

**ON COUNTABLE CONNECTED HAUSDORFF SPACES IN WHICH
THE INTERSECTION OF EVERY PAIR OF CONNECTED
SUBSETS IS CONNECTED**

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(Received July 2, 1996 and in revised form December 28, 1996)

ABSTRACT. We prove that a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, cannot be locally connected, and also that every continuous function from a countable connected, locally connected Hausdorff space, to a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is constant.

KEY WORDS AND PHRASES. Countable connected, locally connected.

1992 AMS SUBJECT CLASSIFICATION CODE. 54G15, 54F55, 54D10, 54D05.

1. INTRODUCTION.

The problem of existence of countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected was posed by Čvid in [1], and was answered in [2]. Recently, Gruenhage [3] assuming the continuum hypothesis constructed a perfectly normal space in which the only non-degenerate connected subsets of it, are the cofinite sets. Also assuming Martin's Axiom he constructed a completely regular and a countable Hausdorff space with this property. Obviously, in these spaces the intersection of every pair of connected subsets is connected. None of the spaces in [2] and [3] is locally connected, or has a dispersion point.

We prove that a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, cannot be locally connected, and also that every continuous function from a countable connected, locally connected Hausdorff space, to a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is con-

stant. Both these results hold in a Hausdorff connected space with a dispersion point: The first is obvious and the second, for not necessarily countable spaces, was proved by Coppin in [4]. Improvements of Coppin's result, as well as results concerning the constancy of functions between two spaces, can be found in the papers by Chew and Doyle [5], and by Sanderson [6].

Let X be a connected topological space. A point t is called a cut point of X if the space $X \setminus \{t\}$ is not connected. Thus, if t is a cut point of X , then the subspace $X \setminus \{t\}$ is the union of two mutually separated sets $A(t), B(t)$. (Two sets A, B are called separated if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$.) Obviously, if $A(t), B(t)$ are connected, the separation is unique. Let $x, y \in X$. A cut point t of X is said to separate the points x, y if the above sets $A(t), B(t)$ can be chosen so that $x \in A(t)$ and $y \in B(t)$. The set of cut points of X separating the points x, y will be denoted by $E(x, y)$. The empty set and the singletons are considered to be connected. All spaces are assumed to have more than one point.

2. RESULTS.

PROPOSITION 1. Let X be a Hausdorff connected space such that $E(a, b) \neq \emptyset$, for every $a, b \in X$. Then there exists a continuous non-constant real valued function on X , separating the points a and b .

PROOF. The proof is reduced to the Urysohn's Lemma in the following manner: For every point $t \in E(a, b)$ there exist two sets $M_a(t), M_b(t)$ such that $a \in M_a(t), b \in M_b(t), \overline{M_a(t)} = M_a(t) \cup \{t\}, \overline{M_b(t)} = M_b(t) \cup \{t\}$ and $X \setminus \{t\} = M_a(t) \cup M_b(t)$. Hence the sets

$$F_1 = \bigcap_{t \in E(a, b)} (M_a(t) \cup \{t\}) \text{ and } F_2 = \bigcap_{t \in E(a, b)} (M_b(t) \cup \{t\})$$

are both closed disjoint containing the points a, b respectively, and not containing any cut point of X separating the points a, b . Consequently, for every point d of the set of positive dyadic rational numbers we can define an open set $(M_a(t))(d)$ such that if $d < r$, then $\overline{(M_a(t))(d)} \subseteq (M_a(t))(r)$. But then the function $f(x) = \inf\{d : x \in (M_a(t))(d)\}$, if $x \notin F_2$, and $f(x) = 1$, if $x \in F_2$ is continuous separating the points a, b .

PROPOSITION 2. Does not exist a countable connected, locally connected Hausdorff space in which the intersection of every pair of connected subsets is connected.

PROOF. As it is proved in [7, Theorem 9.1] a connected locally connected space X is a Hausdorff space in which the intersection of every pair (indeed every collection) of connected sets is connected, if and only if no two point of X are conjugate. That is, $E(x, y) \neq \emptyset$, for every $x, y \in X$. But then, Proposition 1 implies that there exists a non-constant continuous real valued function on X , which is impossible for countable connected spaces.

PROPOSITION 3. Let X be a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected. Then

- (1) The subset D of X at every point of which X is not locally connected, is dense.

(2) The subset L at every point of which X is locally connected is totally disconnected or empty.

PROOF (1). By Proposition 2, $D \neq \emptyset$. Hence at every point $x \in X \setminus \bar{D}$, the space X is locally connected and therefore if U_x is an open connected neighbourhood of x for which $U_x \cap \bar{D} \neq \emptyset$, then U_x is also a locally connected space in which the intersection of every pair of connected subsets is connected, which is impossible, by Proposition 2.

