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EQUICONTINUITY OF ITERATES OF CIRCLE MAPS

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ABSTRACT. Let f be a continuous map of the circle to itself. Necessary and sufficient conditions are given for the family of iterates $\{f^n\}_{n=1}^{\infty}$ to be equicontinuous.

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1. INTRODUCTION.

Let $C^0(X,Y)$ denote the set of continuous maps from X to Y, I a closed unit interval and S^1 the circle. Let $f \in C^0(I,I)$ and suppose that the family of iterates of f, i.e. $\{f^n\}_{n=1}^{\infty}$, is equicontinuous. Let F_1 and F_2 denote the fixed point set of f and f^2 respectively. A. M. Bruckner and T. Hu [4] have shown that $\{f^n\}$ is equicontinuous if and only if $F_2 = \cap_{n=1}^{\infty} f^n(I)$. We show that for maps of the circle the following result holds:

THEOREM. Let $f \in C^0(S^1,S^1)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if one of the following holds:

(1) f is conjugate to a rotation.

(2) F_1 consists of exactly two distinct points and every other point on S^1 has period two.

(3) F_1 consists of single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$.

(4) $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$.

2. PRELIMINARIES.

Let $f \in C^0(S^1,S^1)$. We think of the circle S^1 as R/Z and for $x,y \in S^1$ with $x \neq y$ we denote by $[x,y]$ the closed interval from x counterclockwise to y. Let $d(x,y)$ denote the $min\{|[x,y]|, |[y,x]|\}$ where $|[x,y]|$ is the length of the interval $[x,y]$. For any nonnegative integer n define f^n inductively by $f^n = f \circ f^{n-1}$, where f^0 is the identity map on S^1. A point $x \in S^1$ is a periodic point of f if there is a positive integer n such that $f^n(x) = x$. The least such n is called the period of x. A point of period one is called a fixed point. Let F_n denote the fixed point set of $f^n, \forall n \geq 1$ and $P(f)$ the set of periodic points of f.

If $x \in S^1$ then the trajectory of x is the sequence $\gamma(x,f) = \{f^n(x)\}_{n\geq 0}$ and the ω-limit set of x, $\omega(x,f) = \cap_{m\geq 0} \overline{\cup_{n\geq m} f^n(x)}$. Equivalently, $y \in \omega(x,f)$ if and only if y is a limit point of the trajectory $\gamma(x,f)$, i.e. $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$. Let $\mathcal{F} = \{f, f^2, f^3, ...\}$. The family of functions $\mathcal{F}$ is said to be equicontinuous if given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f^i(x), f^i(y)) < \epsilon$ whenever $d(x,y) < \delta$ for all $x,y \in S^1$ and all $i \geq 1$.

The following theorem is proved by J. Cano [5]:

THEOREM A. Let $f \in C^0(I,I)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then F_1 is connected and if it is non-degenerate then $F_1 = P(f)$.

The next theorem which is given in [4] and is due to A. M. Bruckner and Thakyin Hu (only if) and W. Boyce (if):

THEOREM B. Let $f \in C^0(I,I)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if $\cap_{n=1}^{\infty} f^n(I) = F_2$.

Combining these two theorems we get the following corollary:

COROLLARY. Let $f \in C^0(I,I)$. If f has a periodic point of period $n > 2$, then $\{f^n\}_{n=1}^{\infty}$ cannot be equicontinuous.

3. RESULTS

Let $f \in C^0(S^1,S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. We consider three cases:

(I) f has a fixed point on S^1.

(II) the smallest period of the periodic points of f on S^1 is $n \geq 2$.

(III) f has no periodic points on S^1.

We start with case (I). The basic result of this case is Theorem 1. We first show the following four lemmas:

LEMMA 1. Let $f \in C^0(S^1,S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that there is a fixed point p on S^1, and let J be the component of F_2 containing p. If J is either $\{p\}$ or a proper closed interval containing p then there exists an open interval K containing J such that $\omega(x,f) \subseteq J$, for every x in K.

