

AN APPLICATION OF FIXED POINT THEOREMS IN BEST APPROXIMATION THEORY

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ABSTRACT. In this paper, we give an application of Jungck's fixed point theorem to best approximation theory, which extends the results of Singh and Sahab et al.

KEY WORDS AND PHRASES: Contractive operator, best approximant, compatible mappings, fixed point.

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Let X be a normed linear space. A mapping $T : X \rightarrow X$ is said to be *contractive* on X (resp., on a subset C of X) if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in X (resp., C). The set of fixed points of T on X is denoted by $F(T)$. If \bar{x} is a point of X , then for $0 < a \leq 1$, we define the set D_a of best (C, a) -approximants to \bar{x} consists of the points y in C such that

$$a\|y - \bar{x}\| = \inf\{\|z - \bar{x}\| : z \in C\}.$$

Let D denote the set of best C -approximants to \bar{x} . For $a = 1$, our definition reduces to the set D of best C -approximants to \bar{x} . A subset C of X is said to be *starshaped* with respect to a point $q \in C$ if, for all x in C and all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)q \in C$. The point p is called the *star-centre* of C . A convex set is starshaped with respect to each of its points, but not conversely. For an example, the set $C = \{0\} \times [0, 1] \cup [1, 0] \times \{0\}$ is starshaped with respect to $(0, 0) \in C$ as the star-centre of C , but it is not convex.

In this paper, we give an application of Jungck's fixed point theorem to best approximation theory, which extends the results of Sahab et al. [9] and Singh [10].

By relaxing the linearity of the operator T and the convexity of D in the original statement of Brosowski [1], Singh [10] proved the following:

Theorem 1. Let C be a T -invariant subset of a normed linear space X . Let $T : C \rightarrow C$ be a contractive operator on C and let $\bar{x} \in F(T)$. If $D \subseteq X$ is nonempty, compact and starshaped, then $D \cap F(T) \neq \emptyset$.

In the subsequent paper [11], Singh observed that only the nonexpansiveness of T on $D' = D \cup \{\bar{x}\}$ is necessary. Further, Hicks and Humphries [4] have shown that the assumption $T : C \rightarrow C$ can be weakened to the condition $T : \partial C \rightarrow C$ if $y \in C$, i.e., $y \in D$ is not necessarily in the interior of C , where ∂C denotes the boundary of C .

Recently, Sahab, Khan and Sessa [9] generalized Theorem 1 as in the following:

Theorem 2. Let X be a Banach space. Let $T, I : X \rightarrow X$ be operators and C be a subset of X such that $T : \partial C \rightarrow C$ and $\bar{x} \in F(T) \cap F(I)$. Further, suppose that T and I satisfy

$$\|Tx - Ty\| \leq \|Ix - Iy\| \quad (1)$$

for all x, y in D' , I is linear, continuous on D and $ITx = TIx$ for all x in D . If D is nonempty, compact and starshaped with respect to a point $q \in F(I)$ and $I(D) = D$, then $D \cap F(T) \cap F(I) \neq \emptyset$.

Recall that two self-maps I and T of a metric space (X, d) with $d(x, y) = \|x - y\|$ for all $x, y \in X$ are said to be *compatible* on X if

$$\lim_{n \rightarrow \infty} d(ITx_n, TIx_n) (= \lim_{n \rightarrow \infty} \|ITx_n - TIx_n\|) = 0$$

whenever there is a sequence $\{x_n\}$ in X such that $Tx_n, Ix_n \rightarrow t$, as $n \rightarrow \infty$, for some t in X ([6]-[8]). We shall use N to denote the set of positive integers and $Cl(S)$ to denote the closure of a set S .

For our main theorem, we need the following:

Proposition 3. [8] Let T and I be compatible self-maps of a metric space (X, d) with I being continuous. Suppose that there exist real numbers $r > 0$ and $a \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq rd(Ix, Iy) + a \max\{d(Tx, Ix), d(Ty, Iy)\}.$$

Then $Tw = Iw$ for some $w \in X$ if and only if $A = \bigcap \{Cl(T(K_n)) : n \in N\} \neq \emptyset$, where for each $n \in N$

$$K_n = \left\{x \in X : d(Tx, Ix) \leq \frac{1}{n}\right\}.$$

On the other hand, using this proposition, Jungck [8] proved the following:

Theorem 4. Let I and T be compatible self-maps of a closed convex subset C of a Banach space X . Suppose that I is continuous and linear with $T(C) \subseteq I(C)$. If there exists an $a \in (0, 1)$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a) \max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \quad (2)$$

then I and T have a unique common fixed point in C .

