eigenvalue. These products must be multiplied by λ if zero is a simple eigenvalue, and by λ^2 if zero is a double eigenvalue.

PROOF. Setting

$$G_{i,k}(\lambda) = \int_a^b [\chi_i(x,\lambda) L\chi_i(x,\lambda_{i,k}) - \chi_i(x,\lambda_{i,k}) L\chi_i(x,\lambda)]\, dx, \quad i = 1, 2, 3,$$

and integrating by parts, we obtain

$$G_{i,k}(\lambda) = [\chi_i(x,\lambda), \chi_i(x,\lambda_{i,k})]_a^b, \quad i = 1, 2, 3. \tag{3.7}$$

On the other hand, using (2.1), one gets

$$G_{i,k}(\lambda) = (\lambda - \lambda_{i,k}) \int_a^b \chi_i(x,\lambda) \chi_i(x,\lambda_{i,k})\, dx, \tag{3.8}$$

and therefore

$$G'_{i,k}(\lambda_{i,k}) = \int_a^b |\chi_i(x,\lambda_{i,k})|^2\, dx = \|\chi_i(.,\lambda_{i,k})\|^2. \tag{3.9}$$

Since $f_i \in H_i$, $i = 1, 2, 3$, then

$$f_i(x) = \sum_{k=1}^{\infty} c_{i,k} \frac{\chi_i(x, \lambda_{i,k})}{\|\chi_i(., \lambda_{i,k})\|^2},$$

where

$$c_{i,k} = \int_a^b f_i(x)\chi_i(x, \lambda_{i,k})\, dx = F_i(\lambda_{i,k}). \tag{3.10}$$

Using Parseval's equality, we obtain

$$\begin{aligned} F_i(\lambda) &= \int_a^b f_i(x)\chi_i(x, \lambda)\, dx \\ &= \sum_{k=1}^{\infty} \frac{\int_a^b f_i(x)\chi_i(x, \lambda_{i,k})\, dx \int_a^b \chi_i(x, \lambda)\chi_i(x, \lambda_{i,k})\, dx}{\|\chi_i(., \lambda)\|^2} \\ &= \sum_{k=1}^{\infty} F_i(\lambda_{i,k}) \frac{G_{i,k}(\lambda)}{(\lambda - \lambda_{i,k})G'_{i,k}(\lambda_{i,k})}, \qquad i = 1, 2, 3, \end{aligned}$$

(cf. (3.8), (3.9), and (3.10)). The proof of the uniform convergence can be established as in [8, pp. 113–114].

We now show that $G_{3,k}(\lambda)$, for $k = 1, 2, \ldots$, has no zeros other than the eigenvalues. We use the same technique of [2]. From (3.7) it is clear that each $\lambda_{i,k}$, $i = 1, 2, 3$, $k = 1, 2, \ldots$, is a zero of $G_{3,k}(\lambda)$. Suppose λ^* is another zero of $G_{3,k}(\lambda)$. It will be shown that λ^* is an eigenvalue of (2.1)-(2.2). From (2.4) and (3.7), we obtain

$$G_{3,k}(\lambda) = U_1V_4 + \ldots + U_4V_1,$$

for all λ, where the U_j, $1 \le j \le 4$, are linear forms in $\chi_3(a, \lambda), \chi'_3(a, \lambda), \chi_3(b, \lambda), \chi'_3(b, \lambda)$, and the V_j, $1 \le j \le 4$, are linear forms in $\chi_3(a, \lambda_{3,k}), \chi'_3(a, \lambda_{3,k}), \chi_3(b, \lambda_{3,k}), \chi'_3(b, \lambda_{3,k})$. Since $\chi_3(x, \lambda_{3,k})$ is an eigenfunction, then

$$V_j(\chi_3(x, \lambda_{3,k})) = 0, \qquad j = 1, 2.$$

Obviously $U_1(\chi_3(x, \lambda)) = \alpha\Delta(\lambda)$, $U_2(\chi_3(x, \lambda)) = \Delta(\lambda)$, hence

$$G_{3,k}(\lambda) = \alpha\Delta(\lambda)V_4 + \Delta(\lambda)V_3 = \Delta(\lambda)V(\chi_3(x, \lambda_{3,k})) \tag{3.11}$$

where

$$V(\chi_3(x, \lambda_{3,k})) = \left[\alpha V_4(\chi_3(x, \lambda_{3,k})) + V_3(\chi_3(x, \lambda_{3,k}))\right].$$