PROOF. First suppose that $J = \{p\}$. Let $\epsilon = |S^1|/4 > 0$. By equicontinuity of $\{f^n\}_{n=1}^{\infty}$ there is an open interval K containing p such that for every x in K and for every $n \geq 1$, $d(f^n(x),p) < \epsilon$. Define $L = \overline{\cap_{j=0}^{\infty} f^j(K)}$. Then L is a closed, proper, invariant interval. By previous results on the interval (Theorems A and B), the fixed point set of $f|L$ and $f^2|L$ is connected and therefore, it is $\{p\}$. Moreover, by the above corollary all periodic points of $f|_L$ have period 1 or 2. But the fixed point p is the only periodic point of f in L. Therefore $P(f) = F_1$ and by [2] the ω-limit points coincide with the fixed points. Hence p is the only ω-limit point of f in L and thus $\omega(x,f) = \{p\} = J$, for every x in L. Since $K \subset L$, $\omega(x,f) = \{p\} = J$, for every x in K.

Now suppose that J is a proper closed interval containing p. Let q_1 and q_2 be the endpoints of J, which are fixed points under f^2. Let $\epsilon = |S^1 - J|/4 > 0$. By equicontinuity of $\{f^n\}_{n=1}^{\infty}$, there is an open interval K_1 around q_1 such that for every x in K_1 and for every $n \geq 1$, $d(f^n(x), f^n(q_1)) < \epsilon$. Similarly there is an open interval K_2 around q_2 such that for every x in K_2 and for every $n \geq 1$, $d(f^n(x), f^n(q_2)) < \epsilon$. Define $L = \overline{\cup_{j=0}^{\infty} f^j(K_1 \cup J \cup K_2)}$. Then L is a closed, proper, invariant interval. By previous results on the interval (Theorems A and B), the fixed point set of $f|L$ and $f^2|L$ is connected and therefore it is J. Moreover, by the Corollary all periodic points of $f|_L$ have period 1 or 2, which we know that lie in J. Since $P(f)$ is closed, by [2], it coincides with the set of ω-limit points. Therefore $\omega(x,f) \subset J$, for every x in L. Let $K = K_1 \cup J \cup K_2$. Then $K \subset L$ and $\omega(x,f) \subset J$, for every x in K. □

LEMMA 2. Let $f \in C^0(S^1,S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that there is a fixed point p on S^1, and let J be the component of F_2 containing p. Define $S = \{x \in S^1 : \omega(x,f) \subseteq J\}$. Then $S = S^1$.

PROOF. There are three cases:

(i) $J = \{p\}$, (ii) J is a proper closed interval containing p and (iii) $J = S^1$.

If (iii) holds then obviously $S = J = S^1$.

Therefore assume (i) and (ii) hold. Then by Lemma 1, there exists an open interval K containing J such that $\omega(x,f) \subseteq J$, for every x in K. Note that S is nonempty since $S \supseteq K$. First we show that S is open: Let $x \in S$. Then $\omega(x, f) \subseteq J$, by definition of S. Choose N large enough such that $f^N(x) \in K$. By continuity of f^N there is a neighborhood U of x such that if $y \in U$ then $f^N(y) \in K$. But then $\omega(f^n(y), f) = \omega(y, f) \subseteq J$ and $y \in S$. Therefore S is open.

Let T be the component of S containing J and therefore K, as well. Then T is open and connected. We will show that $T = S^1$. Suppose $T \neq S^1$. Then $S^1 - T$ is a closed interval or a point. Let $J = [q_1, q_2]$ where possibly $q_1 = q_2 = p$.

Suppose first that $S^1 - T$ is a closed interval. Let z_1 and z_2 be the endpoints of this closed interval such that $[z_2, z_1] \cap J = \emptyset$. Let $\epsilon = \frac{1}{2} min\{d(q_1, z_1), d(q_2, z_2)\}$. By equicontinuity of $\{f^n\}$ at z_1, there is an open interval V_1 around z_1 such that for every x in V_1 and for every $n \geq 1$, $d(f^n(x), f^n(z_1)) < \epsilon$. Let $x \in T$ such that $d(x, z_1) < \epsilon$. Since $\omega(x, f) \subseteq J$ and the orbit of z_1 stays by definition out of T, there exists a positive integer k such that $d(f^k(x), f^k(z_1)) > \epsilon$, which is a contradiction.