(2). Obvious.

THEOREM. Every continuous function from a countable connected, locally connected Hausdorff space, to a connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is constant.

PROOF. Let f be a continuous non-constant function from X to Y . Obviously the space $Z = f(X)$ is countable connected Hausdorff in which the intersection of every pair of connected subsets is connected. Let x, y be distinct points of X such that $f(x) \neq f(y)$ and let $U_{f(x)}, U_{f(y)}$ be disjoint open neighbourhoods of $f(x), f(y)$, respectively. Since X is locally connected there exists an open connected neighbourhood U_x of x such that $f(U_x) \subseteq U_{f(x)}$. If $f(U_x) = \{f(x)\}$ then we consider the set $A = \{a \in X : f(a) = f(x)\}$. Since the set $\bar{A} \setminus \overset{\circ}{A}$ is not empty, it follows that there exist a point $a \in \bar{A} \setminus \overset{\circ}{A}$ and a connected open neighbourhood U_a of a , such that $f(U_a) \subseteq U_{f(x)}$ and $f(U_a) \neq \{f(x)\}$. Therefore the component $C_{f(x)}$ of $f(x)$ in $\bar{U}_{f(x)}$ is not a singleton.

Consider the component K of $f(y)$ in $Z \setminus C_{f(x)}$. If $K = \{f(y)\}$ then for the component M of Y in $X \setminus f^{-1}(C_{f(x)})$ it holds that $f(M) = \{f(y)\}$ and $f(\bar{M}) = \{f(y)\}$. Since the subspace $X \setminus f^{-1}(C_{f(x)})$ is locally connected it follows that M is open-and-closed (in $X \setminus f^{-1}(C_{f(x)})$), and hence $\bar{M} \cap f^{-1}(C_{f(x)}) \neq \emptyset$ which is impossible. Therefore the component K of y in $Z \setminus C_{f(x)}$ is not a singleton.

Thus, by [8, Vol. II, Ch. V, Theorem 5, III], for the connected subsets $C_{f(x)}$ and K it follows that the set $Z \setminus K$ is connected and hence either (1) $(\overline{Z \setminus K}) \cap K \neq \emptyset$, or (2) $(Z \setminus K) \cap \bar{K} \neq \emptyset$ or (3) $(\overline{Z \setminus K}) \cap \bar{K} \neq \emptyset$.

In case (1), let $p, q \in (\overline{Z \setminus K}) \cap K$, and $p \neq q$. Then for the connected subsets $(Z \setminus K) \cup \{p, q\}$ and K it holds that $((Z \setminus K) \cup \{p, q\}) \cap K = \{p, q\}$ which is impossible because by assumption the intersection of every pair of connected subsets of Z must be connected. Therefore $(\overline{Z \setminus K}) \cap K$ is a singleton. We set $(\overline{Z \setminus K}) \cap K = \{p\}$. The set K is closed because if a is a limit point of K and $a \notin K$ then for the connected subsets $K \cup \{a\}$ and $(\overline{Z \setminus K})$ the subset $(\overline{Z \setminus K}) \cap (K \cup \{a\}) = \{a, p\}$ must be connected, which is impossible. Hence if we consider the component M of y in $X \setminus f^{-1}(C_{f(x)})$ then $f(\bar{M}) \subseteq K$ which is also impossible because $\bar{M} \cap f^{-1}(C_{f(x)}) \neq \emptyset$.

In case (2) it can be proved in the same manner as in case (1) that $(Z \setminus K) \cap \bar{K}$ is a singleton and that $Z \setminus K$ is closed. We set $(Z \setminus K) \cap \bar{K} = \{q\}$. Since $\bar{K} = K \cup \{q\}$ it follows that $(Z \setminus K) \setminus \{q\}$ is open which implies that q is a cut point of the space Z . Since $q \in Z \setminus K$ it follows that either

$q = f(x)$ or $q \neq f(x)$. If $q = f(x)$ we consider again the component M of y in $X \setminus f^{-1}(C_{f(x)})$, and let $a \in \overline{M} \cap f^{-1}(C_{f(x)})$. Then $f(M) \subseteq K$, the point $f(a)$ is a limit point of K and $f(a) \in C_{f(x)}$. That is $f(a) = q$. But then there exists an open connected neighbourhood U_a of a such that $f(U_a) \subseteq U_{f(x)}$ which implies that $f(U_a) \subseteq C_{f(x)}$. Hence $U_a \subseteq f^{-1}(C_{f(x)})$ which is impossible because $U_a \cap M \neq \emptyset$. If $q \neq f(x)$ then obviously $q \in E(f(x), f(y))$.

Finally, observing that case (3) is reduced to case (1) or (2) we conclude that $E(f(x), f(y)) \neq \emptyset$, which is impossible by Proposition 1.

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