Since λ^* is a zero of $G_{3,k}(\lambda)$, then $G_{3,k}(\lambda^*) = \Delta(\lambda^*)V(\chi_3(x, \lambda_{3,k})) = 0$. Now assume that $V(\chi_3(x, \lambda_{3,k})) = 0$. Since $V(\chi_3(x, \lambda_{3,k}))$ is independent of λ, by (3.11), $G_{3,k}(\lambda) \equiv 0$. Thus $G_{3,k}(\lambda)$ is identically zero, which contradicts the fact that $G'_{3,k}(\lambda_{3,k}) \neq 0$ (cf.(3.9)). So, $V(\chi_3(x, \lambda_{3,k}))$ is not zero for all eigenvalues. Thus we have $\Delta(\lambda^*) = 0$, and so λ^* is an eigenvalue of (2.1) and (2.2).

Finally we show that $G_{3,k}(\lambda)$ may take the from (3.6). Indeed by Hadamard's factorization theorem [4, p. 24] for entire functions and by noting that $G_{3,k}(\lambda)$ is of order $\frac{1}{2}$ [5, p. 55] we can write

$$G_{3,k}(\lambda) = e^{P(z)}G(\lambda),$$

where $P(z)$ is a polynomial whose degree does not exceed $\frac{1}{2}$ (the order of $G_{3,k}$). Thus $P(z) = c(k)$ is a constant depending only on k. The convergence of every product in $G_{3,k}(\lambda)$ is guaranteed since the eigenvalues behaves like $O(k^2)$ as $k \to \infty$. Obviously

$$\frac{G_{3,k}(\lambda)}{G'_{3,k}(\lambda_{3,k})} = \frac{G(\lambda)}{G'(\lambda)},$$

so without loss of generality we may assume that $G_{3,k}(\lambda) = G(\lambda)$. This completes the proof of theorem 3.1.

In some cases the three-series summation (3.5) can be reduced, (see examples 1, 2 below) into a two-series one. The first is written in terms of $F(\lambda_{1,k} = \lambda_{2,k})$ and the second in terms of $F_3(\lambda_{3,k})$. In such cases we may need to reform the integral transform (3.4) into a suitable one as we see in the following corollary

COROLLARY 3.1. Assume that

$$\frac{G_{1,k}(\lambda)}{G_{2,k}(\lambda)} = \mu(k)h(\lambda),$$

where μ is a function depends only on k , and h is an entire function depends only on λ. Let

$$\widetilde{F}_2(\lambda) = \int_a^b f_2(x)\widetilde{\chi}_2(x,\lambda)\,dx,$$

where $\widetilde{\chi}_2(x,\lambda) = h(\lambda)\chi_2(x,\lambda)$. Hence, for the integral transform

$$\widetilde{F}(\lambda) = F_1(\lambda) + \widetilde{F}_2(\lambda) + F_3(\lambda),$$

we have

$$\widetilde{F}(\lambda) = \sum_{k=1}^{\infty} \widetilde{F}(\lambda_{1,k}) \frac{G_{1,k}(\lambda)}{(\lambda-\lambda_{1,k})G'_{1,k}(\lambda_{1,k})} + \sum_{k=1}^{\infty} F_3(\lambda_{3,k}) \frac{G_{3,k}(\lambda)}{(\lambda-\lambda_{3,k})G'_{3,k}(\lambda_{3,k})}.$$