Now suppose that $S^1 - T = \{z\}$. Let $\epsilon = \frac{1}{2} min\{d(z, q_1), d(q_2, z)\}$. By equicontinuity of $\{f^n\}$ at z there is an open interval V around z such that for every x in V and for every $n \geq 1$, $d(f^n(x), f^n(z)) < \epsilon$. Since $\omega(x, f) \subseteq J$ for every $x \in T$ and $f(z) = z$ is a fixed point of f, we get a contradiction.

Hence $T = S^1$. Thus $S = \{x \in S^1 : \omega(x, f) \subset J\} = S^1$. □

LEMMA 3. Let $f \in C^0(S^1, S^1)$. If $F_2 = S^1$ then F_1 cannot consist of exactly one point.

PROOF. Suppose that there is an $f \in C^0(S^1, S^1)$ such that $F_2 = S^1$ and $F_1 = \{p\}$. Let z be a point on $S^1 - \{p\}$ of period two. Let K be the closed interval with endpoints z and $f(z)$ which contains p and let L be the closed interval with the same endpoints that does not contain p. Since f is a homeomorphism, we have two cases:

(i) $f(K) = K$ and $f(L) = L$ or (ii) $f(K) = L$ and $f(L) = K$.

If (i) holds then, since $f(L) = L$, there would be another fixed point of f in L, which is a contradiction since $F_1 \subset K$.

If (ii) holds then $f(K) = L$ implies that p cannot be a fixed point which is again a contradiction. □

LEMMA 4. Let $f \in C^0(S^1, S^1)$. If $F_2 = S^1$ and F_1 consists of more than two distinct points then f is the identity on S^1.

PROOF. Assume that F_1 consists of exactly $k > 2$ distinct fixed points $p_1, p_2, ..., p_k$. Let $L_i = [p_i, p_{i+1}]$ for $i = 1, 2, ..., k-1$ and $L_k = [p_k, p_1]$ so that the interior of each L_i does not contain any fixed points. Then we have two cases: (i) $f(L_i) = L_i$ and (ii) $f(L_i) = S^1 - L_i$.

If (i) holds then pick x in the interior of L_i. Note that $f(x)$ is a point in the interior of L_i and denote by M_x the closed interval with endpoints x and $f(x)$ which is free of fixed points. If $f(M_x) = M_x$ then there would be another fixed point in M_x contradicting that the interior of L_i contains no fixed points. Thus the only choice is $x = f(x)$ for every $x \in L_i$ and $f|L_i$ is the identity map. The same argument applied to every L_i shows that f is the identity map on S^1.

If (ii) holds then there are points in L_i which map onto the other fixed points contradicting that $F_2 = S^1$. □

THEOREM 1. Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that there is a fixed point p on S^1. Then f has periodic points of period at most two and F_2 is connected. Furthermore F_1 is either connected or it consists of exactly two distinct points and every other point on S^1 has period two. Moreover if F_1 is a nondegenerate interval then $F_1 = P(f)$.

PROOF. Let J be the component of F_2 containing p. There are three cases:

(i) $J = \{p\}$, (ii) J is a proper closed interval containing p and (iii) $J = S^1$.

Assume that (i) holds. Then, by Lemma 2, $\omega(x, f) - \{p\}$ for every $x \in S^1$. Thus the fixed point p is the only periodic point of f on S^1 and hence $P(f) = F_1 = F_2 = \{p\}$ is connected.

Assume that (ii) holds. Then, by Lemma 2, $\omega(x, f) \subseteq J$ for every $x \in S^1$ and the periodic points of f on S^1 lie in J. By results on the interval applied to $f|J$, either p is the unique fixed point of f on S^1 or

F_1 is a nondegenerate interval and the fixed points are the only periodic points of f on S^1. In particular, both F_1 and F_2 are connected.