By using this theorem, we extend Theorem 2 as in the following:

Theorem 5. Let X be a Banach space. Let $T, I : X \rightarrow X$ be operators and C be a subset of X such that $T : \partial C \rightarrow C$ and $\bar{x} \in F(T) \cap F(I)$. Further, suppose that T and I satisfy (2) for all x, y in $D_a = D_a \cup \{\bar{x}\} \cup E$, where $E = \{q \in X : Ix_n, Tx_n \rightarrow q, \{x_n\} \subset D_a\}$, $0 < a < 1$, I is linear, continuous on D_a and T, I are compatible in D_a . If D_a is nonempty, compact and convex, and $I(D_a) = D_a$, then $D_a \cap F(T) \cap F(I) \neq \emptyset$.

Proof. Let $y \in D_a$ and hence Iy is in D_a since $I(D_a) = D_a$. Further, if $y \in \partial C$, then Ty is in C since $T(\partial C) \subseteq C$. From (2), it follows that

$$\begin{aligned} \|Ty - \bar{x}\| &= \|Ty - T\bar{x}\| \\ &\leq a\|Iy - I\bar{x}\| + (1 - a) \max\{\|Ty - Iy\|, \|T\bar{x} - I\bar{x}\|\} \\ &\leq a\|Iy - \bar{x}\| + (1 - a)(\|Ty - \bar{x}\| + \|Iy - \bar{x}\|), \end{aligned}$$

which implies $a\|Ty - \bar{x}\| \leq \|Iy - \bar{x}\|$ and so Ty is in D_a . Thus T maps D_a into itself.

By hypothesis, we have $\bar{x} = T\bar{x} = I\bar{x}$. Then Proposition 3 implies that

$$A = \bigcap \{Cl(T(K_n)) : n \in N\} \neq \emptyset.$$

Suppose that $w \in A$. Then for each $n \in N$, there exists $y_n \in T(K_n)$ such that $d(w, y_n) < 1/n$. Consequently, for such n , we can and do choose $x_n \in K_n$ such that $d(w, Tx_n) < 1/n$ and so $Tx_n \rightarrow w$. But since $x_n \in K_n$, $d(Tx_n, Ix_n) < 1/n$ and therefore $Ix_n \rightarrow w$. Thus we have

$$\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Tx_n = w. \quad (3)$$

Therefore, for a sequence $\{x_n\}$ in D_a the existence of (3) is guaranteed whenever $D_a \subset K_n$. Moreover, $w \in E$. Since I and T are compatible and I is continuous, we have $\lim_{n \rightarrow \infty} TIx_n = Iw$ and $\lim_{n \rightarrow \infty} I^2x_n = Iw$. By (2), we have

$$\|TIx_n - \bar{x}\| = \|TIx_n - T\bar{x}\| \leq a\|I^2x_n - I\bar{x}\| + (1 - a)\max\{\|TIx_n - I^2x_n\|, \|T\bar{x} - I\bar{x}\|\},$$

which implies, as $n \rightarrow \infty$,

$$\|Iw - \bar{x}\| \leq a\|Iw - \bar{x}\|.$$

Hence $Iw = \bar{x}$. By (2) again, we have

$$\|Tw - \bar{x}\| = \|Tw - T\bar{x}\| \leq a\|Iw - I\bar{x}\| + (1 - a)\max\{\|Tw - Iw\|, \|T\bar{x} - I\bar{x}\|\},$$

which gives $\|Tw - \bar{x}\| \leq (1 - a)\|Tw - \bar{x}\|$, and so $Tw = \bar{x}$.

Next, we consider

$$\|Tw - Tx_n\| \leq a\|Iw - Ix_n\| + (1 - a)\max\{\|Tw - Iw\|, \|Tx_n - Ix_n\|\},$$

which gives $\|\bar{x} - w\| \leq a\|\bar{x} - w\|$ as $n \rightarrow \infty$, and so $\bar{x} = w$, i.e., $w = Iw = Tw$. By Theorem 4, w must be unique. Hence $E = \{w\}$. Then $D'_a = D_a \cup \{w\} = D'_a$

Let $\{k_n\}$ be a monotonically non-decreasing sequence of real numbers such that $0 \leq k_n < 1$ and $\overline{\lim}_{n \rightarrow \infty} k_n = 1$. Let $\{x_j\}$ be a sequence in D'_a satisfying (3). For each $n \in N$, define a mapping $T_n : D'_a \rightarrow D'_a$ by

$$T_n x_j = k_n T x_j + (1 - k_n)p. \tag{4}$$

It is possible to define such a mapping T_n for each $n \in N$ since D'_a is starshaped with respect to $p \in F(I)$.