PROOF. Instead of $G_{2,k}(\lambda)$, we consider

$$\widetilde{G}_{2,k}(\lambda) = (\lambda-\lambda_{2,k})\int_a^b \widetilde{\chi}_2(x,\lambda)\chi_2(x,\lambda_{2,k})\,dx = h(\lambda)G_{2,k}(\lambda),$$

and $\widetilde{G}'_{2,k}(\lambda_{2,k}) = h(\lambda_{2,k})G'_{2,k}(\lambda_{2,k})$. Then

$$\frac{\widetilde{G}_{2,k}(\lambda)}{\widetilde{G}'_{2,k}(\lambda_{2,k})} = \frac{h(\lambda)G_{2,k}(\lambda)}{h(\lambda_{2,k})G'_{2,k}(\lambda_{2,k})}\frac{\mu(\lambda_{2,k})}{\mu(\lambda_{2,k})} = \frac{G_{1,k}(\lambda)}{G'_{1,k}(\lambda_{2,k})},$$

and

$$\begin{aligned}\widetilde{F}(\lambda) &= F_1(\lambda) + \widetilde{F}_2(\lambda) + F_3(\lambda)\\ &= \sum_{k=1}^{\infty} F_1(\lambda_{1,k}) \frac{G_{1,k}(\lambda)}{(\lambda-\lambda_{1,k})G'_{1,k}(\lambda_{1,k})} + \sum_{k=1}^{\infty} \widetilde{F}_2(\lambda_{2,k}) \frac{\widetilde{G}_{2,k}(\lambda)}{(\lambda-\lambda_{2,k})\widetilde{G}'_{2,k}(\lambda_{2,k})}\\ &\qquad + \sum_{k=1}^{\infty} F_3(\lambda_{3,k}) \frac{G_{3,k}(\lambda)}{(\lambda-\lambda_{3,k})G'_{3,k}(\lambda_{3,k})}\\ &= \sum_{k=1}^{\infty} \{F_1(\lambda_{1,k}) + \widetilde{F}_2(\lambda_{2,k})\} \frac{G_{1,k}(\lambda)}{(\lambda-\lambda_{1,k})G'_{1,k}(\lambda_{1,k})}\\ &\qquad + \sum_{k=1}^{\infty} F_3(\lambda_{3,k}) \frac{G_{3,k}(\lambda)}{(\lambda-\lambda_{3,k})G'_{3,k}(\lambda_{3,k})}\\ &= \sum_{k=1}^{\infty} \widetilde{F}(\lambda_{1,k}) \frac{G_{1,k}(\lambda)}{(\lambda-\lambda_{1,k})G'_{1,k}(\lambda_{1,k})} + \sum_{k=1}^{\infty} F_3(\lambda_{3,k}) \frac{G_{3,k}(\lambda)}{(\lambda-\lambda_{3,k})G'_{3,k}(\lambda_{3,k})}.\end{aligned}$$

4. EXAMPLES.

EXAMPLE 4.1. Consider the periodic eigenvalue problem

$$-y'' = \lambda y = t^2 y, \qquad 0 \le x \le \pi, \tag{4.1}$$

$$\begin{aligned} U_1(y) &= y(0) - y(\pi) = 0, \\ U_2(y) &= y'(0) - y'(\pi) = 0. \end{aligned} \tag{4.2}$$

This is a regular self-adjoint eigenvalue problem. The fundamental set of solutions of (4.1) subject to conditions (2.5) is

$$\left\{\phi_1(x,\lambda) = \cos tx,\ \phi_2(x,\lambda) = \frac{\sin tx}{t}\right\}.$$

Thus

$$\begin{aligned} \Delta(\lambda) &= \begin{vmatrix} 1-\cos\pi t & -\frac{\sin\pi t}{t} \\ t\sin\pi t & 1-\cos\pi t \end{vmatrix} \\ &= 4\sin^2\frac{\pi t}{2}. \end{aligned}$$