Assume that (iii) holds. Then all of the points of $S^1$are periodic with period 1 or 2 and F_2 is connected. By Lemma 3, F_1 cannot consist of one point and by Lemma 4 if F_1 consists of more than two points then f is the identity map. Otherwise F_1 consists of exactly two distinct points and every other point on S^1 has period two. □

We now investigate case (II) where the smallest period of the periodic points of f on S^1 is $n \geq 2$. The main result here is Theorem 2. We use Lemma 5 in the proof of the main theorem.

THEOREM 2. Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that the smallest period of the periodic points of f on S^1 is $n \geq 2$. Then every point on S^1 is periodic with period n.

PROOF. Let p be a periodic point of period n on S^1. Then $f^n(p) = p$ and therefore p is a fixed point of f^n. Applying Theorem 1 to f^n, we conclude that F_n is either connected or it consists of exactly two distinct points and every other point on S^1 has period $2n$.

We claim that there is no continuous map of the circle having two points of period two and every other point periodic of period four. Otherwise, if g is such a map, let $p, g(p)$ be the two points of period two and let $K = [p, g(p)]$ and $L = [g(p), p]$. Since g is a homeomorphism, we have two cases: (i) $g(K) = K$ and $g(L) = L$ or (ii) $g(K) = L$ and $g(L) = K$. In both cases $g^2(K) = K$. Hence if $x \in K$ is a point of period four then $g^2(x) \neq x$ and $g^2(x) \in K$. If M is the closed interval with endpoints x and $g^2(x)$ lying in K then $g^2(M) = M$. Therefore M contains a periodic point of period two, contradicting the assumption that p and $g(p)$ are the only points of period two and every other point has period four.

Hence F_n is connected. Suppose that $F_n \neq S^1$. Then F_n is a proper closed interval containing the orbit of p under f. Moreover $f(F_n) \subset F_n$. This implies that f has a fixed point on S^1, contradicting the hypothesis that the smallest possible period of the periodic points is $n > 1$. Hence $F_n = S^1$. □

For a proof of the following see [7].

LEMMA 5. Let $f \in C^0(S^1, S^1)$. Suppose that there exists a positive integer $n \geq 2$ such that every point on S^1 is periodic with period n. Then f is conjugate to a rational rotation.

Now we consider case (III) where $f \in C^0(S^1, S^1)$ has no periodic points and $\{f^n\}$ is equicontinuous. The main result here is listed in Theorem 3.

Note that f must be onto, since otherwise $f(S^1) = I$ is homeomorphic to a closed interval and $f(I) \subset I$, so f has a fixed point. We shall adapt the techniques and use results due to J. Auslander and Y. Katznelson [1].

Let $x \in S^1$. In [1], J_x is defined to be the largest interval containing x such that $f^m(x) \notin J_x, \forall m \geq 1$. Denote by z_1 and z_2 the endpoints of J_x, where possibly $z_1 = z_2 = x$. The following are showed in [1]: J_x is closed and $z_1, z_2 \neq f^k(x)$ for $k \geq 1$. If $x, y \in S^1$ then $y \in \omega(x, f)$ if and only if y is an endpoint of J_y. If z_1 and z_2 are the endpoints of J_x, then $f(z_1)$ and $f(z_2)$ are the endpoints of $f(J_x)$. Also $f(J_x) \cap J_x = \emptyset$ and $f^m(J_x) = J_{f^m(x)}$, $\forall m \geq 1$. The intervals $J_{f^m(x)}$ $(m = 0, 1, 2, ...)$ are pairwise disjoint and if $f(x) = f(x')$, then $J_x = J_{x'}$. The sets $\{J_x\}$ form a partition of S^1 (that is, if $x, y \in S^1$ then $J_x = J_y$ or $J_x \cap J_y = \emptyset$ and $\cup_{x \in S^1} J_x = S^1$). Finally, at most countably many of the sets J_x are non-degenerate ($J_x \neq \{x\}$).