Since I is linear, we have

$$T_n I x_j = k_n T I x_j + (1 - k_n)p, \quad I T_n x_j = k_n I T x_j + (1 - k_n)p.$$

By compatibility of I and T , we have for each $n \in N$,

$$\begin{aligned} 0 &\leq \lim_{j \rightarrow \infty} \|T_n I x_j - I T_n x_j\| \\ &\leq k_n \lim_{j \rightarrow \infty} \|T I x_j - I T x_j\| + \lim_{j \rightarrow \infty} (1 - k_n) \|p - p\| \\ &= 0 \end{aligned}$$

and so

$$\lim_{j \rightarrow \infty} \|T_n I x_j - I T_n x_j\| = 0$$

whenever $\lim_{j \rightarrow \infty} I x_j = \lim_{j \rightarrow \infty} T_n x_j = w$ since we have

$$\begin{aligned} \lim_{j \rightarrow \infty} T_n x_j &= k_n \lim_{j \rightarrow \infty} T x_j + (1 - k_n)w \\ &= k_n w + (1 - k_n)w \\ &= w. \end{aligned}$$

Thus, I and T_n are compatible on D'_a for each n and $T_n(D'_a) \subset D'_a = I(D'_a)$.

On the other hand, by (2), for all $x, y \in D'_a$, we have, for all $j \geq n$ and n fixed,

$$\begin{aligned} \|T_n x - T_n y\| &= k_n \|Tx - Ty\| \leq k_j \|Tx - Ty\| < \|Tx - Ty\| \\ &\leq a\|Ix - Iy\| + (1 - a)\max\{\|Tx - Ix\|, \|Ty - Iy\|\} \\ &\leq a\|Ix - Iy\| + (1 - a)\max\{\|Tx - T_n x\| + \|T_n x - Ix\|, \\ &\quad \|Ty - T_n y\| + \|T_n y - Iy\|\} \\ &\leq a\|Ix - Iy\| + (1 - a)\max\{(1 - k_n)\|Tx - p\| + \|T_n x - Ix\|, \\ &\quad (1 - k_n)\|Ty - p\| + \|T_n y - Iy\|\}. \end{aligned}$$

Hence for all $j \geq n$, we have

$$\begin{aligned} \|T_n x - T_n y\| &< a\|Ix - Iy\| + (1-a)\max\{(1-k_j)\|Tx - p\| \\ &\quad + \|T_n x - Ix\|, (1-k_j)\|Ty - p\| + \|T_n y - Iy\|\} \end{aligned} \quad (5)$$

Thus, since $\overline{\lim}_{j \rightarrow \infty} k_j = 1$, from (5), for every $n \in N$, we have

$$\begin{aligned} \|T_n x - T_n y\| &= \overline{\lim}_{j \rightarrow \infty} a\|T_n x - T_n y\| \\ &< \overline{\lim}_{j \rightarrow \infty} [a\|Ix - Iy\| + (1-a)\max\{(1-k_j)\|Tx - p\| \\ &\quad + \|T_n x - Ix\|, (1-k_j)\|Ty - p\| + \|T_n y - Iy\|\}], \end{aligned}$$

which implies

$$\|T_n x - T_n y\| = a\|Ix - Iy\| + (1-a)\max\{\|T_n x - Ix\|, \|T_n y - Iy\|\}$$

for all $x, y \in D'_a$. Therefore, by Theorem 4, for every $n \in N$, T_n and I have a unique common fixed point x_n in D'_a , i.e., for every $n \in N$, we have

$$F(T_n) \cap F(I) = \{x_n\}.$$

Now, the compactness of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point z in D_a . Since

$$x_{n_i} = T_{n_i} x_{n_i} = k_{n_i} T x_{n_i} + (1 - k_{n_i}) z \quad (6)$$

and T is continuous, we have, as $i \rightarrow \infty$ in (6), $z = Tz$, i.e., $z \in D_a \cap F(T)$.

Further, the continuity of I implies that

$$Iz = I(\lim_{i \rightarrow \infty} x_{n_i}) = \lim_{i \rightarrow \infty} Ix_{n_i} = \lim_{i \rightarrow \infty} x_{n_i} = z,$$

i.e., $z \in F(I)$. Therefore, we have $z \in D_a \cap F(T) \cap F(I)$ and so

$$D_a \cap F(T) \cap F(I) \neq \emptyset.$$

This completes the proof.

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