The eigenvalues are $\lambda = 4k^2, k = 0,1,2,\ldots$, where $\lambda = 4k^2, k = 1,2,\ldots$ are double eigenvalues and the corresponding eigenfunctions are

$$\left\{\cos 2kx,\ \frac{\sin 2kx}{2k}\right\}_{k=1}^{\infty},$$

and $\lambda = 0$ is the only simple eigenvalue with the eigenfunction $\phi_3(x,0)$, where

$$\begin{aligned} \phi_3(x,\lambda) &= \begin{vmatrix} 1-\cos\pi t & -\frac{\sin\pi t}{t} \\ \cos tx & \frac{\sin tx}{t} \end{vmatrix} + \begin{vmatrix} \cos tx & \frac{\sin tx}{t} \\ t\sin\pi t & 1-\cos\pi t \end{vmatrix} \\ &= \frac{\sin t(\pi - x)}{t} - \cos t(\pi - x) + \frac{\sin tx}{t} + \cos tx. \end{aligned}$$

Hence $\phi_3(x,0) = \pi$. Now

$$G_{1,k}(\lambda) = t\sin\pi t, \quad G_{2,k}(\lambda) = \frac{\sin\pi t}{t}, \quad G_{3,0}(\lambda) = 4\pi\sin^2\frac{\pi t}{2}.$$

So

$$G'_{1,k}(\lambda_{1,k}) = \frac{\pi}{2}, \quad G'_{2,k}(\lambda_{2,k}) = \frac{\pi}{8k^2}, \quad G'_{3,0}(0) = \pi^3,$$

where $\lambda_{1,k} = \lambda_{2,k} = 4k^2$. Let $L^2(a,b) = \sum_{i=1}^{3} \oplus H_i$, where H_1, H_2, H_3 are the subspaces generated by

$$\{\cos 2kx\}_{k=1}^{\infty},\ \left\{\frac{\sin 2kx}{2k}\right\}_{k=1}^{\infty},\ \{\pi\}$$

respectively. Let $F(\lambda) = F_1(\lambda) + F_2(\lambda) + F_3(\lambda)$, where

$$F_i(\lambda) = \int_0^{\pi} f_i(x)\phi_i(x,\lambda)\,dx, \quad f_i \in H_i,\ i = 1,2,3.$$

Then

$$F(\lambda) = \sum_{k=1}^{\infty} F_1(4k^2)\frac{2t\sin\pi t}{\pi(\lambda - 4k^2)} + \sum_{k=1}^{\infty} F_2(4k^2)\frac{8k^2\sin\pi t}{\pi t(\lambda - 4k^2)} + F_3(0)\frac{4\sin^2(\frac{\pi}{2}t)}{\pi^2 t^2}. \tag{4.3}$$

In the following we see that the form (4.3) can be reduced into another form which is similar to those resulting in the case of simple eigenvalues by redefining the sampled integral transform as described in the above corollary. Indeed, $\frac{G_{1,k}(\lambda)}{G_{2,k}(\lambda)} = \lambda = h(\lambda)$ is entire. Let

$$\widetilde{F}(\lambda) = F_1(\lambda) + \widetilde{F}_2(\lambda) + F_3(\lambda),$$

where

$$\widetilde{F}_2(\lambda) = \int_0^{\pi} f_2(x)\lambda\phi_2(x,\lambda)\,dx.$$

Then, noting that $F_3(0) = \widetilde{F}(0)$, we have

$$\widetilde{F}(\lambda) = \sum_{k=1}^{\infty} \widetilde{F}(4k^2)\frac{2t\sin\pi t}{\pi(\lambda - 4k^2)} + \widetilde{F}(0)\frac{4\sin^2(\frac{\pi}{2}t)}{\pi^2\lambda^2}.$$

EXAMPLE 4.2. Consider the anti-periodic eigenvalue problem

$$-y'' = \lambda y = t^2 y, \qquad 0 \le x \le \pi, \tag{4.4}$$

$$\begin{aligned} U_1(y) &= y(0) + y(\pi) = 0, \\ U_2(y) &= y'(0) + y'(\pi) = 0. \end{aligned} \tag{4.5}$$