Before we show our result, we state the following theorem proved in [6] which concerns homeomorphisms.

THEOREM C. Let f be an orientation preserving homeomorphism of S^1 to itself. For $x \in S^1$, let $R_\alpha(x) = x + \alpha$ (*mod* 1) denote irrational rotation by α. Then f is conjugate to some R_α if and only if some (all) orbits of f are dense on S^1.

We are now ready to show the following:

THEOREM 3. Let $f \in C^0(S^1, S^1)$ without periodic points and such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then f is conjugate to an irrational rotation R_α.

PROOF. We first show that $\omega(x, f) = S^1$ for all $x \in S^1$. Since $y \in \omega(x, f)$ if and only if y is an endpoint of J_y, it suffices to show that $\forall y \in S^1$, $J_y = \{y\}$. By the way of contradiction assume that J_{y0} is a non-degenerate interval. Since f is onto, there exists $y_1 \in S^1$ such that $f(J_{y1}) = J_{y0}$. Continuing in this way, we obtain a sequence of intervals $\{J_{yn}\}$ such that $f(J_{yn}) = J_{yn-1}\ \forall n \geq 1$. Since there are countably many non-degenerate such intervals on S^1, $lim_{k\to\infty}|f^{-k}(J_{y0})| = 0$. Hence f^k maps arbitrarily small intervals onto J_{y0} (as $k \to \infty$) which contradicts equicontinuity. Therefore $\omega(x, f) = S^1$ for all $x \in S^1$.

This is equivalent to saying that all orbits of f are dense in S^1. If $f(y) = f(y')$, then $J_y = J_{y'}$ and hence f is a homeomorphism. By Theorem C it follows immediately that f is conjugate to an irrational rotation R_α.

4. PROOF OF THEOREM

We first state the following three lemmas which can be shown to hold on any compact metric space.

LEMMA 6. Let $f, g \in C^0(X, X)$, where X is a compact metric space. Suppose that f is conjugate in X to g and that $\{g^n\}_{n=1}^{\infty}$ is equicontinuous. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

LEMMA 7. Let $f \in C^0(X, X)$, where X is a compact metric space. Let k be a positive integer and $g = f^k$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if $\{g^n\}_{n=1}^{\infty}$ is equicontinuous.

LEMMA 8. Let $f \in C^0(X, X)$, where X is a compact metric space. If $\{(f|_{f(X)})^n\}_{n=1}^{\infty}$ is equicontinuous then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Finally, we summarize the results to the following theorem.

THEOREM. Let $f \in C^0(S^1, S^1)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if one of the following holds:

(1) f is conjugate to a rotation.
(2) F_1 consists of exactly two distinct points and every other point on S^1 has period two.
(3) F_1 consists of a single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$.
(4) $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$.

PROOF. We suppose that $\{f^n\}$ is equicontinuous. First assume that $F_1 = \emptyset$. If f has no periodic points on S^1, then by Theorem 3, f is conjugate to an irrational rotation, so that (1) holds.

If the smallest period of the periodic points of f on S^1 is $n \geq 2$, then By Theorem 2, every point on S^1 is periodic with period n. It follows by Lemma 5 that f is conjugate to a rational rotation, so that (1) holds again.

Now assume that $F_1 \neq \emptyset$. Then by Theorem 1, f has periodic points of period at most two. If F_1 is not connected, then by Theorem 1 it consists of exactly two distinct points and every other point of S^1 has period two, so that (2) holds.

If F_1 is connected then it consists of (i) a single point, (ii) a proper interval or (iii) the whole circle.

(i) First assume that F_1 consists of a single point p. Note that by Theorem1, F_2 is a connected proper interval of S^1. Moreover by Lemma 2, we have that for every $x \in S^1$, $\omega(x, f) \subseteq \overline{F_2}$. As in the proof of Lemma 1, there exists an open interval K containing F_2 such that if $L = \overline{\cup_{j=0}^{\infty} f^j(K)}$ then L is a proper interval. Of course L is also closed and invariant. For $x \in S^1$, since $\omega(x, f) \subseteq F_2 \subset K \subset L$, there exists a positive integer N such that $f^N(x) \in K$. Then $f^m(x) \in L$ for every $m \geq N$. By continuity of f^N there exists an open neighborhood V_x of x such that $f^N(V_x) \in K$ and hence $f^m(V_x) \in L$ for every $m \geq N$. Note that for each $x \in S^1$ the collection $\{V_x\}_{x \in S^1}$ forms an open cover of S^1. By compactness of S^1 there exists a finite subcover, which we denote by $\{V_i\}_{i=1,\dots l}$. Consequently, for every V_i there exists a positive integer N_i such that $f^{N_i}(V_i) \subset K$, for $i = 1, 2, \dots l$

and $f^{m_i}(V_i) \subset L$, for every $m_i \geq N_i$ and for $i = 1, 2, ...l$. Choose $N = max\{N_1, ..., N_l\}$. Then $f^m(V_i) \subset L$, for every $m \geq N$ and $i = 1, 2, ...l$. Thus $f^m(S^1) \subset L$ for every $m \geq N$. By Theorem B, $\cap_{n=1}^{\infty} f^n(L) = F_2$. Since $f^m(S^1) \subset L$, for every $m \geq N$, it follows that $\cap_{n=1}^{\infty} f^n(L) = \cap_{n=1}^{\infty} f^n(S^1) = F_2$. Hence (3) holds.

(ii) Now assume that F_1 is a proper interval of S^1. We know by Theorem 1, that F_1 coincides with the set of periodic points of f. By an argument similar to the above applied to F_1, we can see that $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$ and hence (4) holds.

(iii) If $F_1 = S^1$ then obviously (4) holds again.

This concludes one direction of the proof, namely that if $\{f^n\}_{n=1}^{\infty}$ is equicontinuous then one of (1), (2), (3) or (4) holds. Now we will show that all of these four cases imply that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Suppose that (1) holds i.e. f is conjugate to a rotation R. Then R^n is an isometry for every $n \geq 1$ and therefore $\{R^n\}$ is equicontinuous. It follows by Lemma 6 that $\{f^n\}$ is equicontinuous as well.

Suppose that (2) holds i.e. F_1 consists of exactly two distinct points and every other point on S^1 has period two. Then f^2 is the identity on S^1. Therefore $\{f^{2n}\}$ is equicontinuous. It follows by Lemma 7 that $\{f^n\}$ is equicontinuous as well.

Suppose that (3) holds i.e. F_1 consists of a single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$. Then $f(S^1) \neq S^1$, since otherwise $F_2 = S^1$ and we have seen in Lemma 3 that there is no continuous map of the circle with one fixed point and every other point of period two. Hence $f(S^1)$ is a proper interval of S^1 and $f|_{f(S^1)} : f(S^1) \to f(S^1)$ is a continuous map of the interval with fixed point set of $(f|_{f(S^1)})^2$ equal to F_2. Since $\cap_{n=1}^{\infty} (f|_{f(S^1)})^n (f(S^1)) = F_2$, it follows by Theorem B, that $\{(f|_{f(S^1)})^n\}_{n=1}^{\infty}$ is equicontinuous. By Lemma 8 we get that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Finally suppose that (4) holds i.e. $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$. If $S^1 = f(S^1)$ then $F_1 = S^1$ and the identity map is equicontinuous. If $f(S^1)$ is a proper interval of S^1 then F_1 is a point or a proper interval of S^1. It follows that $f|_{f(S^1)} : f(S^1) \to f(S^1)$ is a continuous map of the interval such that its fixed point set equals the fixed point set of f on S^1. Since $\cap_{n=1}^{\infty} (f|_{f(S^1)})^n (f(S^1)) = F_1$, it follows by Theorem B, that $\{(f|_{f(S^1)})^n\}_{n=1}^{\infty}$ is equicontinuous. By Lemma 8 we get that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. □

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The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

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