This is a regular self-adjoint eigenvalue problem. For the same fundamental solutions in the previous example, we have $\Delta(\lambda) = 4\cos^2\frac{\pi t}{2}$. The eigenvalues are $\{(2k-1)^2\}_{k=1}^{\infty}$, all of them are double, their corresponding eigenfunctions are

$$\left\{\cos(2k-1)x,\ \frac{\sin(2k-1)x}{(2k-1)}\right\}_{k=1}^{\infty}.$$

Now

$$G_{1,k}(\lambda) = -t\sin\pi t, \quad G_{2,k}(\lambda) = -\frac{\sin\pi t}{t};$$

and

$$G'_{1,k}(\lambda_{1,k}) = \frac{\pi}{2}, \quad G'_{2,k}(\lambda_{2,k}) = \frac{\pi}{2(2k-1)^2}.$$

For the corresponding integral transform, $F(\lambda)$, defined as in the theorem, we get the following sampling representation

$$F(\lambda) = \sum_{k=1}^{\infty} F_1((2k-1)^2)\frac{-2t\sin\pi t}{\pi(\lambda-(2k-1)^2)} + \sum_{k=1}^{\infty} F_2((2k-1)^2)\frac{-2(2k-1)^2\sin\pi t}{\pi t(\lambda-(2k-1)^2)}.$$

Also we have, $\frac{G_{1,k}(\lambda)}{G_{2,k}(\lambda)} = \lambda = h(\lambda)$. Let

$$\widetilde{F}(\lambda) = F_1(\lambda) + \widetilde{F}_2(\lambda), \quad \widetilde{F}_2(\lambda) = \int_0^{\pi} f_2(x)\lambda\phi_2(x,\lambda)\,dx.$$

Hence

$$\widetilde{F}(\lambda) = \sum_{k=1}^{\infty} \widetilde{F}((2k-1)^2)\frac{-2t\sin\pi t}{\pi(\lambda-(2k-1)^2)}.$$

REMARK. Unlike the case of simple eigenvalues, as the above two examples show, we do not have the relations $G_{\iota,k}(\lambda) = c_k G_\iota(\lambda)$, $i = 1, 2$, where

$$G_\iota(\lambda) = \begin{cases} \displaystyle\prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{\iota,k}}\right), & \text{if zero is not an eigenvalue,} \\ \displaystyle\lambda\prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{\iota,k}}\right), & \text{if zero is an eigenvalue,} \end{cases}$$

and c_k is a constant depending on k. In fact it can be easily seen that $G_{1,k}$, $G_{2,k}$, in the previous examples, have zeros more than those of G_1, G_2.

ACKNOWLEDGMENT. The authors wish to express their gratitude to Professor M. A. El-Sayed, Cairo University for reading the manuscript and for his constructive comments. The first author wishes to thank Alexander von Humboldt Foundation for supporting his stay in Germany, under the number IV-1039259, when he prepared the revised version of the paper.

REFERENCES

[1] Annaby, M. H., *On sampling expansions associated with boundary value problems*, Proceedings of the 1995 Workshop on Sampling Theory & Applications, Jurmala, Latvia (1995), 137–139.

[2] Butzer, P. L. and Schöttler, G., *Sampling theorems associated with fourth and higher order self-adjoint eigenvalue problems*, J. Comput. Appl. Math. **51** (1994), 159–177.

[3] Kramer, H. P., *A generalized sampling theorem*, J. Math. Phys. **38** (1959), 68–72.

[4] Levin, B., *Distribution of Zeros of Entire Functions*, Translations of Mathematical Monographs, Vol. 5, Amer. Math. Soc. Providence, Rhode Island, 1964.

[5] Naimark, M. A., *Linear Differential Operators*, George Harrap & Co., London, 1967.

[6] Shannon, C. E., *Communications in the presence of noise*, Proc. IRE **37** (1949), 10–21.

[7] Zayed, A. I., *A new role of Green's function in interpolation and sampling theory*, J. Math. Anal. Appl. **175** (1993), 222–238.

[8] Zayed, A. I., *Advances in Shannon's Sampling Theory*, CRC Press, Boca Raton, 1993.

[9] Zayed, A. I., EL-Sayed, M. A. and Annaby, M. H., *On Lagrange interpolations and Kramer's sampling theorem associated with self-adjoint boundary value problems*, J. Math. Anal. Appl. **158, 1** (1991), 269–284.